Chapitre I

On random sets connected to the partial records of Poisson point processes

Abstract

Random intervals are constructed from partial records in a Poisson point process in $]0,\infty[\times]0,\infty[$. These are used to cover partially $[0,\infty[$; the purpose of this work is to study the random set $\mathcal{R}$ that is left uncovered. We show that $\mathcal{R}$ enjoys the regenerative property and identify its distribution in terms of the characteristics of the Poisson point process. As an application we show that $\mathcal{R}$ is almost surely a fractal set and we calculate its dimension.

Key Words. Poisson point process, Extremal Process, Regenerative sets, Subordinators, Fractal dimensions.
A.M.S Classification. 60 D 05

1 Introduction

Mandelbrot [14] introduced a natural and simple random generalization of Cantor's triadic set, as follows:

Let λ be the Lebesgue measure, ν an arbitrary Borel measure on $]0,\infty]$, and $\mathcal{P}\subset]-\infty,\infty[\times]0,\infty]$ a Poisson point process with characteristic measure $\lambda\otimes\nu$. This means that $\mathcal{P}$ is a countable random set with the property that for $A\subset]-\infty,\infty[\times]0,\infty]$ the cardinality of $A\cap\mathcal{P}$ is a Poisson random variable with parameter $\lambda\otimes\nu(A)$; moreover, for disjoint Borel subsets $A_i\subset]-\infty,\infty[\times]0,\infty]$ the cardinalities of $A_i\cap\mathcal{P}$ are independent random variables. For any $(x,y)\in\mathcal{P}$ he associated the open interval $]x,x+y[$. Those intervals plays the role of cut outs of $\mathbb{R}$. He then studied the structure of the so called "uncovered set"

$$\mathcal{M}=\mathbb{R}\setminus\bigcup_{(x,y)\in\mathcal{P}}]x,x+y[,$$

conditioned to contain 0. Mandelbrot has shown that the set $\mathcal{M}$ is equal in distribution to the closure of the image of a subordinator (i.e., an increasing process that has independent and homogeneous increments). He has also raised the problem of determining under which conditions $\mathbb{R}$ is completely covered by the cut outs and gave a partial solution to this problem. In a paper that was published at the same time, Shepp [21] provided a definitive answer showing that $\mathbb{R}$ is completely covered with probability one if

$$\int_0^1 dx \exp\Big\{\int_x^\infty (y-x)\nu(dy)\Big\} = \infty,$$

and with probability zero otherwise. The fact that the closed random set $\mathcal{M}$ is equal in distribution to the closure of the image of a subordinator is equivalent to say that $\mathcal{M}$ is a regenerative random closed set in the sense of Hoffmann–Jørgensen [8], and this leads to study the random set $\mathcal{M}$ through the associated subordinator. This approach was used by Fitzsimmons, Fristedt and Shepp [6] to obtain in a simpler way the necessary and sufficient condition of Shepp and many others characteristics of $\mathcal{M}$. The problem of covering $\mathbb{R}$ or more general sets by random bodies has been studied by several authors with different approaches but we will not consider here and we refer to Kahane [9] and the references therein for an historic account.

In the present work we construct an uncovered random set $\mathcal{R}$, in a different way which is partly inspired by a paper by Marchal [15].

Let $\mathcal{P} \subset]0,\infty[\times]0,\infty[$ be a Poisson point process with characteristic measure $\lambda\otimes\nu$ and $p : [0,\infty[\to [0,1]$ be a measurable function. For every $(x,y)\in\mathcal{P}$ we define x^* as the abscissa of the first point in $\mathcal{P}$ to the right of x with a higher level, say $y^* > y$. In this way for any $(x,y)\in\mathcal{P}$ we associate the interval $[x,x^*[$. We make then a cut out $[x,x^*[$ with probability $p(y)$ and we are interested in the remainder set, $\mathcal{R}$, of points that weren't deleted from $\mathbb{R}^+$.

The class of regenerative sets that arise from our construction differs from that obtained by Mandelbrot. Example belonging to one but not both of such classes is provided (see remarks to Theorems 1 and 3). Regenerative sets that are the image of a stable subordinator can be generated with both methods.

An outline of this note now follows. Section 2 is devoted to present the setting and survey the basic elements on the theory of Extremal Process. In section 3 we obtain some integral test to decide whether $\mathcal{R}$, is bounded, has isolated points, positive Lebesgue measure and further similar properties. In section 4 we recall the definition of regenerative set, preliminaries results on subordinators and regenerative sets and establish that the uncovered random set $\mathcal{R}$ is regenerative. In Section 5 we use the knowledge about subordinators to obtain an explicit formula of the renewal function of the regenerative set $\mathcal{R}$ and an exact formula for the estimation of some fractal dimensions of $\mathcal{R}$.

2 Preliminaries

This section is subdivided in 3 subsections. Subsection 2.1 is devoted to establish mathematically the verbal construction of the uncovered random set $\mathcal{R}$. Once we have built the random set $\mathcal{R}$ we wish to know the probabilities of some related events , such like "$\mathcal{R}$ contains some interval $[0,t[$", "a given point t is in $\mathcal{R}$", "0 is isolated in $\mathcal{R}$", "$\mathcal{R}$ is bounded", etc.. The tools needed for the computation of such probabilities are essentially two well known results: one about Poisson Measures and the other on Extremal Process. These are the subjects of subsection 2.2 and 2.3, respectively.

2.1 Settings

To make precise the construction of the uncovered random set described in the preceding section, let us introduce a Marked Poisson point process, that is, we add a mark to the Poisson point process $\mathcal{P} = \big\{(t,\Delta_t), t>0\big\}$, in the following way: suppose that to each point (t,Δ_t) we associate a random variable u_t independent of the whole Poisson point process $\mathcal{P}$ and that the u_t's are independent identically distributed (i.i.d.) with uniform law over $[0,1]$. We know by the marking Theorem (see [11]) that the process $\mathcal{P}' = \big\{(t, \Delta_t, u_t),\ t > 0\big\}$ is also a Poisson point process with characteristic measure $\mu(dt, dy, du) = dt \otimes \nu(dy) \otimes du$ on $]0,\infty[\times]0,\infty[\times[0,1]$. Let $\big(\mathcal{G}_t\big)_{t\geq 0}$ denote the completed natural filtration generated by $\big((t, \Delta_t, u_t); t \geq 0\big)$. For every $(x,y) \in \mathcal{P}$ define the associated x^* by

$$x^* = \inf\big\{\ x' > x | y' \geq y,\ (x', y') \in \mathcal{P}\ \big\}.$$

Let T be the set of left end points of the intervals $[x, x^*[$ that are deleted from $\mathbb{R}^+$, i.e.,

$$T = \big\{x > 0 \mid p(y) > z,\ (x,y,z) \in \mathcal{P}'\ \big\}.$$

Therefore the uncovered random set, $\mathcal{R}$, is given by

$$\mathcal{R} = [0,\infty) \diagdown \bigcup_{x\in T} [x, x^*[. \tag{1}$$

Clearly $0 \in \mathcal{R}$.

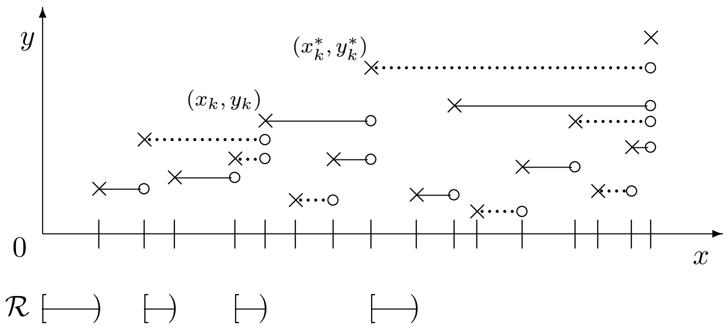

Figure 1 Uncovered set

In Figure 1, the points $\times$ are some points of a P.P.P. So the $\times$ and the $\circ$ denote respectively the left and right extremities of the possible intervals to cover $\mathbb{R}^+$. We have drawn with a doted line the intervals that are not used to cover and with a continuous line those used to cover $\mathbb{R}^+$. Last, under the graph the union of intervals shows the resulting uncovered set.

In order to get explicit and precise formulas we will make a technical assumption but the methods here used can be applied in the general case.

We assume: ν is an atom–less Borel measure such that its tail, $\overline{\nu}(y) = \nu]y,\infty[$, is finite for any $y > 0$, is strictly decreasing and its right limit at zero is infinite, i.e., $\overline{\nu}(0^+) = \infty$. This last has an immediate consequence on the points of the Poisson point process $\mathcal{P}$. If we take any right neighborhood, B_ϵ of zero in $\mathbb{R}^+$ the Poisson random variable card$\{(x,y)|\ (x,y) \in \mathcal{P} \cap \{]0,t] \times B_\epsilon\}\}$ is infinite a.s., for any $t > 0$. More precisely, the points of the Poisson point process are dense in $\mathbb{R}^+$.

It is well known that the distribution of a Poisson point process is determined by its characteristic measure. Let $\mathcal{D}$ and $\mathcal{O}$ be two Poisson point process with the same characteristic measure and a function $p : [0,\infty[\to [0,1]$. By construction we have that two uncovered random sets, say $\mathcal{R}$ and $\mathcal{R}'$, generated via p and the Poisson point process $\mathcal{D}$ and $\mathcal{O}$, respectively, are equal in distribution. To illustrate this and help the reader to become acquainted with the uncovered random sets constructed here, we present the following

Example 1. Let $\mathsf{p} \in [0,1]$ and $\nu(dx)$ an arbitrary Borel measure. Denote by $\mathcal{R}_{\mathsf{p}}$, the uncovered random set generated through the points of a Poisson point process with characteristic measure $\lambda \otimes \nu$ and a constant function p equal to p. It is plain that $\mathcal{R}_0 = \mathbb{R}^+$ a.s. and $\mathcal{R}_1 = \{0\}$ a.s. Later we shall show that the converse also holds, that is, if $\mathcal{R} = \mathbb{R}^+, (\mathcal{R} = \{0\})$ a.s. then the function p is ν–a.s. constant equal to 1 (0) (see Proposition 1 below). The structure of the uncovered random set $\mathcal{R}_{\mathsf{p}}$ with $\mathsf{p} \in]0,1[$ is not so simple; nevertheless, one can show that in this case it has the scaling property, that is, for any $c > 0$ the random sets $\mathcal{R}_{\mathsf{p}}$ and

$$c\,\mathcal{R}_{\mathsf{p}} = \{cx |\ x \in \mathcal{R}_{\mathsf{p}}\}$$

have the same distribution and we say that $\mathcal{R}_{\mathsf{p}}$ is self–similar. To show this we restrict ourselves to the case $\nu(dx) = \alpha x^{-\alpha-1}dx$, a general proof to this fact will be given as a consequence of Theorem 2 below. Indeed, let $f(x,y) = (cx, c^{1/\alpha}y)$. It is well known that

$$f(\mathcal{P}) = \{f(x,y) |\ (x,y) \in \mathcal{P}\},$$

still is a Poisson point process with characteristic measure $\lambda \otimes \nu \circ f$, i.e., for any measurable set $A \subset]0,\infty[\times]0,\infty[$

$$\lambda \otimes \nu \circ f(A) = \lambda \otimes \nu\{(x,y) | f(x,y) \in A\}.$$

Denote by $\mathcal{R}'_{\mathsf{p}}$ the uncovered random set generated via p and $f(\mathcal{P})$. It is straightforward that the measures $\lambda \otimes \nu$ and $\lambda \otimes \nu \circ f$ are equal, thus $\mathcal{R}_{\mathsf{p}}$ and $\mathcal{R}'_{\mathsf{p}}$ have the same distribution. On the other hand, as f scales the x–axis by a factor c it is immediate that $\mathcal{R}'_{\mathsf{p}}$ is equal to $c\,\mathcal{R}_{\mathsf{p}}$.

Remark 1. If a self–similar random set $\mathcal{R}_{\mathsf{p}}$ is regenerative then it must be equal in distribution to the image of a stable subordinator (see example 2 below).

2.2 Campbell's formula

Let N be the Poisson random measure on $]0,\infty[\times]0,\infty[$ defined by

$$N\big(]0,t] \times A\big) = \sum_{\{0<s\leq t; (s,\Delta_s)\in\mathcal{P}\}} 1_{\{\Delta_s \in A\}}$$

for any $t > 0$ and $A \subset]0,\infty[$ measurable. Let $f :]0,\infty[\times]0,\infty[\longrightarrow [0,\infty[$ be a positive measurable function. Define the random variable

$$< N, f >= \sum_{(s,\Delta_s)\in\mathcal{P}} f(s,\Delta_s).$$

The following Lemma provides a criteria to decide whether the random variable $< N, f >$ is finite a.s. as well as a expression of its Laplace transform. This is a classical result and can be found in any text book about Poisson random measures, we refer e.g. to [11] p. 28.

Lemma 1 (Campbell's Theorem and Exponential Formula).
The variable $< N, f >$ is finite a.s. if and only if

$$\int_{]0,\infty[\times]0,\infty[} \min\{1, f(x)\} \lambda \otimes \nu(dx) < \infty.$$

And if this last holds, the Laplace transform of $< N, f >$ is given by

$$\mathbf{E}\left[\exp\{-q < N, f >\}\right] = \exp\Big\{ - \int_{]0,\infty[\times]0,\infty[} (1 - e^{-qf(x)}) \lambda \otimes \nu(dx) \Big\}.$$

2.3 Some facts about Extremal Process

Let F be any distribution function on $\mathbb{R}$. We will say that a process $X(t)$ for $t \geq 0$, is a *Process Extremal–F* if its finite dimensional distribution functions are given by

$$\begin{aligned} &\mathbf{P}\big(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\big) \\ &= F^{t_1}(x_1')F^{t_2-t_1}(x_2') \cdots F^{t_n-t_{n-1}}(x_n'), \end{aligned} \tag{2}$$

for any $0 \leq t_1 < t_2 < \ldots < t_n$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and $x_k' = \bigwedge_k^n x_j$. Define the process $J(t) = \sup\{\Delta_s \,|(s, \Delta_s) \in \mathcal{P}, 0 < s \leq t\}$. It is easy to verify that the process $\{J(t), t \geq 0\}$ is a process extremal–F with $F(x) = \exp\{-\overline{\nu}(x)\}$, for $x \geq 0$. Indeed, take $0 < t_1 < t_2$, and $x_1, x_2 \in \mathbb{R}^+$, the bivariate distribution of J is given by

$$\begin{aligned} &\mathbf{P}\big(J(t_1) \leq x_1, J(t_2) \leq x_2\big) \\ &= \mathbf{P}\big(J(t_1) \leq x_1 \wedge x_2, \sup_{t_1 < u \leq t_2} \{\Delta_u\} \leq x_2\big) \\ &= \exp\big\{-t_1\overline{\nu}(x_1 \wedge x_2)\big\} \exp\big\{-(t_2 - t_1)\overline{\nu}(x_2)\big\}, \end{aligned}$$

where last equality follows from the identity

$$\Big\{\mathrm{card}\big\{0 < s \leq t : \Delta_s \in (u, \infty)\big\} = 0\Big\} = \Big\{J(t) \leq u\Big\}$$

and the independence of the counting processes. Following this pattern we verify that the n–variate distribution function of J satisfies (2). This is the constructive approach of an extremal process given by Resnick [19]. In the remainder of this subsection we recall some properties about general extremal process which can be found in [19] section(4.3), [20] and [18]. Let F be any distribution function on $\mathbb{R}$ with support $[a, b], -\infty \leq a < b \leq \infty$. Then

(i) X is stochastically continuous.

(ii) There is a version in $D(0, \infty)$, the space of right continuous functions on $(0, \infty)$, with left limits.

(iii) X has non-decreasing paths and almost surely

$$\lim_{t\to\infty} \uparrow X(t) = b, \qquad \lim_{t\to 0} \downarrow X(t) = a.$$

(iv) X is a Markov jump processes with

$$\mathbf{P}\big(X(t+s) \leq x \mid X(s) = y\big) = \begin{cases} F^t(x) & \text{if} \quad x \geq y \\ 0 & \text{if} \quad x < y \end{cases}$$

for $t > 0$ and $s > 0$. Set $Q(x) = -\log F(x)$. The parameter of the exponential holding time at x is $Q(x)$, and given that a jump is due to occur the process jumps from x to $]-\infty, y]$ with probability

$$\begin{cases} 1 - \big(Q(y)/Q(x)\big) & \text{if} \quad y > x \\ 0 & \text{if} \quad y \leq x. \end{cases}$$

The definition of extremal process is given for any distribution function but for continuous distribution functions there is essentially only one extremal process because general extremal process generated from a continuous distribution function may be obtained via a change of scale from the process extremal–Λ, where

$$\Lambda(x) = \exp\{-e^{-x}\} \quad \text{for } x \in \mathbb{R}\,.$$

The processes extremal–Λ and any process $X(t)$ extremal–F with F continuous are connected via the following measurable function. Define

$$S(x) = -\log\{-\log F(x)\} \qquad \text{for } x \in \mathbb{R}\,.$$

Note that $S(x)$ is continuous, non–decreasing and $-\infty \leq S(x) \leq \infty$. It can be verified directly from the definition that the process $\{S(X(t))\}_{t\geq 0}$ is extremal and is generated by $\Lambda(x)$. In the case of the process $\{J(t)\}_{t\geq 0}$, defined previously, the corresponding function S, is given by $S(x) = -\ln \overline{\nu}(x)$. The advantage of working with a process extremal–Λ is frequently the calculations are easier thanks to its additive structure (this is maybe the most important special property of this process). More precisely, let X be extremal–Λ. Pick t_0 arbitrary. Let $t_0 < \tau_1 < \tau_2 < \ldots$ be the times of jumps of $X(t)$ in $]t_0, \infty[$ and set $Z_0 = X(t_0)$, $Z_n = X(\tau_n) - X(\tau_{n-1}), n \geq 1$. Then the random variables, $\big\{Z_n, n \geq 1\big\}$, are i.i.d. with common distribution exponential of parameter 1, independent of Z_0 which has the distribution $\Lambda^{t_0}(x)$. Remark that for $s > t_0$ this result yields the representation

$$X(s) = Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j$$

where $\mu]t_0, s]$ is the number of jumps of X in $]t_0, s]$ and it is not independent of $\{Z_j\}$. So for a general process with continuous distribution function F, we have

$$S\big(X(s)\big) = Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j.$$

Let S^{-1} denote the right continuous inverse of S, that is,

$$S^{-1}(x) = \inf\{z|S(z) > x\}.$$

By inversion we obtain

$$\big\{X(s), s \geq t_0\big\} =^d \big\{S^{-1}\big(Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j\big), s \geq t_0\big\}.$$

Define the inverse process $\big\{X^{-1}(x), a \leq x \leq b\big\}$ by

$$X^{-1}(x) = \inf\big\{z \mid X(z) > x\big\}.$$

It is also directly obtained from the definition that if the process X is extremal–Λ then the process $\widetilde{X}(t) = -\log X^{-1}(-\log t)$, is also extremal–$\Lambda$. The following Lemma will be our major tool in the estimation of the probability of the event $t \in \mathcal{R}$.

Lemma 2.
Let X be extremal–F with F a continuous distribution function. Let $t > 0$ fixed. Define $T_1 = inf\{s \mid S(X(s)) = S(X(t))\}$ and for $n \geq 1$ $T_{n+1} = \inf\big\{t \mid S(X(t)) = S(X(T_n^-))\big\}$, so that $\big\{T_j, j \geq 1\big\}$ is the sequence of jump times of $S(X(\cdot))$ on $]0, t]$ ranked in decreasing order. Then

$$\big\{X(T_j^-), j \geq 1\big\} =^d \left\{S^{-1}\Big(S(X(t)) - \sum_{i=1}^{j} Z_i\Big), j \geq 1\right\}$$

Where $\big\{Z_n, n \geq 1\big\}$ are i.i.d. exponential random variables independent of $X(t)$.

The proof of this Lemma is a slight variation to that of Theorem 8 in Resnick [18] so we give a *Sketch of proof* It is clear that it is enough to consider the case $F = \Lambda$. In this case $S(x) = x,\ x \in \mathbb{R}$. As it was noted before the Process $\widetilde{X}(t) = -\log X^{-1}(-\log t)$ is a process extremal–Λ. It is well known that the jump times, $\{\tau_n\}_{n\geq 1}$, after a time $t_0 > 0$ of a extremal process have the same distribution that a function of a sum of independent identically distributed random variables (i.i.d.r.v.'s) with exponential distribution, in fact,

$$\{\tau_n, n \geq 1\} =^d \{\exp\{\log t_0 + W_n\}, n \geq 1\},$$

where $W_n = \sum_1^n Z_i$, and the random variables $Z_n, n \geq 1$ are i.i.d. with exponential distribution (this can be read from [20] p.302). This fact stills true even if t_0 is replaced by a jump time of the extremal process X. So take $\widetilde{T}_0 = \exp\{-X(T_1)\}$, which is clearly a jump time of the process $\widetilde{X}$, thus the process $\widetilde{X}(s)$ remains constant past time $\widetilde{T}_0$ except at times $\tau_1, \tau_2, \dots$ and hence $X^{-1}(s)$ remains constant for $s < X(T_1)$ except at times $-\log \tau_1, -\log \tau_2, \dots$ However

$$\{X(T_j^-), j \geq 1\} = \{-\log \tau_j, j \geq 1\} =^d \Big\{X(t) - \sum_1^j Z_i, j \geq 1\Big\}.$$

3 First properties of $\mathcal{R}$

By the time homogeneity of the Poisson point process we can suppose, and we shall do, that for every $t > 0$, fixed the process

$$Y_t(s) = \sup\big\{\Delta_{t-u} | (t-u, \Delta_{t-u}) \in \mathcal{P}; u \leq s\big\} \quad \text{for } 0 \leq s \leq t,$$

is a process extremal–F restricted to the time interval $[0, t]$, with F given by $F(x) = \exp\{-\overline{\nu}(x)\}$. Thus the law of $Y_t(s)$ for any $s \leq t$ is given by

$$\mathbf{P}(Y_t(s) \leq x) = F^s(x) \quad x \geq 0.$$

Throughout this note the function $S(x)$ will be defined by

$$S(x) = -\log \overline{\nu}(x)$$

and then the distribution function F and S are related by

$$F(x) = \exp\big\{-\exp\{-S(x)\}\big\}.$$

For $t > 0$ fixed, let Γ_t be the set of times between 0 and t where the process $Y_t(\cdot)$ jumps, i.e.,

$$\Gamma_t = \big\{s \mid Y_t(s) > Y_t(s^-),\ 0 \leq s \leq t\big\}$$

with $Y_t(0^-) = 0$. Note the almost sure equivalence

$$s \in \Gamma_t \iff (t-s)^* > t.$$

The proof of the direct implication is straightforward. To prove the converse suppose $0 < s \leq t$ and $s \notin \Gamma_t$. Since the points of the P.P.P. are dense in $\mathbb{R}^+$, there is at least one time r, $0 < r < s$ where the process Y_t jumps, that is, $r \in]0, s[\cap \Gamma_t$. Let

$$v_s = \inf\{r : Y_t(r) = Y_t(s)\}.$$

It is plain that $v_s \in \Gamma_t,\ v_s < s$ and $\Delta_{t-s} \leq \Delta_{t-v_s}$. Moreover, since the measure ν is atom–less the latter is a strict inequality ν–a.s. Therefore $(t-s)^* \leq t$ ν–a.s.

From the preceding equivalence we deduce that for $t > 0$, fixed the only points $x \leq t$ that can be the left extreme of an interval that covers t are those in $\Gamma_t \cap T$ (see figure 2). So we obtain the almost sure equivalence:

$$t \in \mathcal{R} \iff \forall\ s \in \Gamma_t,\ p(\Delta_{t-s}) \leq u_{t-s}, \tag{3}$$

or equivalently

$$t \in \mathcal{R} \iff \forall\ s \in \Gamma_t,\ p(Y_t(s)) \leq u_{t-s}.$$

The equivalence (3) shows two things: that the event "t belongs to $\mathcal{R}$" just depends on the Poisson point process until time t and that we can calculate the probability of the event $t \in \mathcal{R}$, in terms of the process $Y_t(\cdot)$.

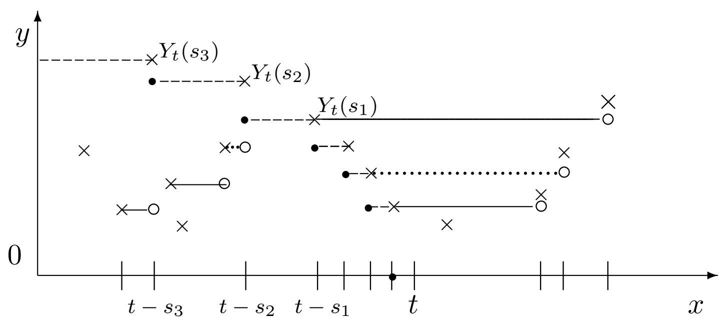

Figure 2 A t fixed that does not belong to $\mathcal{R}$.

We use the same notation as in Figure 1 and for a t fixed we draw with dashed lines the sample path of the process $Y_t(s), 0 \leq s \leq t$. So t does not belong to $\mathcal{R}$ since $t - s_1$ is the left extreme of an interval used to cover $\mathbb{R}$, i.e., $p(Y_t(s_1)) > u_{t-s_1}$.

By means of integral tests in the following results we describe the principal elementary properties of the uncovered random set $\mathcal{R}$.

Proposition 1.

i) Let $Z = \inf\{t > 0, t \notin \mathcal{R}\}$, then $Z > 0$ with probability 1 if and only if

$$\int_0^\infty p(y)\nu(dy) < \infty. \tag{4}$$

In this case Z, follows an exponential law of parameter $\int_0^\infty p(y)\nu(dy)$. In particular, $\mathcal{R} = [0, \infty[$ if and only if $p = 0$, ν–almost surely.

ii) For every $t > 0$

$$\mathbf{P}\big(t \in \mathcal{R}\big) > 0 \iff \int_{0+} p(u)S\big(du\big) < \infty. \tag{5}$$

And if the right hand side of condition (5) holds, then

$$\mathbf{P}(t \in \mathcal{R}) = \int_0^\infty F^t(dy)[1 - p(y)] \exp\Big\{-\int_0^y p(w)S(dw)\Big\}.$$

iii) 0 *is isolated in* $\mathcal{R}$ *a.s. if and only if*

$$\int_{0+} [1-p(y)]S(dy) < \infty. \tag{6}$$

iv) $\mathcal{R}$ *is bounded a.s. if and only if*

$$\int^{\infty+} \big[1-p(y)\big]S(dy) < \infty. \tag{7}$$

v) $\mathcal{R}=\{0\}$ *a.s if and only if* $p=1$, ν*–a.s.*

Proof. We begin by showing *i)*. Note the equivalence,

$$Z > t \Longleftrightarrow p(\Delta_s) \le u_s, \forall s \le t, (s, \Delta_s, u_s) \in \mathcal{P}'\,.$$

This shows in particular that Z is as $(\mathcal{G}_t)_{t\ge 0}$–stopping time. So the event $''Z>0''$ has probability 0 or 1. From the former equivalence we also have that

$$\begin{aligned}\mathbf{P}(Z>t) &= \mathbf{E}\Big[\mathbf{E}\left(\{Z>t\}|\{(s,\Delta_s), s\le t\}\right)\Big]\\ &= \mathbf{E}\Big[\prod_{s\le t}[1-p(\Delta_s)]\Big]\end{aligned} \tag{8}$$

The second equality was obtained using the fact that u's are independent identically distributed with distribution uniform on $[0,1]$. The probability (8) is positive if and only if

$$\prod_{\{(s,\Delta_s),s\le t\}} \big[1-p(\Delta_s)\big] > 0 \quad \text{a.s..}$$

This is also equivalent to the convergence a.s of the series $\sum_{\{s\le t\}} p(\Delta_s)$. We know by Campbell's Theorem that the latter converges a.s. if and only if the condition $\int_0^\infty p(y)\nu(dy) < \infty$ holds. This shows the first assertion of *i)* in Proposition 1. Suppose that $\int_0^\infty p(y)\nu(dy) < \infty$. The fact that Z follows an exponential law with parameter $\int_0^\infty p(y)\nu(dy)$ is a direct application of the exponential formula and the fact that the convergence a.s. of the sum $\sum p(\Delta_s)$ is equivalent to the convergence a.s. of $\sum \log[1-p(\Delta_s)]$. Indeed,

$$\begin{aligned}\mathbf{P}(Z>t) &= \mathbf{E}\Big[\exp\{\sum_{s\le t}\log[1-p(\Delta_s)]\}\Big]\\ &= \exp\Big\{-t\int_0^\infty p(y)\nu(dy)\Big\}\end{aligned}$$

This entails that $\mathbf{P}(Z>t)=1$ for all $t>0$, if and only if $p(y)=0$ for ν–almost every y.

Next, we show *ii)*. Take $\mathcal{H}_t = \sigma\big\{Y_t(s), 0\le s\le t\big\}$. By equivalence (3) and the independence of the random variables $u's$

$$\begin{aligned}\mathbf{P}\left(t\in\mathcal{R}\right) &= \mathbf{E}\left[\mathbf{E}\left(\{t\in\mathcal{R}\}\mid\mathcal{H}_t\right)\right]\\ &= \mathbf{E}\left[\mathbf{E}\left(p(Y_t(s))\le u_{t-s} \text{ for all } s\in\Gamma_t \mid \mathcal{H}_t\right)\right]\\ &= \mathbf{E}\Big[\prod_{y\in A_t}\left(1-p(y)\right)\Big]\\ &= \mathbf{E}\Bigg[\mathbf{E}\Big[\prod_{y\in A_t}\left(1-p(y)\right)\mid Y_t(t)\Big]\Bigg]\end{aligned}$$

with $A_t = \{r < \infty \mid Y_t(s) = r \text{ for some } s,\ 0 \le s \le t\}$. Let $(Z_k)_{k\ge1}$ be a sequence of i.i.d.r.v.'s with common exponential distribution and independent of $Y_t(t)$. Set $W_n = \sum_{k=1}^n Z_k$, for $n \ge 1$. By Lemma 2

$$\mathbf{P}(t \in \mathcal{R}) = \mathbf{E}\Big(\big[1 - p\big(Y_t(t)\big)\big] H\big(Y_t(t)\big)\Big) \tag{9}$$

where $H(Y_t(t)) = \mathbf{E}\left(\prod_{n\ge1}\left[1 - p\Big(S^{-1}\big\{S(Y_t(t)) - W_n\big\}\Big)\right]\right)$. Given that $Y_t(t) = y$, the term under the expectation sign is positive a.s if and only if

$$\sum_{n=0}^{\infty} p\big(S^{-1}\big[S(y) - W_n\big]\big) < \infty \text{ a.s..}$$

Since the points $\{W_n\}_{n\ge1}$ are those of an homogeneous Poisson process (i.e. on $[0,\infty[$ with intensity given by the Lebesgue measure) by Campbell's Theorem the former holds if and only if

$$\int_0^\infty p\big(S^{-1}\big[S(y) - x\big]\big)dx = \int_0^y p(w)S(dw) < \infty.$$

As $|p(\cdot)| \le 1$ and $\overline{\nu}(y) < \infty$ for all $y > 0$, then the integral, $\int_0^y p(w)S(dw)$, is finite for all $y > 0$ if and only if this integral is finite in some neighborhood of 0. As a consequence the convergence of the sum in question does not depend on y. This shows that $H(y)$ is strictly positive for all $y > 0$ if and only if $\int_{0+} p(w)S(dw) < \infty$. The conclusion is straightforward. To obtain the expression for the probability of the event $t \in \mathcal{R}$, suppose that the right hand side of (5) holds, by the equation (9) we just have to calculate $H(y)$ for any $y > 0$. This is a direct application of the exponential formula and the fact that the convergence a.s. of the sum

$$\sum p\big(S^{-1}\big[S(y) - S_n\big]\big)$$

is equivalent to the convergence a.s. of the sum

$$\sum \ln\big[1 - p\big(S^{-1}\big[S(y) - S_n\big]\big)\big],$$

Therefore, $H(y) = \exp\big\{-\int_0^y p(w)S(dw)\big\}$, and the result follows.

The proofs of statement in *iii)* and *iv)* are very similar to that of statement in *ii)*. So we only point out the key arguments. To deal with this task define the process $J(0) = 0$ and for $s > 0$,

$$J(s) = \sup\big\{\Delta_v \mid (v, \Delta_v) \in \mathcal{P}; 0 < v \le s\big\},$$

and its set of jump times $\gamma_0 = \{s \mid J(s) > J(s^-)\}$. It was seen before that a such process is Extremal–F, with $F(x) = \exp -\overline{\nu}(x)$.

Sketch of proof of iii). Let T_1 be the abcissa of the first atom of $\mathcal{P}$ whose ordinate is a local maximum and whose abcissa is the left extremity of an interval that is not used to partially cover $\mathbb{R}^+$. That is,

$$T_1 = \inf\big\{t \in \gamma_0 \mid p(J(t)) \le u_t\big\}.$$

We thus have that

$$T_1 > s \iff p(J(v)) > u_v, \quad \forall\ v \in]0, s[\cap\gamma_0; \tag{10}$$

in words, $T_1 > s$ if and only if all the jump times of J before s are the left extremities of an interval that is used to partially cover $\mathbb{R}^+$. Now we claim that if $T_1 < \infty$ then $T_1 \in \mathcal{R}$. Indeed, if $T_1 = 0$

there is nothing to prove since $0 \in \mathcal{R}$. In the case $0 < T_1 < \infty$, we have, by the way we construct $\mathcal{R}$, that the only intervals that can be used to cover T_1 are those having a left extremity $< T_1$ but, since T_1 is a local maximum, all these intervals have a right extremity $\leq T_1$. Thus no interval with a left extremity to the left of T_1 covers T_1, that is $T_1 \in \mathcal{R}$. Recall that we have assumed that the measure ν has infinite total mass which implies that 0 is an accumulation point for the jump-times of J. Thus, we have furthermore that

$$0 \text{ is isolated in } \mathcal{R} \Longleftrightarrow T_1 > 0.$$

To see this we assume first that $T_1 < \infty$. If $T_1 = 0$ then there exists a random sequence of times $(t_n)_{n\in\mathbb{N}} \subset \{t \in \gamma_0 \mid p(J(t)) \leq u_t\}$ such that $t_n > 0$ and $\lim_{n\to\infty} t_n = 0$. By an argument similar to the one used to prove that $T_1 \in \mathcal{R}$ we have that $t_n \in \mathcal{R}$ for all $n \in \mathbb{N}$. Then 0 is not isolated in $\mathcal{R}$. Now, using that 0 is an accumulation point for the jump-times of J and that every jump–time of J to the left of T_1 is the left extremity of an interval used to partially cover $\mathbb{R}^+$ it is easily seen that if $T_1 > 0$ then the only uncovered point to the left of T_1 is 0, that is 0 is isolated in $\mathcal{R}$. We have proved the claim in the case $T_1 < \infty$, but the latter argument proves also that $T_1 = \infty$ implies that $\mathcal{R} = \{0\}$ and the claim follows.

So the random variable T_1 is an stopping time of the completed σ–field $(\mathcal{G}_t)_{t\geq 0}$, and by the zero-one law the event $\{T_1 > 0\}$ has probability zero or one. Therefore it is enough to show that $\mathbf{P}(T_1 > s) > 0$ for some $s > 0$. To this end we use the equivalence (10) and proceed as in the proof of *(ii)*. We omit the details.

Sketch of proof of iv). Let g_∞ be the largest element of $\mathcal{R}$. That is $g_\infty = \sup\{\ s > 0 :\ s \in \mathcal{R}\}$. It is easy to see that this random variable can be also related to the extremal process J as follows: for any $s > 0$,

$$g_\infty < s \Longrightarrow p\big(J(t)\big) > u_t \quad \text{for all} \quad t \in]s,\infty[\cap\gamma_0 \Longrightarrow g_\infty < \infty. \tag{11}$$

Indeed, let $s > 0$ and $(t_n, n \geq 1)$ be the jump times of J after s ranked in increasing order. By construction we have that $t^*_n = t_{n+1}$ for any $n \geq 1$. To see that if $g_\infty < s$ then every t_n is the left extremity of an interval that is used to partially cover $\mathcal{R}$, suppose that at least one of this times (say t_k) is not so; then by an argument similar to the one given before to prove that $T_1 \in \mathcal{R}$ we see that $t_k \in \mathcal{R}$, which is a contradiction since $g_\infty < s < t_k$. This proves the first claim. To prove the second one we use that $\cup_{n\geq 1}[t_n, t^*_n[$ forms a cover $[t_1, \infty[$ of $\mathbb{R}^+$, which implies that $\mathcal{R} \subset [0, t_1[$ and then that $g_\infty < \infty$ since $t_1 < \infty$ a.s.

Now the proof of (iv) uses the equivalence (11) and the additive structure of the extremal process J after time s stated at subsection (2.3). Indeed, proceeding as in the proof of *(ii)* we get that for any $s > 0$,

$$\begin{aligned}\mathbf{P}(g_\infty < s) &\leq \mathbf{P}(p(J(t)) > u_t, \forall t \in]s,\infty[\cap\gamma_0)\\ &= \mathbf{E}(\widetilde{H}(J(s)))\\ &\leq \mathbf{P}(g_\infty < \infty),\end{aligned}$$

where $\widetilde{H}(y) = \mathbf{E}(\prod_{n=1}^{\infty} p(S^{-1}(S(y) + W_n)))$ for $y > 0$ and $(W_n, n \geq 1)$ as in the proof of *(ii)*. Furthermore, using arguments similar to those given in *(ii)* we prove that $\widetilde{H}(y)$ is strictly positive of every $y > 0$ if and only if $\int^{\infty}(1 - p(w))S(dw) < \infty$. In this case,

$$\widetilde{H}(y) = \exp - \int_y^{\infty} (1 - p(w))S(dw),$$

and for any $s > 0$,

$$0 < \mathbf{E}(\widetilde{H}(J(s))) = \mathbf{E}(\exp\{-\int_{J(s)}^{\infty} (1 - p(w))S(dw)\}) \leq \mathbf{P}(g_\infty < \infty).$$

Thus making $s \to \infty$ we prove that $\mathbf{P}(g_\infty < \infty) = 1$. Now, if $\int^\infty (1-p(w))S(dw) = \infty$, then $H(y) = 0$ for every $y > 0$ and as a consequence $\mathbf{P}(g_\infty < \infty) = 0$.

proof of v) We know that if $p(\cdot) \equiv 1$ then $\mathcal{R} = \{0\}$ a.s.. To show the converse note that

$$\mathbf{P}\left(\mathcal{R} = \{0\}\right) = 1 - \mathbf{P}\left(p(J(s)) \leq u_s \text{ for some } s \in \gamma_0\right).$$

Conditioning by $\{J(s), s > 0\}$ we see that

$$\mathbf{P}\left(p(J(s)) \leq u_s \text{ for some } s > 0\right) = 0 \iff \sum_{s>0}[1 - p(J(s))] = 0 \text{ a.s.}$$

the former can only happen if $p(\cdot) = 1$ ν–a.e.. □

To continue our study of the random set $\mathcal{R}$ we adopt the approach of regenerative sets.

4 Structure of $\mathcal{R}$

In this section we show that the uncovered set $\mathcal{R}$ is regenerative and to make the paper self contained we first outline some relevant results on regenerative sets and subordinators. All the results about subordinators can be found in Bertoin [1] and those regarding regenerative sets in Kingman [10], Maisonneuve [12, 13], Fitzsimmons, Fristedt & Maisonneuve [5], Meyer[16] and Fristedt [7]. This results will be then used to characterize $\mathcal{R}$.

4.1 Regenerative Sets and Subordinators

According to Kingman [10] a random set $\mathcal{M}$ is a *Standard Regenerative Phenomena* if there exists a function $\mathbf{k} :]0,\infty[\to]0,1]$ whose limit at zero is 1 and such that

$$\mathbf{P}\left(t_1, t_2, \cdots, t_n \in \mathcal{M}\right) = \prod_{r=1}^{n} \mathbf{k}(t_r - t_{r-1}),$$

for any $0 = t_0 < t_1 < \cdots < t_n$. The term "regenerative" comes from the following property that is obtained from the former equality. For any $l > 0$ the conditional joint distributions of $\mathcal{M} \cap [l, \infty[$ given that $l \in \mathcal{M}$ and given the past before l are the same as the unconditional joint distributions of $\mathcal{M}$. Kingman has shown that a standard regenerative phenomena is the image of a subordinator with positive drift whose law is characterized by $\mathbf{k}$ (for a proof of the latter properties see Kingman [10]). However this definition is not convenient when $\mathbf{P}(t \in \mathcal{R}) = 0$ for all $t > 0$. An adequate and easy to handle definition was given by Maisonneuve [13]. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, $(\mathcal{Q}_t)_{t\geq 0}$ a filtration in $\mathcal{F}$ and $\mathcal{M} \subset [0, \infty[$ a closed random set in $(\Omega, \mathcal{F})$. $\mathcal{M}$ is a regenerative set relative to $(\mathcal{Q}_t)_{t\geq 0}$ if

(a) $(D_t)_{t\geq 0} = \inf\{\mathcal{M} \cap]t, \infty[\}$ is $(\mathcal{Q}_t)_{t\geq 0}$–adapted;

(b) the law of $\mathcal{M} \circ \theta_{D_t} = \{s - D_t \mid s \in \mathcal{M}, s \geq D_t\}$ given $\mathcal{Q}_t$ and $D_t < \infty$ is the same as $\mathcal{M}$.

See Fitzsimmons et al. [5] for more details. Maisonneuve [12] has shown that the closure of the image of a subordinator is a regenerative set and that any regenerative set is the closure of the image of a subordinator, determined up to *linear–equivalence*, (to be defined below). We are in position to recall some facts about subordinators. The law of a subordinator, σ, is specified by the Laplace transform of its one dimensional distribution. Its Laplace transform can be expressed in the form $\mathbf{E}(\exp\{-\lambda\sigma_t\}) = \exp\{-t\phi(\lambda)\}$ where the function $\phi : [0,\infty[\to [0,\infty[$ is called the Laplace exponent of σ. For each subordinator σ, there exist a unique pair $(k, \mathtt{d})$ of non-negative real numbers and a unique measure Π on $]0,\infty[$ such that $\int \inf\{1,x\}\Pi(dx) < \infty$, and

$$\phi(\lambda) = k + \mathtt{d}\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right)\Pi(dx).$$

Conversely, any function ϕ that can be expressed in the previous form is the Laplace exponent of a subordinator. One calls k the killing rate, $\mathtt{d}$ the drift coefficient and Π the Lévy measure of σ. Let c be a constant strictly positive. Thus σ_{tc} still is a subordinator and its Laplace exponent is characterized by $(ck, c\mathtt{d}, c\Pi)$. So the subordinator $\{\sigma_t, t \geq 0\}$ and $\{\sigma_{tc}, t \geq 0\}$ have the same range. Two such subordinators are called linearly equivalent. The measure potential $U(dx)$ of the subordinator σ is often called the *Renewal Measure* and it is given by

$$\int_{[0,\infty[} f(x)U(dx) = \mathbf{E}\left(\int_0^\infty f(\sigma_t)dt\right).$$

The distribution function of the renewal measure $U(x) = \mathbf{E}\left(\int_0^\infty 1_{\{\sigma_t \leq x\}}dt\right)$ for $x \geq 0$, is called *renewal function*. The Laplace transform of the renewal measure is related to the Laplace exponent of the subordinator by

$$\int_{[0,\infty)} e^{-\lambda x}U(dx) = \frac{1}{\phi(\lambda)}.$$

Denote by $\mathcal{M}$ the closure of the image of a subordinator σ so the renewal measure characterizes the law of the regenerative set $\mathcal{M}$ since ϕ characterizes the law of σ and from the previous identity ϕ is characterized by the renewal measure U. By using Fubini's Theorem we obtain that $\mathcal{M}$ has zero Lebesgue measure a.s. if and only if $\mathtt{d} = 0$, and we then say that $\mathcal{M}$ is light. Otherwise we say that $\mathcal{M}$ is heavy. We will also need the following Lemma that relies the renewal measure with the probability that $x \in \mathcal{M}$ for any $x > 0$, fixed.

Lemma 3.

- *(Kesten) If the drift* $\mathtt{d} = 0$, *then* $\mathbf{P}(x \in \mathcal{M}) = 0$ *for every* $x > 0$.
- *(Neveu) If* $\mathtt{d} > 0$, *then the function* $\mathtt{d}^{-1}\,\mathbf{P}(x \in \mathcal{M})$ *is a version of the renewal density* $dU(x)/dx$ *that is continuous and everywhere positive on* $[0,\infty[$.

Concerning regenerative sets:

$0 \in \mathcal{M}$. If 0 is isolated at $\mathcal{M}$, then $\mathcal{M}$ has only isolated points and we say that $\mathcal{M}$ is discrete. If 0 is not isolated then $\mathcal{M}$ does not have any isolated points, we then say that $\mathcal{M}$ is perfect. A right closed random set is regenerative if and only if its closure is regenerative, this can be read from Fitzsimmons et al. [5], page 158. Let $\underline{\mathcal{M}}$ be the set of isolated points and right accumulation points of $\mathcal{M}$ then for every $t > 0$, $\mathbf{P}\left(t \in \mathcal{M} \setminus \underline{\mathcal{M}}\right) = 0$, then $\mathbf{P}(t \in \mathcal{M}) = \mathbf{P}(t \in \underline{\mathcal{M}})$. Let Z be the first time after 0 when t does not belong to $\mathcal{M}$, then there exist a constant $q \in [0,\infty]$ such that $\mathbf{P}(Z > t) = e^{-qt}$ for all t. If $q = 0$, then $\mathcal{M} = \mathbb{R}^+$ a.s. If $0 < q < \infty$, then $\mathcal{M}$ is a.s. the union of a sequence of closed disjoints intervals. If $q = \infty$, then $\mathcal{M}$ has a.s. empty interior. For us one of the most useful results on regenerative sets will be

Lemma 4. *(Fitzsimmons et al. [5])*
Let $(\mathcal{R}_n, n \geq 1)$ be a decreasing sequence of regenerative sets with corresponding renewal functions U_n. Then $\cap_{n=1}^{\infty} \mathcal{R}_n$ is a regenerative set with a corresponding renewal function equal to the vague limit of $c_n U_n$ as $n \to \infty$, where the $(c_n, n \geq 1)$ is an appropriate sequence of constants, in fact we may choose $c_n = c/U_n(1)$, with $c > 0$, a constant.

4.2 $\mathcal{R}$ as a Regenerative Set

Theorem 1.
The uncovered random set $\mathcal{R}$ is a regenerative set relative to $(\mathcal{G}_t)_{t\geq 0}$.

To prove Theorem 1 we will show that if $\mathbf{P}(t \in \mathcal{R})$ is strictly positive for all $t > 0$, then $\mathcal{R}$ is a *Standard Regenerative Phenomena* with function $\mathbf{k}(t) = \mathbf{P}(t \in \mathcal{R})$ and then we proceed by approximation using Lemma 4.

Proof. Let $p : [0, \infty[\to [0, 1]$ be a measurable function continuous at 0 such that $p(0) = 0$ and $\int_{0+} p(y)S(dy) < \infty$. So by Proposition 1, $\mathbf{P}(t \in \mathcal{R}) > 0$ for all $t > 0$. We begin by showing that

$$\lim_{t \to 0} \mathbf{P}(t \in \mathcal{R}) = 1.$$

We know by *ii)* in Proposition 1 that

$$\mathbf{P}(t \in \mathcal{R}) = \int_0^{\infty} F^t(dy)h(y)$$

with $h(y) = [1 - p(y)] \exp\left\{ -\int_0^y p(w)S(dw) \right\}$. Since the measure $F^t(dy)$ converges weakly to the Dirac mass at zero as t goes to zero, $h(y) \leq 1$ for all $y \geq 0$ and p is continuous at 0, then

$$\lim_{t \to 0} \mathbf{P}(t \in \mathcal{R}) = h(0) = 1.$$

Let $0 < t_1 < t_2$. We next show that

$$\mathbf{P}(t_1, t_2 \in \mathcal{R}) = \mathbf{P}(t_1 \in \mathcal{R})\, \mathbf{P}(t_2 - t_1 \in \mathcal{R}).$$

As $\mathbf{P}(t_1 \in \mathcal{R}) > 0$ then

$$\mathbf{P}(t_1, t_2 \in \mathcal{R}) = \mathbf{P}(t_1 \in \mathcal{R})\, \mathbf{P}(t_2 \in \mathcal{R} \,|t_1 \in \mathcal{R}).$$

Given that $t_1 \in \mathcal{R}$, every interval having left end point in $T \cap]0, t_1[$ can not cover any point $s > t_1$, since it does not do for t_1. So the coverage of any point $s > t_1$ just depends on the points of the Poisson point process $\mathcal{P}$ that fall in $]t_1, \infty[\times]0, \infty[$. Moreover, the shifted point process

$$\mathcal{P}^{t_1} = \left\{ (t_1 + s, \Delta_{t_1+s}, u_{t_1+s}) \in \mathcal{P}'; s > 0 \right\},$$

is independent of $\mathcal{G}_{t_1}$ and still is a Poisson point process with characteristic measure $dt \otimes \nu(dy) \otimes du$ (see Meyer [17]). Let T^{t_1}, be the set of points that are the left end points of the intervals that are deleted from $\mathbb{R}^+$, corresponding to $\mathcal{P}^{t_1}$, that is

$$T^{t_1} = \left\{ r > 0 |\; p(\Delta_{t_1+r}) > u_{t_1+r} \right\}.$$

So

$$\mathcal{R}^{t_1} = [0,\infty[\setminus \bigcup_{x\in T^{t_1}} [x,x^*[,$$

enjoys the property

$$\mathcal{R} =^d \mathcal{R}^{t_1} = \mathcal{R}\circ\theta_{t_1} \mid t_1 \in \mathcal{R},$$

with $\mathcal{R}\circ\theta_t(\omega) = (\mathcal{R}-t)^+(\omega) = \{s-t|\ s\in\mathcal{R}, s\geq t\}$. In particular,

$$\begin{aligned}\mathbf{P}(t_2\in\mathcal{R}\,|t_1\in\mathcal{R}) &= \mathbf{P}(t_2-t_1\in\mathcal{R}\circ\theta_{t_1}|t_1\in\mathcal{R})\\ &= \mathbf{P}(t_2-t_1\in\mathcal{R}).\end{aligned}$$

The argument for any $0<t_1<t_2<\cdots<t_n$, is exactly the same if we note the obvious fact $\mathbf{P}(t_n\in\mathcal{R}\,|t_1,t_2,\cdots t_{n-1}\in\mathcal{R}) = \mathbf{P}(t_n\in\mathcal{R}\,|t_{n-1}\in\mathcal{R})$. So we have showed that $\mathcal{R}$ is the image of a subordinator with positive drift. To conclude the proof, let $p:[0,\infty[\to[0,1]$, be any measurable function. Set

$$p_n(y) = \begin{cases} p(y) & \text{if } y>1/n \\ 0 & \text{if } 0\leq y\leq 1/n \end{cases},$$

and $\mathcal{R}_n$ its associated uncovered set. The function p_n satisfies condition (5) for any Borel measure ν and $n\geq 1$, is continuous at zero and $p_n(0)=0$. Denote by $\overline{\mathcal{R}}_n$ the closure of $\mathcal{R}_n$. So $\big(\overline{\mathcal{R}}_n : n\in\mathbb{N}\big)$ is a decreasing sequence of regenerative closed random sets and $\overline{\mathcal{R}} = \bigcap_{n\in\mathbb{N}}\overline{\mathcal{R}}_n$. Therefore, by Lemma 4 it follows that $\overline{\mathcal{R}}$ is regenerative and by consequence $\mathcal{R}$ is regenerative. □

Remark 2. Let $p:[0,\infty[\to[0,1]$ and ν a Borel measure such that condition (6) holds. Then the associated uncovered random set $\mathcal{R}$ is a discrete regenerative set. This provides an example that does not belong to Mandelbrot's class of regenerative sets, since the latter are always perfect or trivial (equal to $\{0\}$ a.s.), see e.g. Theorem 1 and corollary 1 in Fitzsimmons et al. [6] or Theorem 7.2 in Bertoin [1].

The following statements rephrases Proposition 1 in terms of subordinators.

Let Z be the first time after 0 when t does not belong to $\mathcal{R}$, it was shown that Z follows an exponential law with parameter q, given by $q=\int_0^\infty p(x)\nu(dx)$. As the only Regenerative sets that are union of disjoint closed intervals are those that are the image of a compound Poisson process with drift. Then, $\mathcal{R}$, is the image of a compound Poisson process with drift if and only if $\int_0^\infty p(x)\nu(dx)<\infty$.

Given that the only Regenerative sets that have isolated points are the image of compound Poisson process without drift, $\mathcal{R}$ is the image of a compound Poisson process without drift if and only if $\int_{0+}[1-p(x)]S(dx)<\infty$.

If now we are interested in the Lebesgue measure of the regenerative set $\mathcal{R}$, by applying Fubini's Theorem we obtain that $\mathcal{R}$ is heavy if and only if $\int_{0+}p(y)S(dy)<\infty$. Which is equivalent to $\mathcal{R}$ is light if and only if $\int_{0+}p(y)S(dy)=\infty$.

Last, $\mathcal{R}$ is perfect, equivalently, is the image of a subordinator with Lévy measure Π such that $\Pi]0,\infty[=\infty$ if and only if $\int_{0+}[1-p(y)]S(dy)=\infty$.

5 Further properties of $\mathcal{R}$

Firstly, we will calculate the renewal function of the set $\mathcal{R}$. In the case $\mathcal{R}$ has positive Lebesgue measure a.s., i.e.,

$$\int_{0+} p(y)S(dy)<\infty,$$

from Lemma 3 the function

$$c\int_0^\infty F^t(dy)[1-p(y)]\exp\Big\{\int_0^y p(w)S(dw)\Big\}$$

is a version of the density of the renewal measure of $\mathcal{R}$, for c a positive constant. This means that for $a>0$, the renewal function is given by

$$U[0,a]=c\int_0^a dt\int_0^\infty F^t(dy)[1-p(y)]\exp\big\{-\int_0^y p(w)S(dw)\big\}.$$

We will generalize this result for any measurable function p. Our argument is similar to the analogue of Fitzsimmons et al. [6], Theorem 1. To tackle this problem we will use the following

Lemma 5.
Let v_0, be the first time when $S(x)=0$, that is, $v_0=S^{-1}(0)$. The integral

$$\int_0^a dt\int_0^{v_0} F^t(dy)[1-p(y)]\exp\Big\{\int_y^{v_0} p(w)S(dw)\Big\},$$

is finite for all $a>0$.

Proof. Let $h(y)=[1-p(y)]\exp\Big\{\int_y^{v_0} p(w)S(dw)\Big\}$ and note that $F^t(dy)=te^{-S(y)}F^t(y)S(dy)$. By Fubini's Theorem

$$\begin{aligned}
&\int_0^a dt\int_0^{v_0} F^t(dy)h(y)\\
&=\int_0^{v_0} S(dy)h(y)e^{-S(y)}\Big(\int_0^a dt\ t\exp\big\{-te^{-S(y)}\big\}\Big)\\
&\leq\int_0^{v_0} S(dy)h(y)e^{S(y)}\\
&=e^{S(v_0)}\int_0^{v_0} S(dy)[1-p(y)]\exp\Big\{-\int_y^{v_0}[1-p(w)]S(dw)\Big\}\\
&=\int_0^{v_0} d\Big(\exp\Big\{-\int_y^{v_0}[1-p(w)]S(dw)\Big\}\Big)\\
&=\Big(1-\exp\big\{-\int_0^{v_0}[1-p(w)]S(dw)\big\}\Big)
\end{aligned}$$

the second inequality was obtained from an integration by parts in

$$\int_0^a dt\ t\exp\{-tc_y\}=-\frac{a}{c_y}e^{-ac_y}+\frac{1}{c_y^2}\big(1-e^{-ac_y}\big)\leq\frac{1}{c_y^2},$$

where $c_y=e^{-S(y)}$. □

Just to ease the notation, in the sequel we will suppose that $v_0=S^{-1}(0)=1$. Now we have all the elements to show the

Theorem 2.
Let $p:[0,\infty[\to[0,1]$ be a measurable function, ν be a atom-less measure such that $\overline{\nu}(x)=\nu]x,\infty[$

is finite, strictly decreasing and $\overline{\nu}(0^+) = \infty$. *Set* $F(x) = \exp\{-\overline{\nu}(x)\}$ *and* $S(x) = -\log\{\overline{\nu}(x)\}$ *for all* $x \geq 0$. *Then the renewal function of* $\mathcal{R}$ *is given by*

$$U[0,a] = a\int_0^\infty F^a(dx)\exp\Big\{\int_x^1 p(y)S(dy)\Big\}$$

for all $a > 0$.

Proof. When $p(\cdot) = 1$ ν–a.s., it is the subject of *v)* in Proposition (1), that $\mathcal{R} = \{0\}$ a.s., which implies in particular that $U[0,a] \equiv 1$, for all $a > 0$. On the other hand, for any $a > 0$,

$$a\int_0^\infty F^a(dy)\exp\Big\{\int_y^1 p(x)S(dx)\Big\} = \int_0^\infty dx\ xe^{-x} = 1 = U[0,a]$$

where the first equality was obtained by the change of variables $x = ae^{-S(y)}$. So it remains to study the case $p(\cdot) \not\equiv 1$ in a set of positive ν–measure. For this we build a decreasing sequence of regenerative right closed random sets $\mathcal{R}_n$ as the uncovered random sets generated via

$$p_n(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{n} \\ p(y) & \text{if } y > \frac{1}{n} \end{cases}$$

and note that for this family of functions the condition (5) holds. By *ii)* in Proposition 1 and Lemma 3 the renewal function is given by

$$U_n[0,a] = \frac{1}{\gamma_n}\int_0^a dt\int_0^\infty F^t(dy)h_n(y)$$

with

$$h_n(y) = \big[1 - p_n(y)\big]\exp\Big\{-\int_0^y p_n(w)S(dw)\Big\}.$$

By construction

$$\begin{aligned} U_n[0,a] &= \frac{1}{\gamma_n}\int_0^a dtF^t(1/n) + \frac{1}{\gamma_n}\int_0^a dt\int_{1/n}^\infty F^t(dy)h_n(y) \\ &= I_n + II_n. \end{aligned}$$

Take $\gamma_n = \exp\Big\{-\int_0^1 p_n(y)S(dy)\Big\}$ and note that by monotone convergence

$$II_n \xrightarrow[\text{n}\to\infty]{} \int_0^a dt\int_0^\infty F^t(dy)[1-p(y)]\exp\Big\{\int_y^1 p(w)S(dw)\Big\}.$$

Now if 0 is isolated, i.e., if $\int_{0+}[1-p(w)]S(dw) < \infty$ then

$$\begin{aligned} I_n &= \frac{1}{\gamma_n}\frac{1}{\overline{\nu}(1/n)}\big[1 - e^{-a\overline{\nu}(1/n)}\big] \\ &= \big[1 - e^{-a\overline{\nu}(1/n)}\big]\exp\Big\{-\int_{1/n}^1 [1-p_n(y)]S(dy)\Big\} \\ &\xrightarrow[\text{n}\to\infty]{} \exp\Big\{-\int_0^1 [1-p(y)]S(dy)\Big\}. \end{aligned}$$

Otherwise, $I_n \to 0$ as $n \to \infty$. From the previous calculations we obtain the expression

$$U[0,a] = U\{0\} + \int_0^a dt \int_0^\infty F^t(dy)[1-p(y)] \exp\Big\{ \int_y^1 p(w)S(dw)\Big\}, \tag{12}$$

for any $a > 0$, with $U\{0\} = \exp\Big\{-\int_0^1 [1-p(y)]S(dy)\Big\}$. Next we deduce the result from the identity (12) by means of some relatively elementary calculations. Let dA_y denotes the measure induced by the increasing function $A_y = \exp\Big\{ - \int_y^1 [1-p(w)]S(dw)\Big\}$. From equation (12) and Fubini's Theorem

$$\begin{aligned}
U]0,a] &= \int_0^a dt \int_0^\infty F^t(dy) e^{-S(y)}[1-p(y)]A_y \\
&= \int_0^\infty dA_y e^{-S(y)} \int_0^a dt\ te^{-S(y)} \exp\big\{ - te^{-S(y)}\big\} \\
&= \int_0^\infty dA_y \Big(\big[1 - F^a(y)\big] - ae^{-S(y)} F^a(y)\Big) \\
&= -A_0 + \int_0^\infty S(dy) a^2 e^{-2S(y)} F^a(y) A_y,
\end{aligned}$$

the fourth equality was obtained via an integration by parts using that

$$\begin{aligned}
&d\Big(\big[1-F^a(y)\big] - ae^{-S(y)}F^a(y)\Big) \\
&= -ae^{-S(y)}F^a(y)S(dy) + ae^{-S(y)}F^a(y)S(dy) - a^2 e^{-2S(y)}F^a(y)S(dy),
\end{aligned}$$

and since $F^a(y) \sim 1 - ae^{-S(y)}$, as y goes to ∞,

$$\begin{aligned}
&\Big(\big[1-F^a(y)\big] - ae^{-S(y)}F^a(y)\Big)A_y\Big|_0^\infty \\
&= -A_0 + \lim_{y\to\infty} A_y\big[1 - F^a(y) - ae^{-S(y)}F(y)\big] \\
&= -A_0 + a \lim_{y\to\infty} A_y e^{-S(y)}[1 - F(y)] \\
&= -A_0 + a \lim_{y\to\infty} A_y e^{-2S(y)} \\
&= -A_0 + a \lim_{y\to\infty} \exp\Big\{ - \int_1^y p(w)S(dw) - S(y)\Big\} \\
&= -A_0.
\end{aligned}$$

Therefore,

$$U[0,a] = U\{0\} - A_0 + a\int_0^\infty F^a(dy) \exp\Big\{ \int_y^1 p(w)S(dw)\Big\},$$

which ends the proof since $U\{0\} = A_0$. □

Remark 3. Results iii)–v) in Proposition 1 could be obtained as a corollary to Theorem 2. To see iii), recall that 0 is isolated in $\mathcal{R}$ if and only if the renewal function has an atom at 0. It has been showed at the first stage of the proof of Theorem 2 that U has an atom at 0 if an only if $\int_0^1 [1-p(y)]S(dy) < \infty$. To get iv), recall that $U[0,\infty[< \infty$ if and only if $\mathcal{R}$ is bounded a.s. Use (12) and proceed as in the proof of Lemma 5 to show that

$$U[0,\infty[= \exp\Big\{ \int_1^\infty [1-p(y)]S(dy)\Big\}.$$

Last, if $\mathcal{R} = \mathbb{R}^+$ by Lemma 3 $U(dx) = cdx$, we can suppose without loss of generality that $c = 1$. Use (12) to conclude that $p = 0$ ν-a.s.

Example 2 (continuation example 1). Let the function $p(y) = \mathsf{p}$ for all $y \geq 0$ with $\mathsf{p} \in (0, 1)$. Then the associated set $\mathcal{R}_{\mathsf{p}}$ is indistinguishable of the image of a subordinator stable$(1 - \mathsf{p})$.

To show this we just have to calculate the renewal function. By Theorem 2

$$\begin{aligned} U[0, a] &= a \int_0^\infty F^a(dy) e^{-\mathsf{p}S(y)} \\ &= a \int_0^\infty dS(y) a e^{-(1+\mathsf{p})S(y)} \exp\{-a \exp\{-S(y)\}\} \\ &= a^{1-\mathsf{p}} \int_0^\infty x^{\mathsf{p}} e^{-x} dx \\ &= a^{1-\mathsf{p}} \Gamma(1 + \mathsf{p}) \end{aligned}$$

where $\Gamma(x)$ denotes the function gamma calculated in x. So the Laplace exponent is given by

$$\phi(\lambda) = c_{\mathsf{p}} \lambda^{1-\mathsf{p}},$$

with $c_{\mathsf{p}} = \big(\Gamma(1 + \mathsf{p})\Gamma(2 - \mathsf{p})\big)^{-1}$.

5.1 Fractal Dimensions of $\mathcal{R}$

In this subsection we study some fractal dimensions of the regenerative set $\mathcal{R}$. To this end we next introduce two of the most important notions of fractal indices used in probability *Hausdorff and Packing dimensions*. We refer to Falconer [3] for a detailed account on these and other definitions of dimension.

Hausdorff measures and dimension. Let h be a strictly increasing continuous function on $\mathbb{R}^+$ such that $h(0) = 0$ and $h(\infty) = \infty$ and F be a Borel subset of $\mathbb{R}$. A δ–cover of a subset F is a collection $\{U_i\}$ countable (or finite) of subsets of diameter, $|U_i|$, at most $\delta > 0$ that covers F, i.e., $F \subset \bigcup_i U_i$. For any δ we define

$$\mathcal{H}^h_\delta(F) = \inf \left\{ \sum_{i=1}^\infty h\big(|U_i|\big) : \{U_i\} \text{ is } \delta\text{–cover of } F \right\}.$$

As δ decreases the class of permissible δ–covers of F is reduced. Therefore the number $\mathcal{H}^h_\delta$ increases and so approaches a limit as $\delta \to \infty$. The Hausdorff h–measure of F is the number

$$\mathcal{H}^h(F) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(F) \in [0, \infty].$$

It can be shown that the mapping $F \longrightarrow \mathcal{H}^h(F)$ defines a measure on a σ–field that includes the Borel sets (see Falconer [4]). Of special interest is the case where $h(x) = x^s$, $s > 0$ in which we write $\mathcal{H}^s$ and speak of s–measure. For any F it is clear that $\mathcal{H}^s(F)$ is non–decreasing as s increases. Furthermore, if $t < s$ then

$$\mathcal{H}^s_\delta(F) \leq \delta^{s-t} \mathcal{H}^t_\delta(F),$$

which implies that if $\mathcal{H}^t(F)$ is positive then $\mathcal{H}^s(F)$ is infinite. Thus there exist a critical value, $\dim_H F$, called the Hausdorff dimension of F such that

$$\begin{aligned} \mathcal{H}^s(F) &= \infty \quad \text{if } 0 \leq s < \dim_H(F) \\ \mathcal{H}^s(F) &= 0 \quad \text{if } \dim_H(F) < s < \infty. \end{aligned}$$

Packing measures and dimension. Let F be a Borel subset of $\mathbb{R}$, $s, \delta > 0$ and $B_r(x)$ a ball of radii r with center in x. Consider

$$\mathcal{P}^s_\delta(F) = \sup\left\{ \sum_i |B_{r_i}|^s : \{B_{r_i}(x_i)\} \text{ disjoints such that } x_i \in F, r_i < \delta \right\}$$

Since $\mathcal{P}^s_\delta(F)$ decreases with δ, the limit

$$\mathcal{P}^s_0(F) = \lim_{\delta \to 0} \mathcal{P}^s_\delta(F)$$

exist. It may be shown that the mapping

$$F \longrightarrow \mathcal{P}^s(F) = \inf\left\{ \sum_i \mathcal{P}^s_0(F_i) : F \subset \bigcup_i F_i \right\}$$

defines a measure on $\mathbb{R}$, known as the s–dimensional packing measure. Analogous to the case of the Hausdorff dimension we define the fractal index

$$\mathrm{Dim}_P(F) = \inf\big\{ s > 0 : \mathcal{P}^s(F) = 0 \big\},$$

which is known as the packing dimension. The definition of packing measure and dimension where introduced by Taylor and Tricot [22]. Its well known that for any Borel subset of $\mathbb{R}$

$$0 \leq \dim_H(F) \leq \mathrm{Dim}_P(F) \leq 1,$$

Suitable examples shows that none of the inequalities can be replaced by equality. These fractal indices have the advantage of being defined for any set through measures which are relatively easy to manipulate. A major disadvantage is that in many cases it is hard to calculate or to estimate by computational methods. Although, for regenerative sets there exists some refined results that allow us to obtain its exact Hausdorff and Packing dimension. Let $\phi(\lambda)$ be the Laplace exponent of the regenerative set $\mathcal{R}$, i.e., for any $\lambda > 0$

$$\phi(\lambda) = \left(\int_0^\infty e^{-\lambda t} U(dt) \right)^{-1}$$

with U the renewal function of $\mathcal{R}$ given by Theorem 2. Define the so called lower and upper indices, respectively, of the Laplace exponent ϕ by

$$\underline{\mathrm{Ind}}\ \phi = \sup\big\{ \alpha > 0 : \lim_{\lambda \to \infty} \phi(\lambda)\lambda^{-\alpha} = \infty \big\},$$
$$\overline{\mathrm{Ind}}\ \phi = \inf\big\{ \alpha > 0 : \lim_{\lambda \to \infty} \phi(\lambda)\lambda^{-\alpha} = 0 \big\}.$$

with the usual convention $sup\ \emptyset = 0$. We recall the following results

Lemma 6.
We have a.s for every $t > 0$

$$\underline{Ind}\ \phi = dim_H(\mathcal{R} \cap [0,t])$$
$$\overline{Ind}\ \phi = Dim_P(\mathcal{R} \cap [0,t])$$

For a proof of these facts see chapter 5 section 1 in [1]. In the following Theorem we give formulas to calculate the Hausdorff and Packing dimensions of $\mathcal{R}$ in terms of p and S.

Theorem 3.
Almost surely for every $t > 0$, the Hausdorff and Packing dimensions of $\mathcal{R} \cap [0, t[$ are given by

$$dim_H(\mathcal{R} \cap [0,t]) = \liminf_{y\to 0+} \frac{\int_y^1 (1-p(w))S(dw)}{-S(y)},$$
$$Dim_P(\mathcal{R} \cap [0,t]) = \limsup_{y\to 0+} \frac{\int_y^1 (1-p(w))S(dw)}{-S(y)}.$$

Proof. It is well known that $\phi(\lambda) \asymp (U[0,1/\lambda])^{-1}$, i.e., there exist two positive constants c, c', such that $c(U[0,1/\lambda])^{-1} \le \phi(\lambda) \le c'(U[0,1/\lambda])^{-1}$. (see [1] Proposition 1.8, page 12). So it is immediate that

$$\underline{\text{Ind}}\ \phi = \sup\big\{\alpha > 0 : \lim_{\lambda\to\infty} U[0,1/\lambda]\lambda^\alpha = 0\big\},$$
$$\overline{\text{Ind}}\ \phi = \inf\big\{\alpha > 0 : \lim_{\lambda\to\infty} U[0,1/\lambda]\lambda^\alpha = \infty\big\}.$$

This result will be our major tool in the estimation of the lower and upper indices of ϕ. The conclusion is then obtained by Lemma 6.

When 0 is isolated the affirmation is obvious. Indeed, in one hand, by Proposition 1 we know that $\int_{0+}[1-p(x)]S(dx) < \infty$ so

$$\frac{\int_y^1 [1-p(w)]S(dw)}{-S(y)} \xrightarrow[y\to 0]{} 0 = \rho.$$

On the other hand, in the proof of Theorem 2 we have shown that

$$U[0,a] = U\{0\} + \int_0^a dt \int_0^\infty F^t(dy)[1-p(y)] \exp\Big\{\int_y^1 p(w)S(dw)\Big\}$$

with $U\{0\} = \exp\Big\{-\int_0^1 [1-p(y)]S(dy)\Big\}$, then $\lambda^\alpha U[0,\frac{1}{\lambda}] \to \infty$ as $\lambda \to \infty$ for any $\alpha > 0$. Therefore $\underline{\text{Ind}}\ \phi = \overline{\text{Ind}}\ \phi = 0 = \rho$.

It remains to show the statement of Theorem 3 when 0 is not isolated. To reach our goal, we will first show that the function $\lambda U[0,1/\lambda]$ is related to the Laplace-Stieltjes transform of an increasing extended regularly varying function (say h). Then we will use a Tauberian theorem to determine the behavior at infinity of $\lambda U[0,1/\lambda]$ through that of h. (See e.g. Bingham et al. [2] Chapter 2 for background on extended regularly varying functions.) We first introduce some notation. Let $f_\lambda(x) = (1/\lambda)\exp\{-S(x)\}$ and $f_\lambda^{-1}(\cdot)$ the inverse in the variable x of $f_\lambda(x)$. Observe that $S\big(f_\lambda^{-1}(x)\big) = -\log \lambda x$, for all $x > 0$, thus

$$\lim_{\lambda\to\infty} S\big(f_\lambda^{-1}(x)\big) = -\infty, \quad \text{for all } x > 0,$$

and since S is continuous

$$\lim_{\lambda\to\infty} f_\lambda^{-1}(x) = 0, \quad \text{for all } x > 0.$$

Because of Theorem 2 and making the change of variables $y = f_\lambda(x)$ we get that

$$\begin{aligned}\lambda U[0,1/\lambda] &= \int_0^\infty F^{1/\lambda}(dx) \exp\Big\{\int_x^1 p(w)S(dw)\Big\} \\ &= \int_0^\infty S(dx)\frac{1}{\lambda}e^{-S(x)}F^{1/\lambda}(x)\exp\{\int_x^1 p(w)S(dw)\} \\ &= \int_0^\infty dy e^{-y} \exp\{\int_{f_\lambda^{-1}(y)}^1 p(w)S(dw)\}.\end{aligned}$$

Now, let S^{-1} be the right-continuous inverse of S, that is $S^{-1}(t) = \inf\{x > 0 : S(x) > t\}$. By a change of variables for Stieltjes integrals and a change of variables $u = e^{-w}$ we get that for any $\lambda, y > 0$

$$\begin{aligned} \exp\left\{\int_{f_\lambda^{-1}(y)}^{1} p(w)S(dw)\right\} &= \exp\left\{\int_{S(f_\lambda^{-1}(y))}^{0} p\left(S^{-1}(w)\right) dw\right\} \\ &= \exp\left\{\int_{1}^{\lambda y} p\left(S^{-1}\left(-\ln(u)\right)\right) \frac{du}{u}\right\} := h(\lambda y). \end{aligned} \tag{13}$$

In short, for every $\lambda > 0$,

$$\lambda U[0, 1/\lambda] = \frac{1}{\lambda}\int_0^\infty dy\; e^{-y/\lambda} h(y) = \widehat{h}(1/\lambda),$$

where $\widehat{h}$ denotes the Laplace–Stieltjes transform of h. By the representation theorem for extended regularly varying functions (Theorem 2.2.6 in [2]) we have that the function h is indeed an increasing extended regularly varying function. Furthermore, by a Tauberian theorem (Theorem 2.10.2 in [2]) we have that $h(\lambda) = O(\widehat{h}(1/\lambda))$ and $\widehat{h}(1/\lambda) = O(h(\lambda))$ as $\lambda \to \infty$. We deduce therefrom that

$$\begin{aligned} \underline{\text{Ind}}\ \phi &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} U[0, 1/\lambda]\lambda^\alpha = 0\right\}, \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{\alpha-1}\widehat{h}(1/\lambda) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{\alpha-1} h(\lambda) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \frac{\lambda/h(\lambda)}{\lambda^\alpha} = \infty\right\} \\ &= \liminf_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)}. \end{aligned}$$

Analogously, we get that

$$\overline{\text{Ind}}\ \phi = \limsup_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)}.$$

Last, by the fact that

$$\lambda/h(\lambda) = \exp\left\{\int_1^\lambda \left(1 - p\left(S^{-1}\left(-\ln(u)\right)\right)\right) \frac{du}{u}\right\}, \qquad \lambda > 0,$$

and reversing the change of variables done in equation (13) we deduce that

$$\begin{aligned} \liminf_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)} &= \liminf_{\lambda\to\infty} \frac{\int_{f_1^{-1}(\lambda)}^1 (1 - p(w))S(dw)}{-S(f_1^{-1}(\lambda))} \\ &= \liminf_{y\to 0^+} \frac{\int_y^1 (1 - p(w))S(dw)}{-S(y)}. \end{aligned}$$

Analogously, we prove the claim for the $\limsup$. $\square$

Example 3. Let $p(x) = \beta e^{-x}$ for $x > 0, \beta \in]0, 1]$ and $\overline{\nu}(x) = x^{-\alpha}$ for $\alpha > 0$. So $S(x) = \alpha \ln x$ and the associated uncovered random set $\mathcal{R}$ has zero Lebesgue measure, is perfect if $\beta \in]0, 1[$ and discrete if $\beta = 1$, unbounded and with fractal dimension $1 - \beta$.

Example 4. Let $S(x)$ be as in the previous example and

$$p(x) = \cos^2(1/x).$$

Then the associated uncovered set $\mathcal{R}$ has zero Lebesgue measure, is perfect, bounded and with fractal dimension 1/2.

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Bibliography

[1] J. Bertoin. Subordinators: examples and applications. In *Lectures on probability theory and statistics (Saint-Flour, 1997)*, volume 1717 of *Lecture Notes in Math.*, pages 1–91. Springer, Berlin, 1999.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1989.

[3] K. Falconer. *Fractal geometry.* John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[4] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1986.

[5] P. J. Fitzsimmons, B. Fristedt, and B. Maisonneuve. Intersections and limits of regenerative sets. *Z. Wahrsch. Verw. Gebiete*, 70(2):157–173, 1985.

[6] P. J. Fitzsimmons, B. Fristedt, and L. A. Shepp. The set of real numbers left uncovered by random covering intervals. *Z. Wahrsch. Verw. Gebiete*, 70(2):175–189, 1985.

[7] B. Fristedt. Intersections and limits of regenerative sets. In *Random discrete structures (Minneapolis, MN, 1993)*, volume 76 of *IMA Vol. Math. Appl.*, pages 121–151. Springer, New York, 1996.

[8] J. Hoffmann-Jørgensen. Markov sets. *Math. Scand.*, 24:145–166 (1970), 1969.

[9] J.-P. Kahane. Random coverings and multiplicative processes. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, volume 46 of *Progr. Probab.*, pages 125–146. Birkhäuser, Basel, 2000.

[10] J. F. C. Kingman. *Regenerative phenomena.* John Wiley & Sons Ltd., London-New York-Sydney, 1972. Wiley Series in Probability and Mathematical Statistics.

[11] J. F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability.* The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.

[12] B. Maisonneuve. Ensembles régénératifs, temps locaux et subordinateurs. In *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970)*, pages 147–169. Lecture Notes in Math., Vol. 191. Springer, Berlin, 1971.

[13] B. Maisonneuve. Ensembles régénératifs de la droite. *Z. Wahrsch. Verw. Gebiete*, 63(4):501–510, 1983.

[14] B. B. Mandelbrot. Renewal sets and random cutouts. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:145–157, 1972.

[15] P. Marchal. Stable processes on the boundary of a regular tree. *Ann. Probab.*, 29(4):1591–1611, 2001.

[16] P. A. Meyer. Ensembles régénératifs, d'apres Hoffmann –Jorgensen. In *Séminaire de Probabilités, IV (Univ. Strasbourg, année universitaire 1968–1969)*, pages 133–150. Lecture Notes in Math., Vol. 124. Springer, Berlin, 1970.

[17] P. A. Meyer. Processus de Poisson ponctuels, d'après K. Ito. In *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970)*, pages 177–190. Lecture Notes in Math., Vol. 191. Springer, Berlin, 1971.

[18] S. I. Resnick. Inverses of extremal processes. *Advances in Appl. Probability*, 6:392–406, 1974.

[19] S. I. Resnick. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust.* Springer-Verlag, New York, 1987.

[20] S. I. Resnick and M. Rubinovitch. The structure of extremal processes. *Advances in Appl. Probability*, 5:287–307, 1973.

[21] L. A. Shepp. Covering the line with random intervals. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 23:163–170, 1972.

[22] S. J. Taylor and C. Tricot. Packing measure, and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.*, 288(2):679–699, 1985.