# AVERAGE LINKING NUMBERS

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To the memory of Ricardo Mañé.

## 1. Introduction.

Let [1] $\gamma_1, \gamma_2$ be two disjoint oriented knots in the three dimensional sphere $S^3$ the linking number $\ell(\gamma_1, \gamma_2)$ is defined, as the algebraic intersection number of $\gamma_1$ with an orientable surface $N$ transversal to $\gamma_1$, with oriented boundary $\gamma_2$. For better understanding of our results we need the two following interpretations of the linking number (See [R]):

1) For any regular planar projection it is the algebraic crossing number of $\gamma_1$ *over* $\gamma_2$.

2) It is the degree of the map

$$T : T^2 \to S^2$$
$$(t_1, t_2) \to \frac{\gamma_1(t_1) - \gamma_2(t_2)}{|\gamma_1(t_1) - \gamma_2(t_2)|}$$

where $T^2$ is the two dimensional torus and $S^2$ is the two dimensional sphere. Hence

$$\ell(\gamma_1, \gamma_2) = \frac{1}{\mathrm{vol}(S^2)} \int_{S^1 \times S^1} det(DT) ds dt$$
$$= \frac{1}{4\pi} \int_0^{t_1} \int_0^{t_2} \frac{(\gamma_1' \times \gamma_2')(\gamma_1 - \gamma_2)}{|\gamma_1 - \gamma_2|^3}$$

where $\gamma_i : [0, t_i] \to S^3$ is a parametrization of $\gamma_i$; $i = 1, 2$. This formula is known as the Gauss Formula.

Given a flow $\varphi_t$ generated by a vector field $F$, and two closed orbits $\gamma_1, \gamma_2$ we define the average linking number as $\frac{1}{T_1 T_2} \ell(\gamma_1, \gamma_2)$ where $T_i$ is the period of $\gamma_i$. In order to define the average linking number for general orbits we need a convinient set of short curves.

Define a good set of short curves as a system of piecewise differentiable paths joining points $x, y \in S^3$, depending in a measurable way on $x$ and $y$ such that the Gauss integrals of every pair of nonintersecting pairs of the system, and the Gauss integrals of every pair of nonintersecting { paths of the system, segments of orbits $\varphi_t(p)$, $0 \le t \le 1$ },

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are bounded independently of the paths by a constant $k$ (see [A]). In section 3 we will prove the existence of such a set.

For any $x \in S^3$ and $T \in R$ let $\hat{x}_T$ denote the knot formed by the orbit from$x$ to $\varphi_T(x)$ and then close it by a path $\alpha_{\varphi_T(x),x}$ joining $x$ to $T(x)$ of a good set of short curves. For any two points in different orbits $p, q$ and times $T_1, T_2$ define, when possible, the average linking number as

$$\ell(x,y) := \lim_{T_1,\,T_2\to\infty} \frac{1}{T_1T_2}\ell(\hat{x}_T, \hat{y}_T)$$

The main pourpose of this paper is to decide when does the limit when $T_1$, $T_2$ tend to infinity exists and does not depend on the set of short curves. Our main Theorem is

**Theorem 1.** *Let $\mu, \nu$ be two invariant measures on $S^3$. Then the function $J : S^3 \times S^3 \to R$ defined by*

$$J(p,q) = \frac{F(p) \times F(q) \cdot (p-q)}{|p-q|^3}$$

*is in $L^1(\mu \times \nu)$*

From Birkhoff's Theorem we obtain the following

**Theorem 2.** *a) For $\mu \times \nu$ almost every pair of points the limit*

$$\hat{\ell}(x,y) = \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2} \int_0^{T_1}\int_0^{T_2} \frac{(\gamma_1' \times \gamma_2', \gamma_1 - \gamma_2)}{|\gamma_1 - \gamma_2|^3}$$

*exist.*

*b) If there is no periodic orbit with positive $\mu$ and $\nu$ measure this limit is for $\mu \times \nu$-almost every pair the same as $\ell(x,y)$, in particular the asymptotic linking number does not depend on the choice of the set of short curves. If the measures $\mu$ and $\nu$ are ergodic, the asymptotic linking $\ell(x,y)$ is given by the integral $\int\int J(p,q)\, d\mu(p)\, d\nu(q)$.*

E. Ghys and J.M. Gambaudo have anounced a proof that the average linking is a topological invariant for volume preserving suspension flows. The same assertion of this theorem was proven by Arnold in [A] for smooth measures and in [C] for Gibbs measures. The proof in [A] relays on the fact that the singularity of the formula has order 2 which is less than the codimension of the diagonal, the singular set of $J$. The proof in [C] relays on the observation that the singularity of the Gauss formula has actually order 1 and that the Hausdorff dimension of a Gibbs measure is greater than 1.

The idea of the proof is that the singularity of the Gauss formula disappears when we integrate along the orbits in a small flow box. This is because the integral of the Gauss formula is the integral of the

Jacobian of $T$ therefore in small time intervals the area of the image of $T$ should be contained in one half of the sphere. However since we actually want bound the integral of the absolute value of the Jacobian of $T$ it was easier to carry on an analitical proof, rather than to worry about of regions of injectivity of $T$ and the sign of its Jacobian.

Finally, as an application, in [VV] Verjovsky and Vila following the same philosophy extended other topological invariants, such as Witten's invariant of links, to an average value for an invariant smooth measure of a flow. In their paper, the only obstruction to define the average Witten's invariant for singular invariant measures is the existence of the average linking number for such a measure, which is proven here on theorem 1. We also quote the work of Freedman and He [FH] where they relate the asymptotic crossing number (the integral of $|J(x,y)|$) with the energy of a vector field with a smooth invariant measure.

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## 2. **Proof of the Theorems.**

**Lemma 2.1.** *For any compact subset $K \subset R^3$, there exist $\varepsilon > 0$, $A > 0$, such that if $p \in K$, $|x-p| < \varepsilon$, $|y-p| < \varepsilon$, then*

$$J(x,y) = \frac{H^T \, \mathbf{B} \, H}{|H|^3} + \phi(x,y)$$

*where $|\phi(x,y)| < A$ is bounded, $H = y - x$ and $\mathbf{B}$ is the matrix defined by $\mathbf{B} \cdot H = F(p) \times \Big(DF(p) \cdot H\Big)$ for all $H \in \mathbf{R}^3$.*

**Proof.**
We have that

$$J(x,y) = -\frac{F(x) \times F(y)}{|H|} \cdot \frac{H}{|H|^3},$$

where $y = x + H$. Since $F(y) = F(x) + DF(x) \cdot H + \psi(x,H)$, with $|\psi(x,H)| \leq a \, |H|^2$, uniformly on all $p \in K$, $|x-p| < \varepsilon$, then

$$\begin{aligned} J(x,y)\,|H|^3 &= H \cdot \Big[F(x) \times \Big(F(x) + DF(x) \cdot H + \psi(x,H)\Big)\Big] \\ &= H \cdot F \times DF(x) \cdot H + H \cdot F(x) \times \psi(x,H) \end{aligned}$$

but if $A := a \sup_{d(x,K)\le\varepsilon} |F(x)|$,

$$\frac{H\cdot F(x)\times\psi(x,H)}{|H|^3} \le a\ |F(x)| \le A \qquad \square$$

Given $p\in\mathbf{R}^3$ and $\varepsilon>0$, let $\mathbf{E}(p)$ and $\mathbf{N}(p)$ be defined by

$$\begin{aligned}\mathbf{E}(p)&=\mathbf{E}(p,\varepsilon)=\{\ q\in\mathbf{R}^3\mid(q-p)\cdot F(p)=0\,,\ |q-p|<\varepsilon\ \}\ ,\\ \mathbf{N}(p)&=\mathbf{N}(p,\varepsilon)=\bigcup_{t\in[-\varepsilon,\varepsilon]}\varphi_t\Big(\mathbf{E}(p)\Big)\ .\end{aligned}$$

For $s,t\in\mathbf{R}$ and $q\in\mathbf{E}(p)$, write $p_s=\varphi_s(p)$, $q_t=\varphi_t(q)$. Given $q\in\mathbf{E}(p)$ let $\tau:[-\varepsilon,\varepsilon]\to\mathbf{R}$ be such that $q_{\tau(s)}\in\mathbf{E}(p_s,1)$, i.e. $(q_{\tau(s)}-p_s)\cdot F(p_s)=0$.

**Lemma 2.2.** *For any compact subset $K\subset\mathbf{R}^3$, there exist $\varepsilon>0$, $M>0$, such that if $p\in K$, $q\in\mathbf{E}(p,\varepsilon)$, $|s|$ , $|t|<\varepsilon$, then*

$$J(p_s,q_t)=\frac{Q_s\,\mathbf{B}_s\,Q_s+\Big(t-\tau(s)\Big)\,Q_s\,(F_s\times\mathbf{A}_s\,F_s)}{|H_{st}|^3}+\psi(p,q,s,t)$$

*where $|\psi(p,q,s,t)|\le M$, $F_s=F(p_s)$, $\mathbf{A}_s=DF(p_s)$, $H_{st}=q_t-p_s$, $Q_s=q_{\tau(s)}-p_s$ and $\mathbf{B}_s$ is the matrix defined by $\mathbf{B}_s\,H=F_s\times(\mathbf{A}_s\,H)$ for all $H\in\mathbf{R}^3$.*

**Proof.**
We have that $q_{\tau(s)}=p_s+Q_s$, and

$$\begin{aligned}F\Big(q_{\tau(s)}\Big)\ &=F(p_s)+DF(p_s)\,Q_s+G_1(p,q,s)\ |Q_s|^2\ ,\ \ |G_1(p,q,s)|<a_1\ ,\ \text{for some } a_1>0,\\ q_t-p_s\ &=Q_s+\Big(q_t-q_{\tau(s)}\Big)=Q_s+\Big(t-\tau(s)\Big)\,F\Big(q_{\tau(s)}\Big)+G_2\ |t-\tau(s)|^2\ ,\ \ |G_2|<a_2\\ H_{st}\ &=Q_s+\Big(t-\tau(s)\Big)\,F(p_s)+G_3\,\Big(|t-\tau(s)|^2+|t-\tau(s)|\ |Q_s|\Big)\,,\ \ |G_3|<a_3\,.\end{aligned}$$

Observe that, since $Q_s\perp F(p_s)$, writing $\Delta t:=t-\tau(s)$,

$$\begin{aligned}|H_{st}|^2\ &\ge|Q_{st}|^2+|\Delta t|^2\ |F_s|^2-a_3^2\ \Big(|\Delta t|^2+|\Delta t|\ |Q_s|\Big)^2\\ &\ge|Q_s|^2+\tfrac12\ |\Delta t|^2\ |F_s|^2\\ [\star]\qquad |H_{st}|^2\ &\ge\tfrac12\ \Big[|Q_s|^2+|\Delta t|^2\ |F_s|^2\Big]\qquad ,\end{aligned}$$

if $\varepsilon > 0$ is small enough. Therefore $|Q_s| \le |H_{st}|$, and $|\Delta t| \le \alpha\, |H_{st}|$, for some $\alpha = \alpha(K,\varepsilon) > 0$. Now

$$\begin{aligned} H_{st}\,\mathbf{B}_s\,H_{st} &= \left(Q_s + \Delta t\,F_s + a_3\left(|\Delta t|^2 + |\Delta t|\ |Q_s|\right)\right)\cdot\left(F_s \times \mathbf{A}_s\,H_{st}\right) \\ &= Q_s\,F_s \times \mathbf{A}_s\,H_{st} + a_3\,\mathcal{O}\left(|H_{st}|^2\right)\ \mathcal{O}\left(|H_{st}|\right) \\ &= Q_s\,F_s \times \mathbf{A}_s\left(Q_s + (\Delta t)\,F_s + \mathcal{O}\left(|H_{st}|^2\right)\right) + \mathcal{O}\left(|H_{st}|^3\right) \\ &= Q_s\,\mathbf{B}_s\,Q_s + Q_s\left(F_s \times \mathbf{A}_s\,F_s\right)\Delta t + \mathcal{O}\left(|H_{st}|^3\right)\ . \end{aligned}$$

By lemma 2.1, we have that

$$J(p_s,q_t) = \frac{H_{st}\,\mathbf{B}_s\,H_{st}}{|H_{st}|^3} + \phi(p_s,q_t)$$

with $|\phi(p_s,s,t)| < A$. This completes the proof of the lemma. $\square$

**Lemma 2.3.** *For any compact subset $K \subset R^3$ there exist $\epsilon > 0$, $a,b > 0$, such that if $q \in E(p,\epsilon)$, $|s|,|t| < \epsilon$ then*

$$a|Q_0| \le |Q_s| \le b|Q_0|$$

*where $Q_s$ is defined as in Lemma 2.*

**Proof.**

Define $\tau : \cup_{p\in K}(\{p\}\times\mathbf{E}(p,\epsilon)\times[-\epsilon,\epsilon]) \to R$ by $q_{\tau(p,q,s)} \in \mathbf{E}(p_s,1)$.We first show that $\tau$ is differentiable on $q$. By definition of $\tau$, we have that

$$G(p,q,\tau(p,q,s)) := F(p_s)\cdot(\varphi(q,\tau(p,q,s)) - p_s) = 0,$$

since

$$\frac{\partial G}{\partial \tau} = F(p_s)\cdot\frac{\partial G}{\partial \tau}|_{(q,\tau)} = F(p_s)\cdot F(q_\tau) > 0,$$

by the implicit function theorem we have that $\tau$ is differentiable respect to $q$ and

$$\frac{\partial \tau}{\partial q} = \frac{-F(p_s)\cdot\frac{\partial\varphi}{\partial q}|_{(q,\tau)}}{F(p_s)\cdot F(q_\tau)}.$$

Moreover, there exists $C = C(K,\epsilon) > 0$ such that $|\frac{\partial\tau}{\partial q}| < C$. We have that

$$Q_s = q_{\tau(s)} - p_s = (q_{\tau(s)} - p_s) + (q_s - p_s).$$

Since $q_{\tau(s)} - q_s = (\tau(s) - s)\ F(q_\sigma)$ for some $\sigma$ between $s$ and $\tau(s)$, we have that

$$|\,q_{\tau(s)} - q_s| \le C\ |q-p| \max_q |F(q)| \le B|Q_0|.$$

For $B := C\max\limits_q |F(q)|$. By Gronwall's inequality

$$|q_s - p_s| \le D\ |q-p| = D\ |Q_0|,$$

for some uniform $D = D(K, \epsilon)$. Therefore if $b = C + D$, we have that

$$|Q_s| \le (C + D)\ |Q_0| = b\ |Q_0|.$$

The other inequality is obtained from this one by changing the roles of $p$ and $p_s$ and reversing the time; (observe that $\varphi(q_s, \tau(p_s, q_s, -s) = q)$. Therefore

$$Q_0 \le b\ Q_s.$$

Now take $a = \frac{1}{b}$. $\square$

**Lemma 2.4.** *For any compact subset $K \subset R$ there exist $\varepsilon > 0, M > 0$ such that if $p$ is in $K$ and $q \in \mathbf{E}(p, \varepsilon)$ then*

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} J(p_s, q_t)\, ds\, dt < M\varepsilon$$

**Proof.**

We first prove the case when $p \ne q$. By lemma 2, we have that

$$J(p_s, q_t) = \frac{Q_s\, \mathbf{B}_s\, Q_s}{|H_{st}|^3} + \frac{\Delta t\ Q_s\, (F_s \times \mathbf{A}_s\, F_s)}{|H_{st}|^3} + \psi(p, q, s, t)\ ,$$

with $\psi(p, q, s, t) \le M_1$. We bound the integral of each term.

From lemma 3 and [$\star$], we have that

$$a\ |Q_0| \le |Q_s| \le b\ |Q_0|$$

$$|H_{st}|^2 \ge \frac{1}{2}\left[|Q_s|^2 + |\Delta t|^2 |F_s|^2\right] \ge \frac{1}{2}\left[|Q_s|^2 + |\Delta t|^2 D\right]$$

where $D = \min\limits_{|q-p|<\varepsilon} |F(q)| > 0$. Writing $\alpha = \frac{1}{4}a$ and $\beta = \frac{1}{2}\sqrt{D}$, we have

$$|H_{st}|^2 \ge \frac{1}{2}(a^2|Q_0|^2 + |\Delta t|^2 D),$$

$$|H_{st}| \ge (\alpha^2|Q_0|^2 + \beta^2|\Delta t|^2)^{\frac{1}{2}}.$$

Since $\mathbf{B}_s H = F(p_s) \times (DF(p_s) \cdot H)$, let $B = \max\limits_{p_s} \|\mathbf{B}_s\|$, then

$$\frac{|Q_s B_s Q_s|}{|H_{st}|^3} \le \frac{B\ |Q_s|^2}{|H_{st}|^3} \le \frac{B\, b^2 |Q_0|^2}{(\alpha^2|Q_0|^2 + \beta^2|\Delta t|^2)^{\frac{3}{2}}} = \frac{B\, b^2 |Q_0|^2}{\beta(\ \frac{\alpha^2}{\beta^2}\ |Q_0|^2 + |\Delta t|^2)^{\frac{3}{2}}}$$

Observe that, since $\Delta t = t - \tau(s)$,

$$\int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} |J|\ \ dt\ \ ds = \int\limits_{-\varepsilon}^{\varepsilon} \int\limits_{-\varepsilon-\tau(s)}^{\varepsilon-\tau(s)} |J|\ \ d(\Delta t)\ ds$$

Since $\int \frac{dx}{r^3} = \frac{1}{a^2}\frac{x}{r}$, $r = \sqrt{x^2 + a^2}$ we have that

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{Q_s\, \mathbf{B}_s\, Q_s}{|H_{st}|^3}\, dt\, ds \le \int_{-\varepsilon}^{\varepsilon} 2 \int_0^{2\varepsilon} \frac{B\, b^2\ |Q_0|^2}{\beta \sqrt{\frac{\alpha^2}{\beta^2}\ |Q_0|^2 + |\Delta t|^2}}\ d(\Delta t)\ \ ds$$

$$\leq \frac{2\,\varepsilon\, 2\, B\, b^2\; |Q_0|^2}{\frac{\alpha^2}{\beta}\, |Q_0|^2}\; \frac{\Delta t}{\sqrt{\frac{\alpha^2}{\beta^2}\,|Q_0|^2 + |\Delta t|^2}}\Big|_0^{2\varepsilon}$$

$$\leq \frac{4\,\varepsilon\, B\, b^2\,\beta}{\alpha^2}(1+1) = \frac{8\,\varepsilon\;\, B\, b^2\,\beta}{\alpha^2} = \varepsilon M_2$$

Now we bound the integral of the second term. Let $A := \max_{p_s} \Big| F(p_s) \times \Big( DF(p_s)\cdot F(p_s)\Big)\Big|$. Then, since $\int \frac{x}{r^3}dx = -\frac{1}{r}$, $r = \sqrt{x^2+a^2}$, we have that

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{(t-\tau(s))\, Q_s\,(F_s \times \mathbf{A}_s F_s)}{|H_{st}|^3}\; dt\, ds \leq \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{|\Delta t|\, b\, |Q_0|\, A}{\left(\sqrt{\alpha^2\,|Q_0|^2 + \beta^2\,|\Delta t|^2}\right)^3}\; dt\, ds$$

$$\leq \int_{-\varepsilon}^{\varepsilon}\; 2\int_0^{2\,\varepsilon} \frac{b\,|Q_0|\,A\,|\Delta t|}{\left(\beta\sqrt{\frac{\alpha^2}{\beta^2}\,|Q_0|^2 +\; |\Delta t|^2}\right)^3}\;\; d(\Delta t)\; ds$$

$$\leq \frac{4\,\varepsilon\, b\, A}{\beta}\;\, |Q_0| \int_0^{2\,\varepsilon} \frac{|\Delta t|\;\; d(\Delta t)}{\left(\sqrt{\frac{\alpha^2}{\beta^2}\,|Q_0|^2 +\; |\Delta t|^2}\right)^3}$$

$$\leq \frac{4\,\varepsilon\, b\, A}{\beta^3}|Q_0|\; \frac{1}{\sqrt{\frac{\alpha^2}{\beta^2}\,|Q_0|^2 +\; |\Delta t|^2}}\Bigg|_{2\varepsilon}^{0}$$

$$\leq \frac{4\,\varepsilon\, b\, A}{\beta^3}\;\, \frac{|Q_0|}{\frac{\alpha}{\beta}\,|Q_0|} \leq \frac{4\;\, b\, A}{\alpha\,\beta^2}\;\varepsilon = \varepsilon\;\, M_3\,.$$

From lemma 2, $|\psi(p,q,s,t)| \leq M_1$, therefrore, if $\varepsilon < 1$,

$$\int_{-\varepsilon}^{\varepsilon}\;\int_{-\varepsilon}^{\varepsilon} |J(p_s,q_t)|\;\; ds\;\; dt \leq \varepsilon\, M_2 + \varepsilon\, M_3 + 4\,\varepsilon^2\, M_1 \leq \varepsilon\, M_4\,,$$

where $M_4 := M_1 + M_2 + M_3$ .

Now suppose that $p = q$, we have that

$$\begin{array}{ll} F(q_s) & = F(q) + s\,\mathbf{A}\,F(q) + s^2\,K(q,s) \\ q_s & = q + s\,F(q) + \frac{s^2}{2}\,\mathbf{A}\,F(q) + s^3\,G(q,s) \end{array}$$

where $\mathbf{A} = DF(q)$, $K(q,s)$, and $G(q,s)$ are bounded. Then

$$\begin{array}{ll} |q_s - q|^3\;\; J(q,q_s) & = F(q) \times (F(q) + s\,\mathbf{A}\,F(q) + s^2\,K)\cdot \Big(s\,F(q) + \frac{s^2}{2}\,\mathbf{A}F(q) + s^3\,G\Big) \\ & = (s\,F \times \mathbf{A}\,F + s^2\,F\times K)\cdot \Big(s\,F + \frac{s^2}{2}\,\mathbf{A}F + s^3\,G\Big) \\ & = s^4\,F\times \mathbf{A}\,F\cdot G + \frac{s^4}{2}\,F\times K\cdot \mathbf{A}\,F + s^5\,F\times K\cdot G \\ |q_s - q|^3\;\; |J(q,q_s)| & \leq s^4\;\, D\;, \end{array}$$

for some $D = D(K,\varepsilon) > 0$, for all $q \in K$, $|s| < \varepsilon$. Furthermore,

$$|q_s - q| = \left| s\,F(q) + \frac{s^2}{2}\,\mathbf{A}F(q) + s^3\,G\right| \geq a\,s\,,$$

for $a = \frac{1}{2} \min_{q \in K} |F(q)|$ if $\varepsilon = \varepsilon(K) > 0$ is small enough. Therefore

$$|J(q, q_s)| \le \frac{s^4\, D}{a^3\, s^3} \le \frac{\varepsilon\, D}{a^3}\,,$$

and

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} J(p_s, p_t) \;\; ds \;\; dt \le \frac{\varepsilon^3\, D}{a^3} \le \varepsilon\; M_5 \quad \text{for all} \quad p \in K,$$

for $M_5 := \frac{D}{a^3}$ . $\square$

**Corollary 2.5.** *For any compact subset* $K \subset R$ *there exist* $\delta, \varepsilon > 0, M > 0$ *such that if* $p, q$ *are in* $K$.

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} J(p_s, q_t) ds dt < M\varepsilon$$

**Proof.**

Take $\epsilon$ as in the previous lemma. As the vector field never vanishes $\mathbf{N}(p, \delta)$ is a neighbourhood of , $p$ hence it contains a ball of radius $r$. Moreover this $r$ may be taken independent of the point $p$. For any two points $p, q$ if $\delta$ is small enough then either:

There exist $s_0, t_0$ with $|s_0|, |t_0| < 2\delta$ such that $q_{s_0}$ is in $\mathbf{E}(p_{t_0})$ and in this case

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} J(p_s, q_t) ds dt < \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} J(p_s, q_t) ds dt < M\varepsilon$$

Or the distance between $q_s$ and $p_t$ with $|s_0|, |t_0| < \delta$ is bounded by $r$and hence the integral is bounded. $\square$

**Proof Theorem 1**

Since $J$ is bounded outside of a neighbourhood of the diagonal $\Delta := \{(p, p) \mid p \in S^3\}$, it is enough to prove that $J$ is in $L^1(\mu \times \nu)$ on a neighbourhood of $\Delta$. Since the compact set $\Delta$ may be covered by a finite number of open sets

$$\mathbf{V}(p, \delta) := \{(q_s^1, q_t^2) : (q^1, q^2) \in \mathbf{E}(p, \delta) \times \mathbf{E}(p, \delta) |s|, |t| < \delta\}$$

with $p \in S^3$ and $\delta > 0$ from corollary 5. It is enough then to prove that the restriction $J|_{\mathbf{V}(p,\delta)}$ is in $L^1(\mu \times \nu)$. Let $\hat{\mu}$,$\hat{\nu}$ be the transversal measures on $E(p, \varepsilon)$ defined by $\hat{\mu}(A) = \mu(\cup_{|t|<\varepsilon} \phi_t(A))$, $\hat{\nu}(A) = \nu(\cup_{|t|<\varepsilon} \phi_t(A))$.Then by lemma 4 we have that

$$\int_{V(q,\delta)} |J(x, y)|\, d\mu(x)\, d\nu(y) = \int_{E(q,\delta)} \int_{E(q,\delta)} \left( \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} |J(p_s, y_t)|\, d\hat{\mu}(p)\, d\hat{\nu}(y) \right)$$

$$\le \int_{E(q,\delta)} \int_{E(q,\delta)} M\, \delta\, d\mu'(p)\, d\nu(y) \le M\, \delta\,. \qquad \square$$

**Proof of Theorem 2:** Part (a) is a direct consequence of the Birkhoff theorem and Theorem 1. Part (b) is a consequence of the existence of a good set of short curves because

$$\begin{aligned}\left|\int_0^{T_1+1}\int_0^{T_2+1}-\int_0^{T_1}\int_0^{T_2}\right| &\leq \left|\int_0^{T_1}\int_{T_2}^{T_2+1}\right|+\left|\int_{T_1}^{T_1+1}\int_0^{T_2}\right|+\left|\int_0^1\int_0^1\right| \\ &\leq k\,T_1+k\,T_2+k\end{aligned}$$

and hence

$$\lim_{T_1,T_2\to\infty}\frac{1}{T_1\,T_2}\int_0^{T_1+1}\int_0^{T_2+1}=\lim_{T_1,T_2\to\infty}\frac{1}{T_1\,T_2}\int_0^{T_1}\int_0^{T_2}$$

## 3. Existence of a good set of short curves.

The following lemmas prove the existence of a good set of short curves if they are poligonal curves made of small straight line segments which are always transversal to the direction of the flow.

**Lemma 3.1.** *Let $L(t)=a+t\,b$ be a small line segment in $\mathbf{R}^3$ and let $\gamma(s)$ be a small differentiable curve such that*

*(i) For any $s,\,t\in[-\varepsilon,\varepsilon]$, the vectors $L'(t)=b$, $\gamma'(s)$ and $\gamma(s)-L(t)$ are not coplanar.*

*(ii) The vectors $\frac{\gamma(s)-L(t)}{\|\gamma(s)-L(t)\|}$ are always in a hemisphere of $S^2$.*

*Then the Gauss (absolute) integral*

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}|I(s,t)|\;ds\;dt<\frac{1}{2}\;vol\;(S^2)\,,$$

*where*

$$I(s,t):=\frac{(\gamma'(s)\times L'(t))\cdot(\gamma(s)-L(t))}{|\gamma(s)-L(t)|}\;.$$

**Proof.**

The integral $I(s,t)$ is the Jacobian of the function $T(s,t)=\frac{\gamma(s)-L(t)}{|\gamma(s)-L(t)|}$. The condition (i) implies that $I(s,t)$ has always the same sign. We will see that $T:[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]\to S^2$ is injective and hence

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}|I(s,t)|\;ds\,dt=\text{area }\;(\Gamma\left([-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]\right))<\frac{1}{2}\;\text{vol}\left(S^2\right)\;.$$

We show that $T$ is injective. By the transversality condition (i), if $s_1\neq s_2$, then the planes generated by ($\gamma(s_1)$ and $L$) and ($\gamma(s_2)$ and $L$) are different. Therefore if $\gamma(s_1)-L(t_1)$ is parallel to $\gamma(s_2)-L(t_2)$ then $s_1=s_2$. If $\gamma(\hat{s})-L(t_1)=\lambda\;(\gamma(\hat{s})-L(t_2))$ then $L(t_1)=\lambda\,L(t_2)$ and $t_1=t_2$. $\square$

**Lemma 3.2.** *Let $F$ be a Lipschitz nonsingular vectorfield on $\mathbf{R}^3$ and let $A_1$, $A_2$, $A_3 > 0$ be such that*

$$|F(x) - F(y)| \leq A_1 \, |x - y| \,, \quad A_2 < |F(x)| < A_3 \quad \textit{for all } x, y \in \mathbf{R}^3 \,.$$

*Then there exists $\varepsilon = \varepsilon(A_1, A_2, A_3) > 0$ such that if $L(t) = a + t\,\mathbf{b}$, $\|\mathbf{b}\| = 1$, $|t| < 1$ is a small line segment and $\gamma(s)$, $|s| < \varepsilon$ is a small orbit segment of $F$ such that*

*(i) The angles $\sphericalangle(\gamma'(s), \mathbf{b}) > \frac{\pi}{3}$, $\quad \sphericalangle(\gamma'(s), -\mathbf{b}) > \frac{\pi}{3}$ for all $|s| < \varepsilon$.*

*(ii) The vectors $L'(0) = b$, $\gamma'(0)$ and $\gamma(0) - L(0)$ are not coplanar.*

*Then the Gauss (absolute) integral*

$$\int_{-1}^{1} \int_{-\varepsilon}^{\varepsilon} |I(s,t)| \; ds\, dt < 2 \;\; \textit{vol}(S^2) \,.$$

**Proof.**

It is enough to show that $\int_{-1}^{1} \int_0^{\varepsilon} |I| <$ vol$(S^2)$. Let $g(s) := (\gamma'(s) \times L'(t)) \cdot (\gamma(s) - L(t)) = (\gamma'(s) \times \mathbf{b}) \cdot (\gamma(s) - \mathbf{a})$, and let

$$\begin{aligned}
s_0 \; &:= \min\{\, s > 0 &&|\, g(s) = 0 \,\} \\
t_0 \; &:= \max\{\, s > s_0 &&|\, g(s) = 0 \text{ for all } s_0 \leq s \leq t_0 \,\} \,. \\
s_{n+1} \; &:= \min\{\, s > t_n &&|\, g(s) = 0 \,\} \\
t_{n+1} \; &:= \max\{\, s > s_{n+1} &&|\, g(s) = 0 \text{ for all } s_{n+1} \leq s \leq t_{n+1} \,\} \,.
\end{aligned}$$

By lemma 3.1,

$$\int_{-1}^{1} \int_{0}^{s_0} |I(s,t)| \, ds\, dt < \frac{1}{2} \operatorname{vol}\left(s^2\right) \,.$$

Let $\mathbf{N} := \frac{\mathbf{b} \times \gamma'(0)}{|\mathbf{b} \times \gamma'(0)|}$, $\mathbf{c} := \mathbf{N} \times \mathbf{b}$ and consider the orthonormal basis $\{\, \mathbf{b}, \mathbf{N}, \mathbf{c} \,\}$ of $\mathbf{R}^3$. Given $u_1, u_2 \in \mathbf{T} := \left\{\, w \in S^3 \,|\, \sphericalangle(w, \mathbf{b}) > \frac{\pi}{3} \,,\; \sphericalangle(w, -\mathbf{b}) > \frac{\pi}{3} \,\right\}$, consider their cylindrical coordinates

$$u_i = \lambda_i \, \mathbf{b} + r_i \, \cos\theta_i \; \mathbf{c} + r_i \, \sin\theta_i \; \mathbf{N} \,, \quad \text{some } \lambda_i \,, r_i \,, \theta_i \,.$$

Let $A_4 > 0$ be such that if $u_i \in \mathbf{T}$, $|u_i| > A_2$, $i = 1, 2$, then

$$|u_1 - u_2| > A_4 \, |\theta_1 - \theta_2| \,.$$

Let $\theta_n$, $\varphi_n$ be defined by

$$\begin{aligned}
\gamma'(s_n) \; &= \lambda_n \, \mathbf{b} + r_n \, \cos\theta_n \, \mathbf{c} + r_n \, \sin\theta_n \, \mathbf{N} \,, \\
\gamma'(t_n) \; &= \ell_n \, \mathbf{b} + k_n \, \cos\varphi_n \, \mathbf{c} + k_n \, \sin\varphi_n \, \mathbf{N}
\end{aligned}$$

Observe that since $\gamma(s_n) - L(t)$ is coplanar with $\mathbf{b}$ and $\gamma'(s_n)$ then $\gamma(s_n) - L(t) = \mu_n(t) \, \mathbf{b} + \rho_n(t) \, \cos\theta_n \, \mathbf{c} + \rho_n(t) \mu \sin\theta_n \, \mathbf{N}$ with the same $\theta_n$. Similarly $\gamma(t_n) - L(t)$ has the same angle $\varphi_n$ as $\gamma'(t_n)$.

For $|t|<1$, $t_n<s<s_{n+1}$, the point $G(s,t) := \frac{\gamma(s)-L(t)}{|\gamma(s)-L(t)|}$ remains on the sector

$$H_n := \left\{ v \in S^2 \,|\, v = \lambda\, \mathbf{b} + r\, \cos\theta\, \mathbf{c} + r\, \sin\theta\, \mathbf{N}\,, r>0,\ \lambda \in \mathbf{R},\ \theta \in [\varphi_n,\, \theta_{n+1}] \right\}$$

Moreover, by the argument of Lemma 3.1 since $I(s,t)\neq 0$, for $t_n < s < s_{n+1}$, the map $G|_{[t_n,s_{n+1}]\times[-1,1]}$ is inective. Hence

$$\int_{-1}^{1}\int_{t_n}^{s_{n+1}} |I| < \text{ Area of sector } (H_n) = \frac{\mathrm{vol}\,(S^2)}{2\,\pi}\ |\theta_{n+1}-\varphi_n|\ .$$

Also $\int_{-1}^{1}\int_{s_n}^{t_n}|I| = 0$, because $I(s,t)=0$ on $s_n \le s \le t_n$. By (*) we have that

$$\begin{aligned} A_4\,|\theta_{n+1}-\varphi_n| &< |\gamma'(s_{n+1}-\gamma'(t_n)| = |F\,(\gamma(s_{n+1}))-F\,(\gamma(t_n))| \\ &< A_1\,|\gamma(s_{n+1})-\gamma(t_n)| < A_1\,A_3\,|s_{n+1}-t_n| \\ |\theta_{n+1}-\varphi_n| &< A_5\,|s_{n+1}-t_n|\ , \end{aligned}$$

where $A_5 := \frac{A_1\,A_3}{A_4} > 0$. Therefore

$$\begin{aligned} \int_{-1}^{1}\int_{0}^{\varepsilon} |I(s,t)|\,ds\,dt &= \int_{-1}^{1}\int_{0}^{s_0}|I| + \sum_{n=0}^{\infty}\int_{-1}^{1}\int_{t_n}^{s_{n+1}}|I| \\ &\le \frac{1}{2}\ \mathrm{vol}\left(S^2\right) + \sum_{n=0}^{\infty}\frac{\mathrm{vol}\,(S^2)}{2\,\pi}\,A_5|,|s_{n+1}-t_n| \\ &\le \frac{1}{2}\ \mathrm{vol}\left(S^2\right) + \frac{\mathrm{vol}\,(s^2)}{2\,\pi}\,A_5\,\varepsilon \\ &\le\ \mathrm{vol}\left(S^2\right)\ , \end{aligned}$$

if we take $\varepsilon < \frac{\pi}{A_5}$ . □

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