A law of iterated logarithm for increasing self–similar Markov processes

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Abstract

We consider increasing self–similar Markov processes $(X_t, t \geq 0)$ on $]0,\infty[$. By using the Lamperti's bijection between self–similar Markov processes and Lévy processes, we determine the functions f for which there exists a constant $c \in \mathbb{R}_+ \setminus \{0\}$ such that $\liminf_{t\to\infty} X_t/f(t) = c$ with probability 1. The determination of such functions depends on the subordinator ξ associated to X through the distribution of the Lévy exponential functional and the Laplace exponent of ξ. We provide an analogous result for the self–similar Markov process associated to the opposite of a subordinator.

Key Words. Self–similar Markov processes, Subordinators, Exponential functional of Lévy process, weak duality of Markov processes.

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1 Introduction

Let $X = (X_s, s \geq 0)$ be a strong Markov process with values in $]0,\infty[$ and denote by $\mathbb{P}_x$ its law starting from $X_0 = x > 0$. For $\alpha > 0$, we say that X is α–self–similar (α–ss), whenever it fulfills the scaling property: for any $c > 0$ and $x > 0$

$$\text{the distribution of } \left(cX_{(tc^{-1/\alpha})}, t \geq 0\right) \text{under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}\,. \qquad (1)$$

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Such processes have been introduced by Lamperti [20, 21] under the name of semi–stable processes. We refer to Embrechts and Maejima [12] for some account of their properties and applications.

Recently, Bertoin and Caballero [3] studied the weak behavior of $t^{-\alpha}X_t$ as $t \to \infty$, in the case when X has increasing sample paths (see also Bertoin and Yor [4] for the general case). For any $y > 0$ fixed, they established the weak convergence

$$\mathbb{P}_y\left(t^{-\alpha}X_t \in \cdot\right) \xrightarrow[t\to\infty]{} \mathbb{P}_{0+}\left(X_1 \in \cdot\right),$$

where $\mathbb{P}_{0+}\left(X_1 \in \cdot\right)$ is the so-called entrance law from 0^+. The problem that we consider here concerns the rate at which an increasing α–ss process goes to infinity. More precisely, we should like to determine the functions $f : [0,\infty[\to [0,\infty[$, for which, for any $x > 0$

$$\liminf_{t\to\infty} \frac{X_t}{f(t)} \in]0,\infty[\qquad \mathbb{P}_x\text{–a.s.} \tag{2}$$

Fristedt [13] (see also Breiman [8]) provided an answer to (2) when X has moreover independent and stationary increments, that is X is a stable subordinator. Later, the problem was solved by Watanabe [32] for increasing ss–process with independent increments. In this paper we treat the case that does not assume neither stationarity nor independence of the increments. Namely, under a rather natural hypothesis on the entrance laws, we provide an explicit characterization of the functions that satisfies (2). Our approach is based, essentially on the main result of Lamperti [21] about the existence of a bijection between self–similar and Lévy processes. Specifically, let $\xi = (\xi_t, t \geq 0)$ be a Lévy process and $(\mathcal{F}_t, \geq 0)$ its natural filtration. Denote by $\mathbf{P}$ and $\mathbf{E}$ the probability and expectation with respect to ξ. Suppose that ξ does not drift to $-\infty$. For $\alpha > 0$, define

$$A_t = \int_0^t e^{\xi_s/\alpha} ds, \qquad t \geq 0,$$

and the time change associated to A by

$$\tau(t) = \inf\{s : A_s > t\}.$$

For an arbitrary $x > 0$, write by $\mathbb{P}_x$ the law of the process

$$X_t = x \exp \xi_{\tau(tx^{-1/\alpha})}, \qquad t \geq 0.$$

It is straightforward that under $\mathbb{P}_x$, X has the scaling property defined in (1). A classical result on time changes shows that the process X inherits the strong Markov property from ξ. So X is an α–ss Markov process. Conversely any α-ss Markov process can be obtained in this way.

In our setting X is an increasing process so ξ is a subordinator (see Bertoin [1] § 3, for background). The law of a subordinator is characterized by its Laplace transform,

$$\mathbf{E}(e^{-\lambda\xi_t}) = \exp -t\phi(\lambda) \qquad \lambda \geq 0, t \geq 0$$

where ϕ is the so called Laplace exponent of ξ and can be expressed thanks to the Lévy–Khintchine's formula as

$$\phi(\lambda) = \mathrm{d}\lambda + \int_{]0,\infty[} (1 - e^{-\lambda x})\Pi(dx),$$

The term d is called the drift coefficient and Π is the Lévy measure associated to the subordinator ξ, that is, a positive measure such that $\int_{]0,\infty[}(1 \wedge x)\Pi(dx) < \infty$. We suppose henceforth that the drift coefficient is $\mathrm{d} = 0$, and we shall exclude the case ξ is arithmetic, that is when Π is supported by $k\,\mathbb{N}$, for some $k > 0$.

Bertoin and Caballero [3] showed that if

$$0 < \mu = \mathbf{E}\big(\xi_1\big) = \phi'(0^+) < \infty,$$

then the α–ss Markov process X started at $x > 0$ converges in the sense of finite dimensional distributions when $x \longrightarrow 0^+$ (cf. Bertoin and Yor [4] for the general case). We then denote by $\mathbb{P}_{0+}$ the limiting law. Moreover, the law of X_1 under $\mathbb{P}_{0+}$ is related to the law under $\mathbf{P}$ of the Lévy exponential functional associated to the subordinator ξ, i.e.

$$I = \int_0^\infty e^{-\xi_s/\alpha} ds, \tag{3}$$

by the formula

$$\mathbb{E}_{0^+}\left(f(X_1^{1/\alpha})\right) = \frac{\alpha}{\mu}\mathbf{E}\left(I^{-1}f(1/I)\right), \tag{4}$$

where $f : [0,\infty[\to [0,\infty[$ is a measurable and bounded function. Besides, provided that $\phi'(0^+) < \infty$ Carmona, Petit and Yor [10], showed (c.f. Proposition 2.1 in [10]) that the law of I admits a density ρ which is infinitely differentiable on $]0,\infty[$. Furthermore, Proposition 3.3 op. cit. establishes that the law of I is determined by its integral moments, which in turn are given by the formulae

$$\mathbf{E}\left(I^n\right) = \prod_{k=1}^{n} \frac{k}{\phi(k/\alpha)} \quad n \in \mathbb{N},$$

and that

$$\mathbf{E}(e^{rI}) < \infty,$$

for every $0 < r < \phi(\infty)$. Let us introduce the following technical hypothesis

(H) The density ρ is decreasing in a neighborhood of ∞, and bounded.

Examples which satisfy hypothesis **(H)** are given in Section 5.

Recall that a Borel function $f:\mathbb{R}_+\to\mathbb{R}_+$, is regularly varying at infinity (resp. at 0) with index β if

$$\frac{f(xt)}{f(t)}\longrightarrow x^{\beta},\qquad \text{as}\qquad t\to\infty\quad (\textit{resp. as } t\to 0)$$

for every $x>0$. We refer to Bingham et al. [7] for a complete account of the theory of regular variation.

It is well known in the theory of subordinators that the regular variation at infinity (resp. at 0) of the Laplace exponent ϕ, is related to the behavior at 0 (resp. at ∞) of the subordinator ξ associated to it. So it is natural to expect that the regular variation at ∞ of the Laplace exponent should also be related to the local behavior of any α–ss process associated to ξ. This is indeed the case, but we first need to recall a result on subordinators in order to give a precise statement: let ϕ be regularly varying at infinity with index $\beta\in]0,1[$, let ψ be the inverse of ϕ and

$$g(t)=\frac{\log|\log t|}{\psi(t^{-1}\log|\log t|)},\qquad 0<t<e^{-1},$$

then

$$\liminf_{t\to0}\frac{\xi_t}{g(t)}=(1-\beta)^{(1-\beta)/\beta}\qquad \mathbf{P}\text{–a.s.},$$

see e.g. Bertoin [1], section III.4. It follows easily that g is regularly varying at 0 with index $1/\beta$ and

$$\lim_{t\to0}\frac{\tau(t)}{t}=1,\qquad \mathbf{P}\text{–a.s.}$$

This being said, it is straightforward that for any $x>0$ and X an α–ss process associated to ξ we have that

$$\liminf_{t\to0}\frac{X_t-X_0}{g(t)}=X_0^{(\alpha\beta-1)/\alpha\beta}(1-\beta)^{(1-\beta)/\beta}\qquad \mathbb{P}_x\text{–a.s.}\tag{5}$$

On the other hand, contrary to what we might expect, it is also the regular variation at infinity of the Laplace exponent that gives us the means to determine the behavior at infinity of an increasing self–similar Markov process. Indeed, we have the following

Theorem 1. *Let ξ be a subordinator such that $0<\mu=\mathbf{E}(\xi_1)<\infty$ and whose Laplace exponent ϕ is regularly varying at infinity with index $\beta\in]0,1[$. Suppose that the density ρ, of the Lévy exponential* *functional I of ξ satisfies hypothesis (**H**). For $\alpha > 0$, let X be the α–ss process associated to the subordinator ξ. Define*

$$\mathrm{f}(t) = \frac{\phi(\log\log t)}{\log\log t}, \qquad t > e.$$

Then for any $x > 0$

$$\liminf_{t\to\infty} \frac{X_t}{(t\mathrm{f}(t))^{\alpha}} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)} \qquad \mathbb{P}_x\text{ -a.s.}$$

This result also holds true under $\mathbb{P}_{0^+}$.

From the equation (5) only the local behavior of X under $\mathbb{P}_{0^+}$ remains to be determined. In the next result we fill this gap.

Theorem 2. *Under the hypothesis and notation of Theorem 1, we have that*

$$\liminf_{t\to 0} \frac{X_t}{(t\mathrm{f}(1/t))^{\alpha}} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)} \qquad \mathbb{P}_{0^+}\text{ -a.s.}$$

The rest of this note is organized as follows. In section 2 we state two propositions that enable us to prove Theorem 1. Section 3 is devoted to the proof of these propositions. The proof of Theorem 2 is given in section 4 where we obtain some results on time reversal for a self–similar Markov process. There we also obtain a result analog to Theorem 1 for the self–similar Markov process associated to the opposite of a subordinator near the first time that it hits 0. Finally in section 5 we give some examples.

2 Preliminaries

Let X be an α-ss Markov process with $\alpha > 0$. It is plain that the process $Y = X^{1/\alpha}$, is a 1-ss Markov process, in fact it is the 1–ss process associated to $(1/\alpha)\xi$. Conversely if Y is a 1-ss Markov process then, for any $\alpha > 0$, the process $X = Y^{\alpha}$ is an α–ss Markov process. So we can assume henceforth, without loss of generality, that $\alpha = 1$.

We can deduce from equation (4) that the entrance law $\mathbb{P}_{0^+}(X_1 \in dx)$ has a density

$$p_1(x) = \begin{cases} (\mu x)^{-1}\rho(x^{-1}) & \text{if } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

with ρ the density of the law of I.

Denote by $U = (U_s, s \geq 0)$ the Ornstein–Uhlenbeck (OU) process associated to the 1–ss Markov process X, (or to the underlying subordinator ξ through Lamperti's transformation if $X_0 = x$ for some $x > 0$) that is

$$U_t = e^{-t} X_{e^t - 1} \qquad t \geq 0.$$

This process inherits the homogeneity and strong Markov property from X, has transition probabilities

$$\widetilde{P}_s f(x) = \mathbb{E}_x\left(f(e^{-s} X_{e^s - 1})\right) \qquad s \geq 0,$$

for every Borel function f. Moreover, it has a unique invariant probability measure given by the entrance law $p_1(x)dx$. See e.g. Carmona, Petit and Yor [10] for a proof of these facts.

The asymptotic behavior of the OU process U, defined above is described in the next

Proposition 1. *Let ξ be a subordinator such that $0 < \mu = \mathbf{E}(\xi_1) < \infty$ and whose Laplace exponent is regularly varying at infinity with index $\beta \in]0,1[$. Suppose that the density, $\rho(\cdot)$, of the exponential functional I satisfies **(H)**. Let U be the Ornstein–Uhlenbeck process associated to ξ. If $h :]0,\infty[\to]0,\infty[$, is a decreasing function then for every $x > 0$*

$$\mathbb{P}_x\Big(U_s < h(s) \quad i.o. \quad s \to \infty\Big) = 0 \qquad or \qquad = 1$$

according whether

$$\int^{\infty} \rho(1/h(s))ds < \infty \qquad or \qquad = \infty.$$

This result also holds true if we suppose that the Lévy measure is finite, $\Pi]0,\infty[< \infty$, instead of the regular variation at infinity of ϕ.

Remark Of course one can derive an integral test from Proposition 1 for the 1–ss Markov process associated to ξ. Indeed, if h is a decreasing function then

$$\mathbb{P}_x\Big(X_s < sh(s) \quad \text{i.o. } s \to \infty\Big) = 0 \qquad \text{or} \qquad 1$$

according whether

$$\int^{\infty} \rho(1/h(s))\frac{ds}{s} < \infty \qquad \text{or} \qquad = \infty.$$

However this result is not really satisfactory unless one has good estimates of ρ.

Despite the characterization of the law of the exponential functional I it is not always possible get an explicit representation of its density. But to obtain the result stated at Theorem 1 we will only need estimations of the behavior of $\log \rho(\cdot)$ near infinity. That is the purpose of the following

Proposition 2. *Let I be the exponential functional associated to a subordinator $(\xi_s, s \geq 0)$ whose Laplace exponent ϕ, varies regularly at infinity with index $\beta \in]0,1[$. Then*

$$-\log \mathbf{P}(I > t) \sim (1-\beta)\varphi^{\leftarrow}(t), \qquad t \to \infty, \tag{6}$$

where

$$\varphi^{\leftarrow}(t) = \inf\Big\{s > 0\ ,\ \frac{s}{\phi(s)} > t\Big\}.$$

If moreover, the density $\rho(\cdot)$ of the law of I, is decreasing on some neighborhood of ∞, then

$$-\log \rho(t) \sim (1-\beta)\varphi^{\leftarrow}(t), \qquad t \to \infty. \tag{7}$$

Remark The fact that the tail distribution of I has this asymptotic form implies that the law of I cannot be infinitely divisible (see e.g. Steutel [30] or Bingham et al. [7] section 8.2.8).

If we take for granted Propositions 1 and 2 the proof of Theorem 1 follows by standard arguments.

Proof of Theorem 1. By Proposition 2 and the fact that $\varphi^{\leftarrow}$ is regularly varying with index $\frac{1}{1-\beta}$ we have that for any constant $c > 0$

$$-\log \rho\Big(\frac{1}{\mathrm{cf}(t)}\Big) \sim (1-\beta)c^{-\frac{1}{1-\beta}}\varphi^{\leftarrow}\Big(\frac{1}{\mathrm{f}(t)}\Big) \quad \text{as} \quad t \to \infty.$$

Since $\varphi^{\leftarrow}$ is the inverse of $s/\phi(s)$ we then have that

$$-\log \rho\Big(\frac{1}{\mathrm{cf}(t)}\Big) \sim (1-\beta)c^{-\frac{1}{1-\beta}}\log\log t \quad \text{as} \quad t \to \infty. \tag{8}$$

The statement in Theorem 1 is equivalent to the property (to be proven) that for any $\epsilon > 0$,

$$\mathbb{P}_x\left(X_s < (1-\epsilon)c_\beta s\mathrm{f}(s) \quad \text{i.o. } s \to \infty\right) = 0,$$

and

$$\mathbb{P}_x\left(X_s < (1+\epsilon)c_\beta s\mathrm{f}(s) \quad \text{i.o. } s \to \infty\right) = 1,$$

where $c_\beta = (1-\beta)^{(1-\beta)}$. From the remark after Proposition 1 the former and later equations hold if for any $\epsilon > 0$,

$$\int^{\infty} \rho(1/\mathrm{f}_{1,\epsilon}(s))\frac{ds}{s} < \infty,$$
$$\int^{\infty} \rho(1/\mathrm{f}_{2,\epsilon}(s))\frac{ds}{s} = \infty,$$

where

$$\mathrm{f}_{1,\epsilon}(s) = (1-\epsilon)c_\beta \mathrm{f}(s), \qquad \mathrm{f}_{2,\epsilon}(s) = (1+\epsilon)c_\beta \mathrm{f}(s),$$

respectively. Indeed, let $\epsilon > 0$, by equation (8) there exists an s_ϵ such that for every $s > s_\epsilon$,

$$\begin{aligned} -\log \rho\Big(\frac{1}{\mathrm{f}_{1,\epsilon}(s)}\Big) &\geq (1-\beta)(1-\epsilon)(1-\epsilon)^{-\frac{1}{1-\beta}} c_\beta^{-\frac{1}{1-\beta}} \log\log s \\ &= (1-\epsilon)^{-\frac{\beta}{1-\beta}} \log\log s. \end{aligned}$$

Therefore, taking $k_\epsilon = (1-\epsilon)^{-\frac{\beta}{1-\beta}}$, we have

$$\int_{s_\epsilon}^{\infty} \rho(1/\mathrm{f}_{1,\epsilon}(s))\frac{ds}{s} \leq \int_{s_\epsilon}^{\infty} (\log s)^{-k_\epsilon}\frac{ds}{s} < \infty,$$

since $k_\epsilon > 1$. Similarly, one shows the divergence of

$$\int^{\infty} \rho(1/\mathrm{f}_{2,\epsilon}(s))\frac{ds}{s}.$$

We have showed the statement of Theorem 1 for $\alpha = 1$, to show that the result holds for any α, consider the 1–ss process Y associated to the subordinator $\alpha^{-1}\xi$. This subordinator has Laplace exponent $\phi_\alpha(\lambda)$, such that

$$\phi_\alpha(\lambda) = \phi(\alpha^{-1}\lambda) \sim \alpha^{-\beta}\phi(\lambda) \qquad \lambda \to \infty,$$

owed to the regular variation of ϕ. Then one obtain the result readily by means of the α–ss process $X = Y^\alpha$.

$\square$

3 Proofs

This section contains two parts. In the first one, we give the proof of the Proposition 1, which is rather technical so that we decompose it in to several Lemmas. The second part contains the proof of the Proposition 2.

3.1 Proof of Proposition 1

Let $\widetilde{U}$ be process

$$\left\{\widetilde{U}_s = e^{-s} X_{e^s}, s \in \mathbb{R}\right\}.$$

Under $\mathbb{P}_{0+}$ the process $\widetilde{U}$ is a stationary strong Markov process, whose transition probabilities are those of the OU process U defined in the preceding section. In fact, the law of the process $(\widetilde{U}_s, s \geq 0)$ under $\mathbb{P}_{0+}$ is the same as that of the OU process $(U_s, s \geq 0)$ with initial measure the entrance law $\mathbb{P}_{0+}(X_1 \in dx) = p_1(x)dx$. This process will enable us to describe the local behavior of the OU process U and in section 4 prove the Theorem 2.

The first ingredient in the proof of Proposition 1 is the following

Lemma 1. *For any $x > 0$*

$$\mathbb{P}_{0+}\left(\lim_{h \to 0} \frac{\widetilde{U}_h - \widetilde{U}_0}{h} = -\widetilde{U}_0 \quad \Big| \widetilde{U}_0 = x\right) = 1.$$

Remark In Lemma 1 we do not impose any constraint in the way we make h tend to 0. That is why we postpone its proof until section 4.

We suppose in the sequel that the starting point of the OU process U is fixed, $U_0 = x > 0$, unless otherwise stated. The main argument in the proof of Proposition 1 is that of Breiman's [8] proof of a law of iterated logarithm for stable subordinators, which in turn is an adaptation of Motoo's [24] proof of Kolmogorov's test for diffusions. Here is an outline of such a method, see e.g. Ito and McKean [18] for Motoo's proof of Kolmogorov's test. Let $\{R_n, n \geq 0\}$ be the successive return times of the OU process U to its starting point, i.e.,

$$R_{n+1} = \inf\{t > R_n : U_t = U_0\},$$

with $R_0 = 0$. Denote by $R = R_1$ and T_y the first hitting time of a level $y > 0$ by the OU process U, i.e.,

$$T_y = \inf\{t > 0 : U_t = y\}.$$

Define the function $g(x, y)$ by

$$g(x, y) = \mathbb{P}_x(T_y < R) = \mathbb{P}_x\left(\inf_{t \in [0,R]} U_t < y\right), \quad y < x.$$

By the homogeneity and strong Markov property of U the random variables

$$\{R_{n+1} - R_n, n \geq 0\}$$

are independent and identically distributed with the same law as R. The fact that the OU process U has a unique invariant probability implies that

$$\mathbb{E}_x(R) < \infty.$$

Then, by the strong law of large numbers

$$\frac{R_n}{n} \xrightarrow[\mathrm{n}\to\infty]{} \mathbb{E}_x(R), \qquad \mathbb{P}_x\text{–a.s.}$$

Besides, we can deduce from the Lemma 1, using the homogeneity and the strong Markov properties of the OU process $\widetilde{U}$ that the OU process U hits points from above and it leave it from below and more importantly the range of the excursion process $\{U_s, s \in [0, R]\}$ is a compact interval with U_0 in its interior. Thanks to these facts Motoo's arguments apply to show that for any decreasing function $h :]0, \infty[\to]0, \infty[$ we have the $\mathbb{P}_x$–a.s. inclusion of sets

$$\big\{U_s < h(s) \quad \text{i.o.} \quad s \to \infty\big\} \subseteq \big\{\inf_{s\in[R_n,R_{n+1}]} U_s < h(c_1 n) \quad \text{i.o.} \quad n \to \infty\big\}$$

$$\big\{\inf_{s\in[R_n,R_{n+1}]} U_s < h(c_2 n) \quad \text{i.o.} \quad n \to \infty\big\} \subseteq \big\{U_s < h(s) \quad \text{i.o.} \quad s \to \infty\big\}$$

with $c_1, c_2 > 0$ constants that depend only on $\mathbb{E}_x(R)$. Therefore, by a standard application of the Borel–Cantelli Lemma we get that if the integral

$$\int^{\infty} g(x, h(s))ds \tag{9}$$

converges then

$$\mathbb{P}_x\left(\inf_{t\in[R_n,R_{n+1}]} U_t < h(c_1 n) \quad \text{i.o.} \quad n \to \infty\right) = 0,$$

whereas if (9) diverges then

$$\mathbb{P}_x\left(\inf_{t\in[R_n,R_{n+1}]} U_t < h(c_2 n) \quad \text{i.o.} \quad n \to \infty\right) = 1.$$

The proof reduces then to estimate the function $g(x, y)$, that is, estimate the distribution of the depth of the excursion and to show that the criterion does not depend of x. Namely that

$$g(x, y) \asymp \rho(1/y) \qquad \text{as} \qquad y \to 0, \tag{10}$$

that is there exists two positive constants b_1, b_2 such that

$$b_1\rho(1/y) \le g(x,y) \le b_2\rho(1/y) \qquad \text{as} \qquad y \to 0.$$

In [2] Bertoin gets an estimate for the function g when the underlying self–similar process is a stable subordinator. His proof provides the key steps for our estimation of the function g.

Lemma 1 enable us to follow the arguments of section 3 in [2] and this yields

Lemma 2. *Assume ρ is bounded. For every $x, y > 0$ and $q > 0$ we have*

(i)

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^R 1_{\{U_s\in[y-\epsilon,y]\}} ds = \frac{1}{y} Card\big\{t \in [0,R[: U_t = y\big\}$$

both $\mathbb{P}_x$–a.s. and in $L^1(\mathbb{P}_x)$.

(ii)

$$\lim_{\epsilon\to 0} \mathbb{E}_y\left(\frac{1}{\epsilon}\int_0^R e^{-qs} 1_{\{U_s\in[y-\epsilon,y]\}} ds\right) = \frac{1}{y}.$$

(iii)

$$\mathbb{E}_x(R) = \frac{\mu}{\rho(1/x)}$$

(iv)

$$\mathbb{E}_x\Big(\sum_{t\in[0,R[} 1_{\{U_t=y\}}\Big) = \frac{\rho(1/y)}{\rho(1/x)}.$$

Proof. First note that an application of Dynkin's formula shows that the measure

$$\nu(f) = \frac{\mathbb{E}_x(\int_0^R f(U_s)ds)}{\mathbb{E}_x(R)},$$

is an invariant law for the OU process. Moreover, by the uniqueness of the invariant law we have that

$$\nu(f) = \int_0^\infty f(z)(\mu z)^{-1}\rho(1/z)dz, \tag{11}$$

for every function f non–negative and measurable. Next, if we take for granted Lemma 1, then we may simply repeat the arguments of [2] to prove (i–iii). The statement in (iv) follows from (i), (iii) and the identity in equation (11). $\square$

A standard application of the strong Markov property at time R shows that for every $y > 0$

$$\begin{aligned}\mathbb{E}_x\Big(\sum_{t\in[0,R[} 1_{\{U_t=y\}}\Big) &= \mathbb{P}_x(T_y < R)\big(1 + \mathbb{P}_y(R < T_x) + (\mathbb{P}_y(R < T_x))^2 + \cdots\big) \\ &= \frac{\mathbb{P}_x(T_y < R)}{\mathbb{P}_y(T_x \le R)}.\end{aligned} \tag{12}$$

Therefore, by comparing (iv) in Lemma 2 and equation (12) we get that

$$\mathbb{P}_x(T_y < R) = \mathbb{P}_y(T_x \le R)\frac{\rho(1/y)}{\rho(1/x)}.$$

Since by hypothesis **(H)** we have that

$$\lim_{y\to 0} \rho(1/y) = 0,$$

then we may conclude that the statement in (10) is equivalent to

$$\liminf_{y\to 0} \mathbb{P}_y(T_x \le R) > 0. \tag{13}$$

We next focus in the proof of (13). To that end, we will obtain more precise information on the duration R of the excursion as the starting point tends to 0 using the well known fact that the distribution of R can be characterized in terms of the resolvent density. We introduce some notation.

Define by $\{L_t^y, t > 0\}$ the "local time" at $y > 0$ of the OU process U, that is

$$L_t^y = \frac{1}{y}\sum_{0<s\le t} 1_{\{U_s=y\}}.$$

Let $x \ge 0, y > 0$, and $u_1(x,y)$ the 1-potential of L_t^y under $\mathbb{P}_x$, for $x > 0$ and $\mathbb{P}_{0+}$ for $x = 0$ i.e.,

$$u_1(x,y) = \mathbb{E}_x\Big(\int_{]0,\infty[} e^{-s} dL_s^y\Big).$$

We have by the strong Markov property that

$$u_1(x,y) = y^{-1}\frac{\mathbb{E}_x(e^{-T_y})}{1-\mathbb{E}_y(e^{-R})}. \tag{14}$$

Lemma 3. *For every $y > 0$, we have*

$$u_1(0,y) = \frac{1}{\mu}\int_0^{1/y} dz\rho(z).$$

In particular $u_1(0,y)$ is a bounded and continuous function.

Proof. Let $\mathcal{R}_1$ denote the 1-resolvent operator of the OU process, that is,

$$\mathcal{R}_1 f(x) = \mathbb{E}_x \Big(\int_0^\infty e^{-s} f(U_s) ds \Big) = \int \mathcal{R}_1(x, dy) f(y),$$

for any Borel positive function f, and $x \geq 0$. Our first aim is to show that the measure $\mathcal{R}_1(0, dy)$ has a density that coincides with $u_1(0, y)$. Indeed, by a change of variables, an application of Fubini's Theorem and the self–similarity of X, we get

$$\begin{aligned}
\mathcal{R}_1 f(0) &= \mathbb{E}_{0^+} \Big(\int_0^\infty e^{-s} f(U_s) ds \Big) \\
&= \mathbb{E}_{0^+} \Big(\int_0^1 f(uX_{(1-u)/u}) du \Big) \\
&= \int_0^1 \mathbb{E}_{0^+} \big(f(uX_1) \big) du. \\
&= \int_0^1 du \int_0^\infty dx (\mu x)^{-1} \rho(1/x) f(xu).
\end{aligned}$$

Straightforward calculations shows that

$$\mathcal{R}_1 f(0) = \int_0^\infty dy f(y) v(y),$$

with $v(y) = \mu^{-1} \int_0^{1/y} dx \rho(x)$. This shows that $\mathcal{R}_1(0, dy)$ has a density $v(y)$, that is continuous and bounded. In particular

$$v(y) = \lim_{\epsilon \to 0} \epsilon^{-1} \int_{y-\epsilon}^y v(x) dx.$$

On the other hand,

$$\begin{aligned}
\int_{y-\epsilon}^y v(x) dx &= \mathbb{E}_{0^+} \Big(\int_0^{T_y} e^{-s} 1_{\{U_s \in [y-\epsilon, y]\}} ds \Big) + \mathbb{E}_{0^+} \Big(\int_{T_y}^\infty e^{-s} 1_{\{U_s \in [y-\epsilon, y]\}} ds \Big) \\
&= I_\epsilon + II_\epsilon.
\end{aligned}$$

By the strong Markov property

$$II_\epsilon = \frac{\mathbb{E}_{0^+}(e^{-T_y})}{1 - \mathbb{E}_y(e^{-R})} \mathbb{E}_y \Big(\int_0^R e^{-s} 1_{\{U_s \in [y-\epsilon, y]\}} ds \Big).$$

Using (ii) in Lemma 2 and equation (14) we get that

$$\lim_{\epsilon \to 0} \epsilon^{-1} II_\epsilon = y^{-1} \frac{\mathbb{E}_{0^+}(e^{-T_y})}{1 - \mathbb{E}_y(e^{-R})} = u_1(0, y).$$

Thus the proof will be completed if we show that, $\epsilon^{-1} I_\epsilon \to 0$ as $\epsilon \to 0$. Let $H_y = \inf\{s > 0 : U_s > y\}$ be the first time that the OU process jumps above the level $y > 0$. Indeed, by the Markov property applied at the first passage time of the OU process above the level $y - \epsilon$ we get

$$I_\epsilon \leq \mathbb{E}_{0^+} \big(1_{\{H_{y-\epsilon} < H_y\}} e^{-H_{y-\epsilon}} \big) \sup_{z \in [y-\epsilon, y]} \mathbb{E}_z \big(\int_0^{H_y} e^{-s} 1_{\{U_s \in [y-\epsilon, y]\}} ds \big).$$

Applying repeatedly the Markov property at the stopping time R we get for every $z > 0$,

$$\mathbb{E}_z\Big(\int_0^{H_y} e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\Big) \leq \frac{\mathbb{E}_z\left(\int_0^R e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\right)}{1-\mathbb{E}_z\left(e^{-R}1_{\{R<H_y\}}\right)}.$$

The claimed result now follows from an application of (ii) in Lemma 2 and the fact that $H_{y-\epsilon} \longrightarrow H_y$ as $\epsilon \to 0$ a.s. $\square$

We assume throughout the rest of this section that either ϕ is regularly varying at ∞ with index $\beta \in]0,1[$ or the Lévy measure is finite, $\Pi]0,\infty[< \infty$.

Lemma 4. *One has*

$$\liminf_{y\to 0^+} y^{-1}\,\mathbb{E}_{0+}\left(e^{-T_y}\right) > 0.$$

Before proving this Lemma let us define a function that will be used in the sequel. Since the function $\phi(\lambda)/\lambda$ is decreasing, there exists a function β_y such that

$$\phi(\beta_y)/\beta_y = y,$$

we denote $\delta_y = e^{1/\beta_y} - 1$.

Proof. The statement in Lemma 4 means that

$$\liminf_{y\to 0^+} y^{-1}\,\mathbb{P}_{0+}(T_y < \mathbf{e}) > 0,$$

with $\mathbf{e}$ an exponential random variable independent of the OU process. To show this fact we will need to introduce some notation and recall some results. Let H^X_y be the first passage time above the level y by the 1-ss process X, that is

$$H^X_y = \inf\big\{s > 0 : X_s > y\big\}.$$

Bertoin and Caballero [3] showed that under the entrance law $\mathbb{P}_{0+}$, the law of the pair

$$(H^X_y, X_{H^X_y})$$

is the same as that of

$$(yI\exp\{-VZ\}, y\exp\{(1-V)Z\}),$$

where V, Z and I are independent and V is uniformly distributed on $[0,1]$ and the law of Z is given by

$$\mathbf{P}(Z \in dz) = \mu^{-1}z\Pi(dz) \qquad z > 0.$$

So by taking $S_y = \log(1 + H_y^X)$ we get that

$$\big(U_{S_y}/y, S_y\big) \xrightarrow[y\to 0]{\mathfrak{D}} (e^K, 0),$$

where K is a random variable with law

$$\mathbf{P}(K \in dk) = \mu^{-1}\overline{\Pi}(k)dk,$$

and $\overline{\Pi}(k) = \Pi(k,\infty)$. Recall that H_y is the first time that the OU process U jumps above the level y. It is plain that the OU process hits a level $[y,\infty[$ only if the ss process X is already at this level, i.e.

$$\log(1 + H_y^X) \le H_y,$$

for every $y > 0$. Moreover, the weak convergence of U_{S_y}/y implies that

$$\mathbb{P}_{0+}\left(\log(1 + H_y^X) < H_y\right) \le \mathbb{P}_{0+}(U_{S_y} \in [0, y[) \longrightarrow 0 \quad \text{ as } y \to 0.$$

So we can suppose henceforth that $\log(1 + H_y^X) = H_y$, for all y small enough.

Let $t > 0$ fixed and ϵ_y an arbitrary function vanishing at 0. For every $y > 0$ such that $t > \epsilon_y$ we have by the strong Markov property applied at time S_y, that

$$\begin{aligned}
&\mathbb{P}_{0+}(T_y < t)\\
&= \int_y^\infty \int_0^t \mathbb{P}_{0+}(U_{S_y} \in dz, S_y \in dr)\, \mathbb{P}_z(\exists s \in [0, t-r], U_s = y)\\
&\ge \int_y^{y(1+\delta_y)} \int_0^{t-\epsilon_y} \mathbb{P}_{0+}(U_{S_y} \in dz, S_y \in dr)\, \mathbb{P}_z(\exists s \in [0, \epsilon_y], U_s \le y),
\end{aligned} \tag{15}$$

the inequality in the former equation is owed to the fact that the OU process does not have negative jumps and hits the points from above. Using the Lamperti's transformation, it is straightforward that for every $z \in]y, y(1+\delta_y)[$

$$\begin{aligned}
\mathbb{P}_z\left(\exists s \in [0, \epsilon_y], U_s \le y\right) &= \mathbf{P}\left(\exists s \in [0, \epsilon_y],\ ze^{-s}\exp\{\xi_{\tau((e^s-1)/z)}\} \le y\right)\\
&\ge \mathbf{P}\left(\exists s \in [0, \epsilon_y],\ y(1+\delta_y)e^{-s}\exp\{\xi_{\tau((e^s-1)/y)}\} \le y\right)\\
&= \mathbb{P}_y\left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \le y\right).
\end{aligned} \tag{16}$$

The weak convergence of $(U_{S_y}/y, S_y)$ as $y \to 0$, implies that

$$\mathbb{P}_{0+}(U_{S_y} \in [y, y(1+\delta_y)], S_y \in [0, t-\epsilon_y]) \sim \mathbf{P}\left(e^K \in]1, 1+\delta_y[\right),$$

as $y \to 0$. Putting together equations (15) & (16) and the later fact we get the estimation

$$\begin{aligned} &\mathbb{P}_{0+}(T_y < t) \\ &\geq \mathbb{P}_{0+}(U_{S_y} \in [y, y(1+\delta_y)], S_y \in [0, t-\epsilon_y])\, \mathbb{P}_y\left(\exists s \in [0,\epsilon_y], (1+\delta_y)U_s \leq y\right) \\ &\sim \mathbf{P}\left(e^K \in]1, 1+\delta_y[\right) \mathbb{P}_y\left(\exists s \in [0,\epsilon_y], (1+\delta_y)U_s \leq y\right), \end{aligned} \tag{17}$$

as $y \to 0$. Furthermore, by the definition of δ_y we have that

$$\begin{aligned} \mathbf{P}(e^K \in [1, 1+\delta_y]) &= \mathbf{P}\left(K \in [0, \beta_y^{-1}]\right) \\ &= \frac{1}{\mu}\int_0^{\beta_y^{-1}} \overline{\Pi}(r)dr, \end{aligned} \tag{18}$$

and the last term in the former equation can be estimated in terms of the Laplace exponent. Specifically, there exist two constant c_1, c_2 depending only on ϕ such that

$$c_1 \frac{\phi(\beta_y)}{\beta_y} \leq \frac{1}{\mu}\int_0^{\beta_y^{-1}} \overline{\Pi}(r)dr \leq c_2 \frac{\phi(\beta_y)}{\beta_y},$$

see e.g. [1] Proposition III.1. Since β_y is the inverse of $\phi(z)/z$ we have by equations (17) & (18) that

$$\mathbb{P}_{0+}(T_y < t) \geq y c_1\, \mathbb{P}_y\left(\exists s \in [0,\epsilon_y], (1+\delta_y)U_s \leq y\right),$$

for every y small enough. Now, we shall show in Lemma 5 below that the function ϵ_y can be chosen such that

$$\liminf_{y \to 0} \mathbb{P}_y\left(\exists s \in [0,\epsilon_y], (1+\delta_y)U_s \leq y\right) = \vartheta > 0. \tag{19}$$

Tacking for granted this statement we end the proof since we have showed that for all $t > 0$,

$$\begin{aligned} \mathbb{P}_{0+}(T_y < \mathbf{e}) &\geq e^{-t}\, \mathbb{P}_{0+}(T_y < t) \\ &\geq e^{-t} C y \qquad \text{as} \qquad y \to 0, \end{aligned}$$

where $\mathbf{e}$ is an exponential random variable independent of U and $C = c_1\vartheta$. □

Lemma 5. *We may choose ϵ_y such that (19) holds true.*

Proof. Recall that β_y is determined by $\phi(\beta_y)/\beta_y = y$ and that $\delta_y = e^{1/\beta_y} - 1$. The regular variation at infinity of ϕ will enable us to show that the functions

$$\epsilon_y = \frac{e^{d/\beta_y} - 1}{y} \qquad \text{and} \qquad a_y = (d-1)/\beta_y,$$

with $d > 1$ arbitrary, are such that

(i)

$$\epsilon_y,\ a_y, \longrightarrow 0,\ \text{as}\ y \to 0,$$

(ii)

$$\lim_{y\to 0} \mathbf{P}(\xi_{\epsilon_y} \le a_y) = \vartheta_1 > 0.$$

The reason why we require the functions ϵ_y and a_y to have this behavior is the following. Let

$$s_y = \log(1 + y\epsilon_y),\ y > 0$$

and note that $\tau(s) \le s$, for every $s \ge 0$, since $A_s \ge s$ for every $s \ge 0$. Then on the event

$$\xi_{\epsilon_y} \le a_y,$$

we have the inequalities

$$\exp\{\xi_{\tau((e^{s_y}-1)/y)}\} \le \exp\{\xi_{\epsilon_y}\} \le e^{a_y},$$

due to the fact that ξ is an increasing process and

$$\tau\left((e^{s_y} - 1)/y\right) \le (e^{s_y} - 1)/y = \epsilon_y.$$

So, on this event, we have also the inequalities

$$y(1+\delta_y)e^{-s_y}\exp\{\xi_{\tau\left((e^{s_y}-1)/y\right)}\} \le y(1+\delta_y)e^{-s_y+a_y} \le y,$$

from the definition of the functions s_y and a_y. Since $s_y \le \epsilon_y$ for every y small enough and the OU process does not have negative jumps, we can conclude by (ii) that

$$\liminf_{y\to 0} \mathbf{P}\left(\exists\ s \in [0,\epsilon_y],\ y(1+\delta_y)e^{-s}\exp\{\xi_{\tau((e^s-1)/y)}\} \le y\right) \ge \lim_{y\to 0}\mathbf{P}(\xi_{\epsilon_y} \le a_y) > 0.$$

Then the proof reduces to show that the functions ϵ_y and a_y so defined satisfies (i,ii).

Let $\phi'(\cdot)$ be the derivative of ϕ,

$$\Lambda(u) = \phi(u) - u\phi'(u), \qquad u > 0,$$

and λ_y the function determined by the relation

$$\phi'(\lambda_y) = \frac{a_y}{\epsilon_y}.$$

Since $\phi(\lambda)$ is concave and regularly varying with index $\beta \in]0,1[$ then $\phi(\lambda)/\lambda$ is regularly varying with index $\beta - 1$ and $\phi'(\lambda) \sim \beta\phi(\lambda)/\lambda$. This implies in turn that $\beta_y \longrightarrow \infty$ and $y\beta_y \longrightarrow \infty$ as $y \to 0$. Thus it is straightforward that ϵ_y, and a_y satisfy (i), and moreover $a_y = O(y\epsilon_y)$. This and the regular variation of ϕ imply that $\lambda_y = O(\beta_y)$.

According to Jain and Pruitt [19] Theorem 5.1 the statement in (ii) is equivalent to

$$\lim_{y\to 0} \epsilon_y \Lambda(\lambda_y) < \infty.$$

The former is indeed true in our construction,

$$\begin{aligned}\Lambda(\lambda_y) &= \lambda_y\Big(\frac{\phi(\lambda_y)}{\lambda_y} - \phi'(\lambda_y)\Big)\\ &\sim \lambda_y\phi'(\lambda_y)\Big(\frac{1-\beta}{\beta}\Big)\\ &= \lambda_y\frac{a_y}{\epsilon_y}\Big(\frac{1-\beta}{\beta}\Big).\end{aligned}$$

Therefore

$$\epsilon_y\Lambda(\lambda_y) \sim \frac{\lambda_y}{\beta_y}(d-1)\Big(\frac{1-\beta}{\beta}\Big) = O(1).$$

This ends the proof in the case ϕ is regularly varying at infinity. When the Lévy measure is a finite measure, that is ξ is a compound Poisson process, we can take $a_y \equiv 0$ and

$$\epsilon_y = (e^{1/\beta_y} - 1)/y \qquad y > 0.$$

This choice of the functions a_y, ϵ_y is due to the fact that a compound Poisson process remains at zero during an exponential time and a fortiori

$$\lim_{y\to 0^+} \mathbf{P}\left(\xi_{\epsilon_y} \le a_y\right) > 0.$$

The rest of the proof follows as in the case ϕ is regularly varying at ∞. □

The last ingredient in the proof of Proposition 1 is the following result.

Lemma 6. *One has*

(i)

$$\limsup_{y\to 0^+} \mathbb{E}_y(e^{-R}) < 1,$$

(ii)

$$\liminf_{y\to 0} \mathbf{P}_y(T_x \le R) > 0.$$

Proof. (i) We know from equation (14) that

$$u_1(0,y) = y^{-1}\frac{\mathbb{E}_{0+}\left(e^{-T_y}\right)}{1-\mathbb{E}_y(e^{-R})} \ge \mathbb{E}_{0+}\left(e^{-T_y}\right) u_1(y,y).$$

Moreover, by Lemma 3 one has

$$\lim_{y\to 0} u_1(0,y) = \lim_{y\to 0} \frac{1}{\mu}\int_0^{1/y} dz\rho(z) = \frac{1}{\mu}.$$

Thus Lemma 4 implies

$$\limsup_{y\to 0^+} y u_1(y,y) = \theta < \infty.$$

In particular, using equation (14) one gets

$$\limsup_{y\to 0} \mathbb{E}_y(e^{-R}) = \frac{\theta}{1+\theta}.$$

(ii) The statement in (i) shows that for every $t>0$

$$\limsup_{y\to 0} \mathbb{P}_y(R \le t) \le \frac{\theta}{1+\theta}.$$

Since the OU process U hits the points continuously from above, it is plain that for every $y<x$

$$\mathbb{P}_y(T_x < R) = \mathbb{P}_y(H_x < R).$$

Thus, for every $t>0$

$$\mathbb{P}_y(H_x < R) \ge \mathbb{P}_y(H_x < t) - \mathbb{P}_y(R \le t),$$

and as a consequence

$$\begin{aligned}\liminf_{y\to 0^+} \mathbb{P}_y(H_x < R) &\ge \mathbb{P}_{0^+}(H_x < t) - \limsup_{y\to 0} \mathbb{P}_y(R \le t)\\ &\ge \mathbb{P}_{0^+}(H_x < t) - \frac{\theta}{1+\theta}.\end{aligned}$$

Since the OU process U is recurrent and without negative jumps we can ensure that

$$\mathbb{P}_{0^+}(H_x < \infty) = 1.$$

Then there exists a $t>0$ such that the right hand term in the former inequality is strictly positive. $\square$

Lemma 6 ends the proof of Proposition 1 since we have noted that (10) is equivalent to (13).

3.2 Proof of Proposition 2

This proof is based on the fact that one can relate the behavior of $\mathbf{P}(I > t)$ to that of the Laplace exponent ϕ of ξ by using connections between the behavior of $\mathbf{E}(e^{\lambda I})$ as $\lambda \to \infty$ and that of $\mathbf{P}(I > t)$ as $t \to \infty$. This result can be proved using the results in Geluk [16]. However, for ease of reference we provide a complete proof based on a result due to Kasahara. We note that a similar result has been obtained in Haas [17] Proposition 11.

Proof of Proposition 2. Since the moment generating function of I, is well defined, that is,

$$\widehat{\rho}(s) = \mathbf{E}(e^{sI}) < \infty \qquad \forall s > 0,$$

we have the conditions to use Kasahara's Tauberian Theorem (Bingham et al. [7] Theorem 4.12.3), it links the regular variation of $\log \widehat{\rho}(s)$ as $s \to \infty$ with that of $-\log \mathbf{P}(I > t)$ as $t \to \infty$. On the other hand the characteristic function of I, say f, is an entire function, admits a Taylor series

$$f(z) = \sum_n a_n z^n, \quad \text{with} \quad a_n = i^n \frac{\mathbf{E}(I^n)}{n!} = \frac{i^n}{\prod_{k=1}^n \phi(k)} \qquad \forall n \in \mathbb{N},$$

and its maximum modulus,

$$M(s,f) = \sup\left\{|f(z)| : |z| \le s\right\},$$

coincides with $\widehat{\rho}(s)$, that is

$$M(s,f) = \widehat{\rho}(s), \qquad \forall s > 0,$$

e.g. Lukacs [23] Theorem 7.1.2.

In order to apply Kasahara's Theorem we must check that $\log \widehat{\rho}(s)$, i.e. $\log M(s,f)$, is asymptotically regularly varying. To this end, we recall that we can estimate the behavior of $\log M(s,f)$ in terms of the coefficients of the Taylor expansion of f. More precisely, suppose that

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} = \frac{1}{\beta}. \tag{20}$$

By Levin [22] (section 1.13), if there exists a regularly varying function with index β, say ψ, such that

$$\lim_{n\to\infty} |a_n|^{1/n} \psi(n) = e^{\beta}, \tag{21}$$

then

$$\lim_{s\to\infty} \frac{\log M(s,f)}{\psi^{\leftarrow}(s)} = \beta, \tag{22}$$

with $\psi^{\leftarrow}$ the asymptotic inverse of ψ. A version of $\psi^{\leftarrow}$ is

$$\psi^{\leftarrow}(s) = \inf\{r > 0 | \psi(r) > s\}.$$

With the aim of obtaining the asymptotic behavior of $-\log \mathbf{P}(I > t)$, let $\varphi(s) = s/\psi(s)$. Then φ is a regularly varying function with index $1-\beta$ and its asymptotic inverse, $\varphi^{\leftarrow}$, varies regularly with index $(1-\beta)^{-1}$. Using equation (22), a straightforward application to Theorem 4.12.7 in Bingham et al. [7] leads to

$$-\log \mathbf{P}(I > t) \sim (1-\beta)\varphi^{\leftarrow}(t), \qquad t \to \infty,$$

and, provided that ρ decreases in some neighborhood of ∞, we can apply Theorem 4.12.10 op. cit. to get

$$-\log \rho(t) \sim (1-\beta)\varphi^{\leftarrow}(t), \qquad \text{as} \qquad t \to \infty.$$

The rest of the proof is devoted to the proof of (20) and the fact that ϕ satisfies the equation (21). With this aim, recall that

$$|a_n| = \mathbf{E}(I^n)/n! = \left(\prod_{k=1}^{n} \phi(k)\right)^{-1}.$$

As ϕ is regularly varying with index β, it can be expressed as $\phi(s) = s^{\beta} l(s)$, with l a slowly varying function. Moreover, there exist two functions ε and c and a positive constant a, such that

$$l(t) = \exp\left\{c(t) + \int_a^t \varepsilon(s)\frac{ds}{s}\right\}$$

and $\varepsilon(t) \to 0$ and $c(t) \to c$ with $c \in \mathbb{R}$, as $t \to \infty$. Therefore

$$\log 1/|a_n| = \sum_{k=1}^{n} \log \phi(k) = \beta \sum_{k=1}^{n} \log k + \sum_{k=1}^{n} \log l(k).$$

Since

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log k}{n \log n} = 1,$$

and for every slowly varying function l we have

$$\lim_{t\to\infty} \frac{\log l(t)}{\log t} = 0,$$

it is straightforward that the lim sup in (20) is in fact a limit and equals $1/\beta$. Next, we show that

$$\lim_{n\to\infty} \phi(n)|a_n|^{1/n} = e^{\beta}.$$

To do this, observe that due to the fact that $(n!)^{1/n} \sim ne^{-1}$ we get

$$|a_n|^{1/n} \sim (ne^{-1})^{-\beta} \exp\{-\frac{1}{n}\sum_{k=1}^{n} \log l(k)\}.$$

Moreover,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} \log l(k) \\
=&\frac{1}{n}\Big(n\int_a^1 \varepsilon(s)\frac{ds}{s} + \sum_{k=1}^{n-1}(n-k)\int_k^{k+1} \varepsilon(s)\frac{ds}{s}\Big) + \frac{1}{n}\sum_{k=1}^{n} c(k) \\
=&\int_a^n \varepsilon(s)\frac{ds}{s} - \frac{1}{n}\sum_{k=1}^{n-1} k\int_k^{k+1} \varepsilon(s)\frac{ds}{s} + \frac{1}{n}\sum_{k=1}^{n} c(k) \\
\sim&\int_a^n \varepsilon(s)\frac{ds}{s} + c,
\end{aligned}$$

the last line is a consequence of Cesaro's theorem since $c(k) \to c$, and

$$k\int_k^{k+1} \varepsilon(s)\frac{ds}{s} \longrightarrow 0,$$

as $k \to \infty$. Therefore

$$|a_n|^{1/n} \sim e^{\beta}(\phi(n))^{-1}.$$

□

4 On time reversal of X.

The aim of this section is to obtain a result on time reversal for a self–similar process and then use it to prove Lemma 1 and Theorem 2.

Let $z > 0$ and $\widehat{\mathbb{P}}_z$ the law of the process $\widehat{X}$ defined by

$$\widehat{X}_t = z \exp -\xi_{\widehat{\tau}(t/z)}, \quad t \geq 0$$

with the time change

$$\widehat{\tau}(t) = \inf\{s > 0, \int_0^s e^{-\xi_r} dr > t\},$$

and the convention that $\widehat{X}_t = 0$ if $\widehat{\tau}(t/z) = \infty$. Define $\widehat{\mathbb{P}}_0$ the law of the process identical to 0. Then, under the family $(\widehat{\mathbb{P}}_z, z \geq 0)$ the process $\widehat{X}$ is Markovian and has the scaling property defined in equation (1) with $\alpha = 1$. We will say that $\widehat{X}$ is the dual 1–self–similar Markov process, cf. Bertoin and Yor [4] and the reference therein. Observe that 0 is an absorbing state for $\widehat{X}$ and let $\mathfrak{J}$ be its lifetime, i.e.,

$$\mathfrak{J} = \inf\{t \geq 0 : \widehat{X}_t = 0\}.$$

It should be clear that the distribution of $\mathfrak{J}$ under $\widehat{\mathbb{P}}_z$ is that of zI, where I is the Lévy exponential functional defined in (3). Last, denote $(\widehat{\mathcal{F}}_t, t \geq 0)$ the natural filtration and $\widehat{P}_t(z, dy)$ the semigroup of the dual 1–ss Markov process.

Lemma 2 in [4], states that the q–resolvents R_q and $\widehat{R}_q$ of the processes X and $\widehat{X}$, respectively, are in weak duality with respect to the Lebesgue measure (cf. Vuolle-Apiala and Graversen [31] for a related discussion). Thus duality also holds for the respective semigroups. We will refer to this result as the *"duality Lemma"* and we will use it to show, roughly speaking, that the law of the process

$$\left(X_{(r-t)^-}, 0 \leq t < r \mid X_{r^-} = x\right),$$

under $\mathbb{P}_{0+}$ is the same as that

$$\left(\widehat{X}_t, 0 \leq t < r \mid \mathfrak{J} = r\right),$$

under $\widehat{\mathbb{P}}_x$, with $r, x > 0$ fixed. A rigorous statement will be done by using the method of h–transform of Doob, see e.g. Sharpe [29] section 62, Fitzsimmons et al. [15],... To this end, note that

- by the self similarity of X, for any $s > 0$ the law of the random variable X_s under $\mathbb{E}_{0+}$ has a density

$$p_s(z) = (\mu z)^{-1}\rho(s/z), \qquad z > 0,$$

- for every $s, t > 0$ and $z > 0$

$$\widehat{P}_t p_s(z) = p_{t+s}(z). \tag{23}$$

The identity (23) follows from the duality Lemma and the fact that $(\mathbb{P}_{0+}(X_s \in dz), s > 0)$ is a family of entrance laws for the semigroup P_t, of the 1–ss Markov process X.

Equation (23) and the Markov property of $\widehat{X}$ implies that for any $r, x > 0$ the process

$$h_t^r = \frac{p_{r-t}(\widehat{X}_t)}{p_r(\widehat{X}_0)} 1_{\{t<r\}}, \qquad t > 0,$$

is a $\widehat{\mathbb{P}}_x$ martingale. Let Ω be the space of càdlàg maps from $[0, \infty[$ to $[0, \infty[$ killed at the first hitting time of 0. After Sharpe [29] Theorem 62.19, there exists a unique probability measure $\mathbb{Q}_x^r$ on Ω equipped with its natural filtration, rendering the process $\widehat{X}$ an inhomogeneous Markov process with semigroup

$$Q^r_{t,t+s}(z,dy) = \frac{\widehat{P}_s(z,dy)p_{r-t-s}(y)}{p_{r-t}(z)}, \tag{24}$$

and such that $\mathbb{Q}^r_x(\widehat{X}_0 = x) = 1$. The measure $\mathbb{Q}^r_x$ has the property that for any $s > 0$

$$\mathbb{Q}^r_x(F1_{\{s<\mathfrak{J}\}}) = \widehat{\mathbb{P}}_x\left(Fh^r_s\right), \tag{25}$$

for every F in $\widehat{\mathcal{F}}_s$.

Lemma 7. *(i) If F is $\widehat{\mathcal{F}}_{\mathcal{J}^-}$–measurable and $g \geq 0$ is a Borel function, then*

$$\widehat{\mathbb{E}}_x\left(Fg(\mathcal{J})\right) = \mu\int drp_r(x)g(r)\mathbb{Q}^r_x\left(F\right). \tag{26}$$

Thus,

$$(\mathbb{Q}^r_x)_{r>0}$$

is a regular version of the family of conditional probabilities

$$\widehat{\mathbb{P}}_x\left(\,\cdot\,|\mathfrak{J} = r\right), \qquad r > 0.$$

(ii) Let $r > 0$ fixed and $G \geq 0$ a bounded functional then

$$\mathbb{E}_{0^+}\left(G\left(X_{(r-t)^-}, 0 \leq t < r\right)\right) = \mathbb{Q}^r_{p_r}\left(G(\widehat{X}_t, 0 \leq t < r)\right), \tag{27}$$

where $\mathbb{Q}^r_{p_r}$ denotes the law of the process $\widehat{X}$ under $\mathbb{Q}^r_x$ with initial measure $\mathbb{P}_{0^+}(X_r \in dx)$.

It is implicit in the statement in (ii) of Lemma 7 that $\mathbb{Q}^r_x$ is the image under time reversal of a measure $\widetilde{\mathbb{Q}}^r_x$ on Ω corresponding to $(X_t, 0 \leq t < r)$ under the conditional law

$$\mathbb{P}_{0^+}(\,\cdot\,|X_{r^-} = x).$$

So the support of $\mathbb{Q}^r_x$ is the set Ω_r of càdlàg paths that start at x and are absorbed at 0 at time r.

Proof. (i) By the Monotone class Theorem, to prove (26), it suffices to check that for any $s \geq 0$ the formula holds for every element of the form $F = F' \cap \{\mathfrak{J} > s\}$ with F' in $\widehat{\mathcal{F}}_s$. Indeed, note that on the set $\{s < xI\}$ we have that

$$xI = x\int_0^{\widehat{\tau}(s/x)} e^{-\xi_t}dt + xe^{-\xi_{\widehat{\tau}(s/x)}}I' = s + xe^{-\xi_{\widehat{\tau}(s/x)}}I',$$

with I' independent of $(\xi_{\widehat{\tau}(u/x)}, u \leq s)$ and equal in law to I, owed to the strong Markov property of ξ. Using the fact that under $\widehat{\mathbb{P}}_x$ the law of $\mathcal{J}$ is that of xI, the former equality and the strong Markov property of $\widehat{X}$ we get that

$$\widehat{\mathbb{E}}_x \left(g(\mathfrak{J}) 1_{\{s<\mathfrak{J}\}} | \widehat{\mathcal{F}}_s \right) = W(s, \widehat{X}_s),$$

where

$$\begin{aligned} W(s,z) &= \widehat{\mathbb{E}}_z \left(1_{\{0<\mathfrak{J}\}} g(s+\mathfrak{J}) \right) \\ &= \mathbf{E}\left(g(s+zI) \right) \\ &= \int_s^\infty dr z^{-1} \rho\left((r-s)/z \right) g(r) \\ &= \mu \int_s^\infty dr p_{r-s}(z) g(r). \end{aligned}$$

Note that

$$\frac{\widehat{\mathbb{P}}_z(\mathcal{J} \in dr)}{dr} = \mu p_r(z).$$

Thereby an application of formula (25) gives

$$\begin{aligned} \widehat{\mathbb{E}}_x \left(F g(\mathcal{J}) \right) &= \widehat{\mathbb{E}}_x \left(F' \ W(s, \widehat{X}_s) \right) \\ &= \mu \widehat{\mathbb{E}}_x \left(F' \int_s^\infty dr \ p_{r-s}(\widehat{X}_s) g(r) \right) \\ &= \mu \int_0^\infty dr p_r(x) g(r) \widehat{\mathbb{E}}_x \left(F' h^r_s \right) \\ &= \int_0^\infty dr \mu p_r(x) g(r) \mathbb{Q}^r_x(F). \end{aligned}$$

(ii) We first verify that under $\mathbb{P}_{0+}$ the process $Y_t = X_{r-t}, 0 < t < r$, admits the semigroup defined in equation (24). Let $a, b : [0,\infty[\to]0,\infty[$ be Borel functions and $t, t+s \in [0, r[$. Indeed, by the duality lemma for ss Markov processes, we have that

$$\begin{aligned} \mathbb{E}_{0+} \left(a(Y_t) b(Y_{t+s}) \right) &= \int dz \ p_{r-t-s}(z) b(z) \, \mathbb{E}_z(a(X_s)) \\ &= \int dz a(z) \widehat{\mathbb{E}}_z (p_{r-t-s}(\widehat{X}_s) b(\widehat{X}_s)) \\ &= \int dz a(z) p_{r-t}(z) \frac{\widehat{\mathbb{E}}_z (p_{r-t-s}(\widehat{X}_t) b(\widehat{X}_t))}{p_{r-t}(z)} \\ &= \mathbb{E}_{0+} (a(Y_t) Q^r_{t,t+s} b(Y_t)), \end{aligned}$$

with $Q^r_{t,t+s}$ the semigroup defined in equation (24).

By the Monotone class theorem, to prove (ii) it suffices to check that equation (27) holds for every G of the form $f_1(X_{(r-t_1)^-}) \cdots f_n(X_{(r-t_n)-})$ with $f_1, \ldots, f_n$ positive bounded Borel functions and $0 \leq t_1 < \cdots < t_n < r$. Using the fact that the ss process X does not have fixed jumps we get that for $n = 2$

$$\begin{aligned}\mathbb{E}_{0^+}\left(f_1(X_{(r-t_1)^-})f_2(X_{(r-t_2)^-})\right) &= \mathbb{E}_{0^+}\left(f_1(X_{(r-t_1)})f_2(X_{(r-t_2)})\right)\\ &= \mathbb{E}_{0^+}\left(Q^r_{0,t_1}fQ^r_{t_1,t_2}f_2(X_r)\right)\\ &= \int dx p_r(x)\mathbb{Q}^r_x\left(f_1(\widehat{X}_{t_1})f_2(\widehat{X}_{t_2})\right),\end{aligned}$$

the general case follows by iteration. □

Now we have all the elements to provide a

Proof of Lemma 1. When $h \to 0^+$, thanks to the Markov property of X, applied at time 1, our problem reduces to show that for every $x > 0$

$$\mathbb{P}_x\left(\lim_{h\to 0^+}\frac{U_h - U_0}{h} = -U_0\right) = 1.$$

To this end, we recall that since ξ is a subordinator we have

(i)
$$\lim_{s\to 0}\frac{\xi_s}{s} = 0,$$

(ii) ξ at time $\tau(1/x)$ is $\mathbf{P}$–a.s continuous and

(iii)
$$\lim_{s\to 0}\frac{\tau(s)}{s} = 1 \qquad \mathbf{P}\text{–a.s.}$$

Using these facts and Lamperti's transformation it is straightforward that

$$\lim_{\epsilon\to 0^+}\frac{X_\epsilon - X_0}{\epsilon} = 0, \qquad \mathbb{P}_x\text{–a.s.}$$

The rest of the proof, in the case $h \to 0^+$, follows by standard arguments.

Next we use Lemma 7 to study the case $h \to 0^-$. By equation (27) we know that

$$\mathbb{P}_{0^+}\left(\lim_{h\to 0^-}\frac{\widetilde{U}_h - \widetilde{U}_0}{h} = -\widetilde{U}_0 \;\middle|\widetilde{U}_0 = x\right) = \mathbb{Q}^1_x\left(\lim_{h\to 0^+}\frac{e^h\widehat{X}_{(1-e^{-h})} - \widehat{X}_0}{-h} = -\widehat{X}_0\right).$$

Since for any $x > 0$ and $\epsilon > 0$ the measure $\mathbb{Q}^1_x$ is absolutely continuous with respect to $\widehat{\mathbb{P}}_x$ on the trace of $\{\epsilon < \mathfrak{J}\}$ in $\widehat{\mathcal{F}}_\epsilon$, the result follows as in the case $h \to 0^+$ but this time for the dual self–similar process $\widehat{X}$. □

Other interesting results on time reversal can be deduced from the duality Lemma by using the classical Theorem on time reversal of Nagasawa or its generalized version in Theorem 47 chapter XVIII Dellacherie et al. [11]. We will content ourselves with the following result and refer to Bertoin and Yor [4] and the reference therein for a related discussion.

Proposition 3. *Let $x > 0$ fixed. Under $\mathbb{Q}_x^1$ the dual Ornstein–Uhlenbeck process*

$$\widehat{U} = \{e^t \widehat{X}_{1-e^{-t}}, t > 0\},$$

is an homogeneous strong Markov process with semigroup

$$Q^1_{0,1-e^{-s}} H_{e^s} f(\cdot),$$

where H_t is the dilatation $H_t f(z) = f(tz)$.

Proof. The homogeneity is obtained from the expression of the semigroup in (24) using the self–similarity enjoyed by $\widehat{X}$ under $\widehat{\mathbb{P}}_x$. Indeed, let f, g positive Borel functions then

$$\mathbb{Q}_x^1 \left(f(e^t \widehat{X}_{1-e^{-t}}) g(e^{t+s} \widehat{X}_{1-e^{-(t+s)}}) \right) = \mathbb{Q}_x^1 \left(f(e^t \widehat{X}_{1-e^{-t}}) Q^1_{1-e^{-t},1-e^{-(t+s)}} H_{e^{t+s}} g(\widehat{X}_{1-e^{-t}}) \right).$$

The expression of the semigroup can be reduced to

$$\begin{aligned} Q^1_{1-e^{-t},1-e^{-(t+s)}} H_{e^{t+s}} g(z) &= (p_{e^{-t}}(z))^{-1} \,\widehat{\mathbb{E}}_z \left(g\left(e^{t+s} \widehat{X}_{e^{-t}(1-e^{-s})} \right) p_{e^{-(t+s)}} \left(\widehat{X}_{e^{-t}(1-e^{-s})} \right) \right) \\ &= \left(p_1(e^t z) \right)^{-1} \widehat{\mathbb{E}}_{e^t z} \left(g(\widehat{U}_s) p_{e^{-s}}(\widehat{X}_{1-e^{-s}}) \right), \\ &= Q^1_{0,1-e^{-s}} H_{e^s} g(e^t z) \end{aligned}$$

where the second equality is owed to the self–similarity and the obvious identity

$$c p_{rc}(u) = p_r(c^{-1} u).$$

The strong Markov property follows from (25) by the optional stopping theorem using standard arguments. $\square$

We have now the elements to prove the Theorem 2.

Proof of Theorem 2. The statement in (ii) in Lemma 7 shows that for every positive and bounded functional F,

$$\mathbb{Q}_x^1 \left(F(\widehat{U}_t, 0 \le t \le \widehat{R}) \right) = \mathbb{E}_{0+} \left(F(e^t X_{e^{-t}}, 0 \le t \le R') \mid X_1 = x \right),$$

with R' (resp. $\widehat{R}$) the first return time of the process $\{e^t X_{e^{-t}}, t \geq 0\}$ (resp. of $\widehat{U}$), to its starting point. Moreover, by the stationarity of the OU process $\widetilde{U}$ defined at the beginning of the subsection 3.1, one gets that

$$\mathbb{E}_{0+}(R'|X_1 = x) = \mathbb{E}_x(R),$$

and

$$\mathbb{P}_{0+}\left(\inf_{0<t<R'} e^t X_{e^{-t}} > y \mid X_1 = x\right) = \mathbb{P}_x\left(\inf_{0<t<R} e^{-t} X_{e^t - 1} > y\right).$$

Recall that our proof of Proposition 1 is based on the fact that the OU process U is homogeneous and strong Markov and the probabilities that we considered there depend only on the excursion away its starting point. It should be then clear that thanks to Proposition 3 one can repeat the arguments in the proof of Proposition 1 to show that for any decreasing Borel function h we have

$$\mathbb{Q}_x^1\left(\widehat{U}_t < h(t) \quad \text{i.o.} \quad t \to \infty\right) = 0 \quad \text{or} \quad 1,$$

according whether

$$\int^{\infty} \rho(1/h(s))ds < \infty \quad \text{or} \quad = \infty.$$

We deduce from this criterion, the equation (27) and a time change that for any increasing Borel function ℓ such that $\ell(0) = 0$ we have

$$\mathbb{P}_{0+}\left(X_t < t\ell(t) \quad \text{i.o.} \quad t \to 0\right) = 0 \quad \text{or} \quad 1,$$

according whether

$$\int_{0+} \rho(1/\ell(s))\frac{ds}{s} < \infty \quad \text{or} \quad = \infty.$$

Rewriting the arguments in the proof of Theorem 1 we obtain the result. □

The former proof provides further information on the behavior of the dual 1–ss Markov process near its lifetime.

Corollary 1. *Let ξ be a subordinator such that its Laplace exponent ϕ is regularly varying at infinity with index $\beta \in]0,1[$ and $0 < \phi'(0^+) < \infty$. Suppose that the density of the Lévy exponential functional associated to ξ satisfies hypothesis **(H)**. If $\widehat{X}$ is the dual 1–ss process associated to ξ with lifetime $\mathfrak{I}$ and f is the function defined in Theorem 1 then for any $x > 0$*

$$\liminf_{s \to 0} \frac{\widehat{X}_{r(1-s)}}{s\mathrm{f}(1/s)} = r(1-\beta)^{(1-\beta)} \qquad \mathbb{Q}_x^r\text{–}a.s.$$

Proof. In the previous proof we showed that for any $x > 0$ and ℓ an increasing Borel function we have

$$\mathbb{Q}^1_x\left(\widehat{X}_{(1-s)} < s\ell(s) \quad \text{i.o.} \quad s \to 0\right) = 0 \qquad \text{or} \qquad = 1$$

according whether

$$\int_{0+} \rho(1/\ell(s))\frac{ds}{s} < \infty \quad \text{or} \quad = \infty.$$

Moreover, a straightforward verification of the finite dimensional distributions shows that the scaling property of $\widehat{X}$ under $\widehat{\mathbb{P}}$ is translated for the dual OU process in the form: under $\mathbb{Q}^r_x$ the law of the process

$$\frac{1}{r}e^t\widehat{X}_{r(1-e^{-t})}, t > 0$$

is that of the dual OU under $\mathbb{Q}^1_{x/r}$. The result follows as in the proof of Theorem 1. □

5 Examples

Example (Watanabe process) Let ξ be a subordinator with zero drift and Lévy measure $\nu(dx) = abe^{-bx}dx$, with $a, b > 0$. That is, ξ a compound Poisson process with jumps having an exponential distribution. Carmona et al. [10] §2 showed that in this case the density of the law of $I = \int_0^\infty e^{b\xi_s}ds$ is given by

$$\rho(x) = a^2xe^{-ax}, \qquad x > 0.$$

So $\rho(x)$ satisfies the hypothesis **(H)**. The $(1/b)$–ss Markov process associated to ξ by Lamperti transformation is a process that arises in the study of extremes. More precisely, the $(1/b)$–ss Markov process associated to ξ is a Q–Extremal process with

$$Q(x) = \begin{cases} \infty & x \leq 0, \\ ax^{-b} & x > 0 \end{cases}.$$

See Resnick [25]. This family of process is usually called generalized Watanabe process in honor to Watanabe S. who studied them, when $b = 1$, using the theory of Brownian excursions, see e.g. Revuz et Yor [26] pp. 504. We refer also to Carmona et al. [9] and the reference therein for the study of this process as a ss Markov process and its generalizations. Hence, thanks to Proposition 1 we obtain

Corollary 2. *Let X be a generalized Watanabe process and h an increasing function such that $(h(s))^b/s$ is a decreasing function. Then*

$$\mathbb{P}_x\left(X_s < h(s) \quad i.o.\ s \to \infty\right) = 0 \quad or \quad 1$$

according whether

$$\int^{\infty} (1/h(s))^b e^{-as(h(s))^{-b}} ds < \infty \quad or \quad = \infty.$$

This result appears in Yimin Xiao [33] Corollary 4.1 in the case $b = a = 1$.

With the aim of providing a larger class of examples, in the following construction we make some assumptions on the subordinators that ensure that the density of I satisfies hypothesis **(H)**. It uses the recent results of Bertoin and Yor [5, 6].

Let $U(dx)$, be the renewal measure of ξ, i.e.

$$\mathbf{E}\left(\int_0^{\infty} f(\xi_s)ds\right) = \int_{[0,\infty)} f(x)U(dx).$$

If the renewal measure is absolutely continuous with respect to Lebesgue measure, the function $u(x) = U(dx)/dx$, is usually called the renewal density.

Proposition 4. *Let ξ be a subordinator. Suppose that its renewal measure is absolutely continuous with respect to Lebesgue measure and that its renewal density $u(x)$, is a decreasing and convex function such that*

$$\lim_{t\to\infty} u(t) = \frac{1}{\mu} \in]0, \infty[,$$

*i.e., $\mathbf{E}(\xi_1) = \mu$. Then the density ρ, of the exponential functional associated to ξ satisfies the hypothesis **(H)**.*

Examples of such subordinators are those arising in Mandelbrot's construction of regenerative sets (see e.g. Fitzsimmons et al. [14]).

Proof. It is well known, that the renewal measure and the Laplace exponent of ξ are related by the formula

$$\frac{1}{\phi(\lambda)} = \int_0^{\infty} e^{-\lambda x} u(x)dx. \tag{28}$$

An integration by parts in the former equation leads

$$\kappa(\lambda) = \frac{\lambda}{\phi(\lambda)} = \frac{1}{\mu} + \int_0^\infty (1 - e^{-\lambda x}) g(x) dx,$$

where $-g(x)$ is the left hand derivative of $u(x)$. That is, κ is the Laplace exponent of a subordinator with killing term $\frac{1}{\mu}$, zero drift and Lévy measure with density $g(x)$. Integrating by parts, once more, we obtain that

$$\psi(\lambda) = \lambda\kappa(\lambda) = \frac{\lambda}{\mu} + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x)\nu(-dx),$$

with $\nu(dx) = -dg(x)$ a Stieltjes measure. Specifically, $\psi(\lambda)$ is the Laplace exponent of a Lévy process, say $(\zeta_s, s \geq 0)$, with no-positive jumps, drift term $1/\mu$ and no Gaussian component. We have furthermore, that

$$\mathbf{E}(\zeta_1) = \psi'(0^+) = \frac{1}{\mu} \in]0, \infty[,$$

then ζ drifts to ∞. This implies that the law of the exponential functional, I_ψ, associated to ζ, is self–decomposable, i.e., for every $0 < a < 1$, there exists an independent random variable J_a such that $J_a + aI_\psi$ has the same law as I_ψ, we refer to Sato [27] for background on self–decomposable laws. To see this, consider the first passage time above the level $-\log a$, that is $\varrho_a = \inf\{s > 0 : \zeta_s > -\log a\}$. By the strong Markov property of ζ we have that

$$\zeta'_s = \zeta_{\varrho_a + s} - \zeta_{\varrho_a}$$

is a Lévy process independent of $\{\zeta_r, r < \varrho_a\}$ and the same law as ζ. Moreover, by the absence of positive jumps and the fact that $\mathbf{E}(\zeta_1) \in]0, \infty[$, we have that $\zeta_{\varrho_a} = -\log a$ a.s. Therefore,

$$\begin{aligned}\int_0^\infty e^{-\zeta_s} ds &= \int_0^{\varrho_a} e^{-\zeta_s} ds + e^{-\zeta_{\varrho_a}} \int_0^\infty e^{-\zeta'_s} ds \\ &= J_a + aI'_\psi.\end{aligned}$$

As a consequence the density ρ_ψ, of the law of I_ψ, is unimodal, i.e., there exists a $b > 0$ such that $\rho_\psi(x)$ is increasing on $]0, b[$ and decreasing on $]b, \infty[$, see e.g. Sato [27] Theorem 53.1. Besides, Bertoin and Yor [6] section 3, showed that

$$\frac{1}{\mu}\mathbf{E}\left(f(I_\phi)\right) = \mathbf{E}\left(I_\psi^{-1} f(I_\psi^{-1})\right),$$

for every positive measurable function f, in the obvious notation. In particular, the densities of I_ψ and I_ϕ are related by

$$\frac{1}{\mu}\rho_\phi(x) = \frac{1}{x}\rho_\psi\left(\frac{1}{x}\right), \qquad \text{for every} \quad x > 0. \tag{29}$$

We derive from this that ρ_ϕ is a bounded and decreasing function on some neighborhood of ∞. $\square$

Remark Equation (29) and the uniqueness of the invariant law for the OU process show that the law of I_ψ is the invariant law of the OU process associated to the subordinator with Laplace exponent ϕ.

Remark Since every self–decomposable law is infinitely divisible, then the law of I_ψ is infinitely divisible. According to Steutel [30] its tail distribution is of the form

$$-\log \mathbf{P}(I_\psi > x) = O(x \log x),$$

and since its density is decreasing on a set $]b, \infty[$ it follows by Theorem 4.12.10 in Bingham et al. [7] that its density has the same behavior at infinity, i.e.

$$-\log \rho_\psi(x) = O(x \log x) \qquad x \to \infty.$$

This provides a complementary result to Proposition 2,

$$-\log x\rho_\phi(x) = O(x^{-1} \log(1/x)), \qquad x \to 0.$$

We lift the following examples after Fitzsimmons et al. [14] and Bertoin and Yor [6], respectively.

Example Let ξ be a subordinator without killing term, with zero drift and Lévy measure

$$\Pi(dx) = \frac{\beta e^x}{\Gamma(1-\beta)(e^x - 1)^{1+\beta}} dx,$$

with $\beta \in]0, 1[$. An integration by parts in the Lévy–Khintchine formula and a use of the beta integral show that the Laplace exponent of ξ is given by

$$\phi(\lambda) = \frac{\Gamma(\lambda + \beta)}{\Gamma(\lambda)}.$$

Using equation (28) we get that the potential measure of ξ is absolutely continuous with respect to Lebesgue measure and that the renewal density is given by

$$u(x) = \frac{1}{\Gamma(\beta)} \left(\frac{e^x}{e^x - 1} \right)^{1-\beta} \qquad x > 0.$$

Therefore u is a convex decreasing function. Moreover,

$$\phi(\lambda) \sim \lambda^\beta \quad \text{as} \quad \lambda \to \infty.$$

According to Lamperti [21], the increasing $1/\beta$–ss Markov process X, associated to ξ is a β–stable subordinator. Then by Theorem 1 one gets

$$\liminf_{t\to\infty}\frac{X_t}{t^{1/\beta}\big(\log\log t\big)^{(\beta-1)/\beta}}=\beta(1-\beta)^{\frac{(1-\beta)}{\beta}}.$$

That is we recover the law of iterated logarithm for stable subordinators of Fristedt [13]. Furthermore, since under $\mathbb{P}_{0+}$ the law of $X(1)$ is that of an β–stable random variable one can use Proposition 1 and the estimations of the stable density, see e.g. Zolotarev [34], to recover the Breiman's [8] test for stable subordinators.

Example Let $\beta\in]0,1[$ and ξ be a subordinator with zero drift and Lévy measure

$$\Pi(dx)=\frac{e^{-x/\beta}}{\Gamma(1-\beta)(1-e^{-x/\beta})^{1+\beta}}dx.$$

By straightforward calculations we get that its Laplace exponent, say ϕ, can be expressed as

$$\phi(\lambda)=\frac{\Gamma(\beta\lambda+1)}{\Gamma(\beta(\lambda-1)+1)},$$

and by the Stirling formula

$$\phi(\lambda)\sim\beta^{\beta}\lambda^{\beta}\qquad\text{as}\qquad\lambda\to\infty.$$

Proceeding as in the former example we get that the renewal density of ξ is given by

$$u(x)=\frac{1}{\Gamma(1+\beta)}(e^{x/\beta}-1)^{-(1-\beta)},$$

and is a convex decreasing function. Besides, since the law of the exponential functional I associated to this subordinator is characterized by its entire moments it is immediate that its Laplace transform is given by

$$\mathbf{E}(e^{-sI})=E_{\beta}(-s)=\sum_{n=0}^{\infty}\frac{(-s)^n}{\Gamma(n\beta+1)}.$$

The function $E_{\beta}(x)$ is the so called Mittag-Leffler function. Hence, I follows the Mittag–Leffler distribution, that is, I follows the same distribution as $\gamma_{\beta}^{-\beta}$ with γ_{β} a β–stable random variable. Furthermore, it can be showed without the use of Proposition 2 that

$$-\log\mathbf{P}(I>x)\sim(1-\beta)\beta^{\frac{\beta}{(1-\beta)}}x^{\frac{1}{(1-\beta)}},\qquad x\to\infty,$$

see e.g. Bingham et al. [7] Theorem 8.1.12 or Sato [27] solution to exercise 29.19. This fact can be considered as a motivation for our proof of Proposition 2.

More generally, one can consider the subordinator with Laplace exponent

$$\phi_\theta(\lambda) = \frac{\Gamma(\beta\lambda + \theta)}{\Gamma(\beta(\lambda - 1) + \theta)}, \tag{30}$$

for $\beta \in]0,1[$ and $\theta \geq \beta$. See Bertoin and Yor [6] for a description of the Lévy measure corresponding to this Laplace exponent. The renewal density associated to this Laplace exponent admits the expression

$$u_\theta(x) = \frac{1}{\Gamma(\theta + 1)} e^{-x(\theta-1)/\beta}\left(e^{x/\beta} - 1\right)^{-(1-\beta)}, \qquad x \geq 0.$$

Which is easily seen to be a decreasing and convex function. The entire moments of the exponential functional I_θ associated to this subordinator are given by

$$\mathbf{E}(I_\theta^n) = \frac{n!\Gamma(\theta)}{\Gamma(\beta n + \theta)}, \qquad n \geq 1.$$

We recognize in this formula the entire moments of a generalized Mittag–Leffler distribution see e.g. Schneider [28]. Schneider showed that this distribution admits a density $\rho_{\beta,\theta}(x)$, whose behavior at infinity is

$$\rho_{\beta,\theta}(x) \sim Bx^\delta \exp\{c_\beta x^\sigma\}, \qquad x \to \infty,$$

with

$$\sigma = 1/(1-\beta), \qquad \delta = \frac{(\beta - \theta + 1/2)}{1 - \beta}, \qquad c_\beta = (1-\beta)\beta^{\frac{\beta}{1-\beta}}, \tag{31}$$

and $B = (2\pi)^{-1/2}\Gamma(\theta)\sigma^{1/2}\beta^\delta$. This fact enables us to state the sharper result

Corollary 3. *Let X be the 1–ss process associated to a subordinator ξ with Laplace exponent defined by (30). If $h : [0,\infty[\to [0,\infty[$ is a decreasing function then*

$$\mathbb{P}_x(X_s < sh(s) \quad i.o\ s \to \infty) = 0 \quad or \quad 1,$$

according whether

$$\int^\infty (h(s))^{-\delta} \exp\left\{ - c_\beta(h(s))^{-\sigma}\right\}\frac{ds}{s} < \infty \quad or \quad = \infty$$

with σ, c_β and δ as in (31).

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