Chapitre III

Recurrent extensions of self–similar Markov processes and Cramér's condition

Abstract

Let ξ be a real valued Lévy process that drifts to $-\infty$ and satisfies Cramér's condition, and X a self–similar Markov process associated to ξ via Lamperti's [22] transformation. In this case, X has 0 as a trap and fulfills the assumptions of Vuolle-Apiala [34]. We deduce from [34] that there exists a unique excursion measure $\mathbf{n}$, compatible with the semigroup of X and such that $\mathbf{n}(X_{0+} > 0) = 0$. Here, we give a precise description of $\mathbf{n}$ via its associated entrance law. To this end, we construct a self–similar process $X^{\natural}$, which can be viewed as X conditioned to never hit 0, and then we construct $\mathbf{n}$ in a similar way to the way in which the Brownian excursion measure is constructed via the law of a Bessel(3) process. An alternative description of $\mathbf{n}$ is given by specifying the law of the excursion process conditioned to have a given length. We establish some duality relations from which we determine the image under time reversal of $\mathbf{n}$.

Key words. Self–similar Markov processes, description of excursion measures, weak duality, Lévy processes.
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1 Introduction

Let $X = (X_t, t \geq 0)$ be a strong Markov process with values in $[0, \infty[$ and for $x \geq 0$, denote by $\mathbb{P}_x$ its law starting from x. Assume that X fulfills the scaling property: there exists some $\alpha > 0$ such that

$$\text{the law of } \ (cX_{tc^{-1/\alpha}}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}, \tag{1}$$

for any $x \geq 0$ and $c > 0$. Such processes were introduced by Lamperti [22] under the name of semi–stable processes, nowadays they are called α–self–similar Markov processes. We refer to Embrechts and Maejima [14] for a recent account on self–similar processes.

Lamperti established that for each fixed $\alpha > 0$, there exists a one to one correspondence between α–self–similar Markov processes on $]0, \infty[$ and Lévy processes which we now sketch. Let $(\mathbb{D}, \mathcal{D})$ be the space of càdlàg paths $\omega : [0, \infty[\to] - \infty, \infty[$ endowed with the σ–algebra generated by the coordinate maps and the natural filtration $(\mathcal{D}_t, t \geq 0)$ satisfying the usual conditions of right continuity and completeness. Let $\mathbf{P}$ be a probability measure on $\mathcal{D}$ such that under $\mathbf{P}$ the coordinate process ξ is a Lévy process that drifts to $-\infty$, i.e. $\lim_{s\to\infty} \xi_s = -\infty$. Set for $t \geq 0$

$$\tau(t) = \inf\{s > 0, \int_0^s e^{\xi_r/\alpha} dr > t\},$$

with the usual convention that $\inf\{\emptyset\} = \infty$. For an arbitrary $x > 0$, let $\mathbb{P}_x$ be the distribution on $\mathbb{D}^+ = \{\omega : [0, \infty[\to [0, \infty[\text{ càdlàg}\}$, of the time–changed process

$$X_t = x \exp\left(\xi_{\tau(tx^{-1/\alpha})}\right), \qquad t \geq 0,$$

where the above quantity is assumed to be 0 when $\tau(tx^{-1/\alpha}) = \infty$. We agree that $\mathbb{P}_0$ is the law of the process identical to 0. Classical results on time change yield that under $(\mathbb{P}_x, x \geq 0)$ the process X is Markovian with respect to the filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)}, t \geq 0)$. Furthermore, X has the scaling property (1). Thus, X is a self–similar Markov process on $[0, \infty[$ having 0 as trap or absorbing point. It should be clear that the distribution of the first hitting time of 0 for X,

$$T_0 = \inf\{t > 0 : X_t = 0\}$$

under $\mathbb{P}_x$ is the same as that of $x^{1/\alpha} I$ under $\mathbf{P}$, with I the so–called Lévy exponential functional associated to ξ and α, that is

$$I = \int_0^\infty \exp\{\xi_s/\alpha\} ds. \tag{2}$$

Since ξ drifts to $-\infty$ we have that $I < \infty$, $\mathbf{P}$–a.s. and

$$\mathbb{P}_x(X_{T_0-} = 0, T_0 < \infty) = 1 \quad \text{for all} \quad x > 0.$$

We will say that X hits 0 continuously. Besides, if in the former construction we use a Lévy process killed at an independent exponential time the resulting process is a self–similar Markov process X that hits 0 by a jump

$$\mathbb{P}_x(X_{T_0-} > 0, T_0 < \infty) = 1 \quad \text{for all} \quad x > 0.$$

Conversely, any self–similar Markov process that has 0 as a trap and hits 0 continuously (resp. by a jump) is the exponential of Lévy process (resp. killed at an independent exponential time) time changed, cf. [22]. In this chapter we will restrict ourselves to the case where X hits 0 continuously and we will devote the Chapter IV to study the case where X hits 0 by a jump.

Denote P_t and V_q the semigroup and resolvent for the process X killed at time T_0, say (X, T_0),

$$P_t f(x) = \mathbb{E}_x(f(X_t), t < T_0), \qquad x > 0,$$

$$V_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt, \qquad x > 0,$$

for non–negative or bounded measurable functions f. It is customary to refer to (X, T_0) as the minimal process.

Given that the preceding construction enables us to describe the behavior of the self–similar Markov process X until its first hitting time of 0, Lamperti [22] raised the following question: *What are the self–similar Markov processes $\widetilde{X}$ on $[0, \infty[$ which behave like (X, T_0) up to the time $\widetilde{T}_0$?* Lamperti solved this problem in the case where the minimal process is a Brownian motion killed at 0. Then Vuolle-Apiala [34] tackled this problem using excursion theory for Markov processes and assuming that the following hypotheses hold. There exists $\kappa > 0$ such that

(H1-a) the limit

$$\lim_{x\to 0} \frac{\mathbb{E}_x(1-e^{-T_0})}{x^{\kappa}},$$

exists and is strictly positive;

(H1-b) the limit

$$\lim_{x\to 0} \frac{V_q f(x)}{x^{\kappa}},$$

exists for all $f \in C_K]0,\infty[$ and is strictly positive for some such functions,

with $C_K]0,\infty[= \{f : \mathbb{R} \to \mathbb{R},$ continuous and with compact support on $]0,\infty[\}$. The main result in [34] is the existence of an unique entrance law $(\mathbf{n}_s, s > 0)$ such that

$$\lim_{s\to 0} \mathbf{n}_s\, B^c = 0,$$

for every neighborhood B of 0 and

$$\int_0^\infty e^{-s}\, \mathbf{n}_s\, 1 ds = 1.$$

This entrance law is determined by its q–potential via the formula

$$\int_0^\infty e^{-qs}\, \mathbf{n}_s\, f ds = \lim_{x\to 0} \frac{V_q f(x)}{\mathbb{E}_x(1-e^{-T_0})}, \qquad q > 0, \tag{3}$$

for $f \in C_K]0,\infty[$. Then, using the results of Blumenthal [7], Vuolle-Apiala proved that associated to the entrance law $(\mathbf{n}_s, s > 0)$ there exists a unique recurrent Markov process $\widetilde{X}$ having the scaling property (1) which is an extension of the minimal process (X, T_0), that is $\widetilde{X}$ killed at time $\widetilde{T}_0$ is equivalent to (X, T_0) and 0 is a recurrent regular state for $\widetilde{X}$, i.e.

$$\widetilde{\mathbb{P}}_x(T_0 < \infty) = 1, \quad \forall x > 0, \quad \widetilde{\mathbb{P}}_0(T_0 = 0) = 1,$$

with $\widetilde{\mathbb{P}}$ the law on $\mathbb{D}^+$ of $\widetilde{X}$. Furthermore, we know from [7] that there exists a unique excursion measure say $\mathbf{n}$, on $(\mathbb{D}^+, \mathcal{G}_\infty)$ compatible with the semigroup P_t such that its associated entrance law is $(\mathbf{n}_s, s > 0)$; the property $\lim_{s\to 0} \mathbf{n}_s\, B^c = 0$, for any B neighborhood of 0, is equivalent to $\mathbf{n}(X_{0+} > 0) = 0$, that is the process leaves 0 continuously under $\mathbf{n}$. Then the excursion measure $\mathbf{n}$ is the unique excursion measure having the properties $\mathbf{n}(X_{0+} > 0) = 0$ and $\mathbf{n}(1 - e^{-T_0}) = 1$. See subsection 2.1 for the definitions.

The first aim of this paper is to provide a more explicit description of the excursion measure $\mathbf{n}$ and its associated entrance law $(\mathbf{n}_s, s > 0)$. To this end, we shall mimic a well known construction of the Brownian excursion measure via the Bessel(3) process that we next sketch for ease of reference. Let P (respectively R) be a probability measure on $(\mathbb{D}^+, \mathcal{G}_\infty)$ under which the coordinate process is a Brownian motion killed at 0 (respectively a Bessel(3) process). The probability measure R appears as the law of the Brownian motion conditioned to never hit 0. More precisely, for $u > 0, x > 0$

$$\lim_{t\to\infty} P_x(A \mid T_0 > t) = R_x(A),$$

for any $A \in \mathcal{G}_u$, see e.g. McKean [23]. Moreover, the function $h(x) = x^{-1}, x > 0$ is excessive for the semigroup of the Bessel(3) process and its h–transform is the semigroup of the Brownian motion killed at 0. Let n be the h–transform of R_0 via the function $h(x) = x^{-1}$, i.e. n is the unique measure in $(\mathbb{D}^+, \mathcal{G}_\infty)$ with support on $\{0 < T_0 < \infty\}$ such that under n the coordinate process is Markovian with semigroup that of Brownian motion killed at 0, and for every $\mathcal{G}_t$–stopping time T and any $\mathcal{G}_T$–measurable variable F_T,

$$\mathrm{n}(F_T, T < T_0) = R_0(F_T \frac{1}{X_T}).$$

Then the measure n is a multiple of the Itô's excursion measure for Brownian motion, see e.g. Imhof [20] § 4.

In order to carry out this program, through this chapter, unless otherwise stated, we will assume that ξ is a Lévy process with infinite lifetime that satisfies the following hypotheses

(H2-a) ξ is not arithmetic, i.e. the state space is not a subgroup of $k\mathbb{Z}$ for any real number k;

(H2-b) There exists $\theta > 0$ such that $\mathbf{E}(e^{\theta\xi_1}) = 1$;

(H2-c) $\mathbf{E}(\xi_1^+ e^{\theta\xi_1}) < \infty$, with $a^+ = a \vee 0$.

The condition (H2-c) can be stated in terms of the Lévy measure Π of ξ as

(H2-c') $\int_{\{x>1\}} xe^{\theta x}\Pi(dx) < \infty$;

cf. Sato [32] Theorem 25.3. Such hypotheses are satisfied by a wide class of Lévy processes, in particular by those associated with self–similar diffusions and stable processes with no negative jumps. In the sequel we will refer to these hypotheses as (H2) hypotheses.

The condition (H2-b) is called *Cramér's condition* for the Lévy process ξ and force ξ to drift to $-\infty$ or equivalently $\mathbf{E}(\xi_1) < 0$. Cramér's condition enables us to construct a law $\mathbf{P}^\natural$ on $\mathbb{D}$, such that under $\mathbf{P}^\natural$ the coordinate process $\xi^\natural$ is a Lévy process that drifts to ∞ and $\mathbf{P}^\natural|_{\mathcal{D}_t} = e^{\theta\xi_t}\,\mathbf{P}|_{\mathcal{D}_t}$. Then, we will show that the self–similar Markov process $X^\natural$ associated to the Lévy process $\xi^\natural$ plays the rôle of a Bessel(3) process in our construction of the excursion measure $\mathbf{n}$.

The rest of this paper is organized as follows. In Subsection 2.1 we recall the Itô's program as established by Blumenthal [7]. The excursion measure $\mathbf{n}$ that interests us is the unique (up to a multiplicative constant) excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. Nevertheless, this is not the only excursion measure compatible with the semigroup of the minimal process, which is why in Subsection 2.2 we review some properties that should be satisfied by any excursion measure corresponding to a self–similar extension of the minimal process. There we also obtain necessary and sufficient conditions for the existence of an excursion measure n^j such that $n^j(X_{0+} = 0) = 0$, which are valid for any self–similar Markov process having 0 as a trap, regardeless if it hits 0 continuously or by a jump. In Subsection 2.3 we construct a self–similar Markov process $X^\natural$ which is related to (X, T_0) in an analogous way to that in which the Bessel(3) process is related to Brownian motion killed at 0. We also prove that the conditions (H1) are satisfied under the hypothesis (H2), give a more explicit expression for the limit in equation (3) and show that the hypotheses (H1) imply the conditions (H2-b,c). Next, in Section 3 we give our main description of the excursion measure $\mathbf{n}$ and give an answer to the question raised by Lamperti that can be sketched as follows: given a Lévy process ξ satisfying the hypotheses (H2), then an α–self–similar Markov process X associated to ξ admits a recurrent extension that leaves 0 continuously a.s. if and only if $0 < \alpha\theta < 1$. The purpose of Section 4 is to give an alternative description of the measure $\mathbf{n}$ by determining the law of the excursion process conditioned by its length (for Brownian motion this corresponds to the description of the Itô excursion measure via the law of a Bessel(3) bridge). In Section 5 we study some duality relations for the minimal process and in particular we determine the image under time reversal of $\mathbf{n}$. Finally, in

Appendix A we establish that the extensions of any two minimal processes which are in weak duality are still in weak duality as might be expected.

Last, the development of this work uses the theory of h–transforms of Doob, we refer to Sharpe [33] or Walsh [35] for background.

2 Preliminaries and first results

This section contains several parts. In the first one, we recall the Itô's program and the results in Blumenthal [7]. The purpose of Subsection 2.2 is study the excursion measures compatible with the semigroup of the minimal process (X, T_0). Finally, in Subsection 2.3 we establish the existence of a self–similar Markov process $X^{\natural}$ which bears the same relation to the minimal process (X, T_0) as the Bessel(3) process does to Brownian motion killed at 0. The results in Subsections 2.1 and 2.2 do not require hypotheses (H2).

2.1 Some general facts on recurrent extensions of Markov processes

A measure n on $(\mathbb{D}^+, \mathcal{G}_\infty)$ having infinite mass is called a *pseudo excursion measure* compatible with the semigroup P_t if the following are satisfied:

(i) n is carried by
$$\{\omega \in \mathbb{D}^+ \mid 0 < T_0(\omega) < \infty \text{ and } X_t(\omega) = 0, \forall t \geq T_0\};$$

(ii) for every bounded $\mathcal{G}_\infty$–measurable H and each $t > 0$ and $\Lambda \in \mathcal{G}_t$
$$n(H \circ \theta_t, \Lambda \cap \{t < T_0\}) = n(\mathbb{E}_{X_t}(H), \Lambda \cap \{t < T_0\}),$$
where θ_t denotes the shift operator.

If moreover

(iii) $n(1 - e^{-T_0}) < \infty$,

we will say that n is an *excursion measure*. A normalized excursion measure n is an excursion measure n such that $n(1 - e^{-T_0}) = 1$. The rôle played by condition (iii) will be explained below.

The entrance law associated to a pseudo excursion measure n is defined by
$$n_s(dy) := n(X_s \in dy, s < T_0), \quad s > 0.$$

A partial converse holds: given an entrance law $(n_s, s > 0)$ such that
$$\int_0^\infty (1 - e^{-s}) dn_s 1 < \infty,$$
there exists a unique excursion measure n such that its associated entrance law is $(n_s, s > 0)$, see e.g. [7].

It is well known in the theory of Markov processes that one way to construct recurrent extensions of a Markov process is the Itô's program or pathwise approach that can be described as follows. Assume that there exists an excursion measure n compatible with the semigroup of the minimal process P_t. Realize a Poisson point process $\Delta = (\Delta_s, s > 0)$ on $\mathbb{D}^+$ with characteristic measure n. Thus each atom Δ_s is a path and $T_0(\Delta_s)$ denotes its lifetime, i.e.

$$T_0(\Delta_s) = \inf\{t > 0 : \Delta_s(t) = 0\}.$$

Set

$$\sigma_t = \sum_{s \leq t} T_0(\Delta_s), \qquad t \geq 0.$$

Since $n(1 - e^{-T_0}) < \infty$, $\sigma_t < \infty$ a.s. for every $t > 0$. It follows that the process $\sigma = (\sigma_t, t \geq 0)$ is an increasing càdlàg process with stationary and independent increments, i.e. a subordinator. Its law is characterized by its Laplace exponent ϕ, defined by

$$\mathbf{E}(e^{-\lambda \sigma_1}) = e^{-\phi(\lambda)}, \qquad \lambda > 0,$$

and $\phi(\lambda)$ can be expressed thanks to the Lévy–Khintchine formula as

$$\phi(\lambda) = \int_{]0,\infty[} (1 - e^{-\lambda s}) \nu(ds),$$

with ν a measure such that $\int s \wedge 1 \ \nu(ds) < \infty$, called the Lévy measure of σ; see e.g. Bertoin [1] § 3 for background. An application of the exponential formula for Poisson point processes gives

$$\mathbf{E}(e^{-\lambda \sigma_1}) = e^{-n(1 - e^{-\lambda T_0})}, \qquad \lambda > 0,$$

i.e. $\phi(\lambda) = n(1 - e^{-\lambda T_0})$ and the tail of the Lévy measure is given by

$$\nu[s, \infty[= n(s < T_0) = n_s 1, \qquad s > 0.$$

Observe that if we assume $\phi(1) = n(1 - e^{-T_0}) = 1$ then ϕ is uniquely determined. Since n has infinite mass, σ_t is strictly increasing in t. Let L_t be the local time at 0, i.e. the continuous inverse of σ

$$L_t = \inf\{r > 0 : \sigma_r > t\} = \inf\{r > 0 : \sigma_r \geq t\}.$$

Define a process $(\widetilde{X}_t, t \geq 0)$ as follows. For $t \geq 0$, let $L_t = s$, then $\sigma_{s-} \leq t \leq \sigma_s$, set

$$\widetilde{X}_t = \begin{cases} \Delta_s(t - \sigma_{s-}) & \text{if } \ \sigma_{s-} < \sigma_s \\ 0 & \text{if } \ \sigma_{s-} = \sigma_s \text{ or } s = 0. \end{cases} \tag{4}$$

That the process so constructed is a Markov process has been established in full generality by Salisbury [30, 31] and under some regularity hypotheses on the semigroup of the minimal process by Blumenthal [7]. See also Rogers [29] for its analytical counterpart. In our setting, the hypotheses in [7] are satisfied as is shown by the following lemma.

Lemma 1. *Let $C_0]0, \infty[$, be the space of continuous functions on $]0, \infty[$ vanishing at 0 and ∞.*

(i) if $f \in C_0]0, \infty[$, then $P_t f \in C_0]0, \infty[$ and $P_t f \to f$ uniformly as $t \to 0$.

(ii) $\mathbb{E}_x(e^{-qT_0})$ is continuous in x for each $q > 0$ and

$$\lim_{x \to 0} \mathbb{E}_x(e^{-T_0}) = 1 \quad \textit{and} \quad \lim_{x \to \infty} \mathbb{E}_x(e^{-T_0}) = 0.$$

This Lemma is an easy consequence of Lamperti's transformation. Alternatively a proof can be found in [34] pp. 549–550. Therefore we have from [7] that $\widetilde{X}$ is a Markov process with Feller semigroup and its resolvent $\{U_q, q > 0\}$ satisfies

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0}) U_q f(0), \quad x > 0,$$

for $f \in C_b(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R},$ continuous and bounded$\}$. That is $\widetilde{X}$ is an extension of the minimal process. Furthermore, if $\{X'_t, t \geq 0\}$ is a Markov process extending the minimal one with Itô excursion measure n and local time at 0, say $\{L'_t, t \geq 0\}$, such that

$$\mathbb{E}'_0(\int_0^\infty e^{-s} dL'_s) = 1,$$

where $\mathbb{E}'$ is the law for X'. Then the process $\widetilde{X}$ and X' are equivalent and the Itô's excursion measure for $\widetilde{X}$ is n.

Thus, the results in [7] establish a one to one correspondence between excursion measures and recurrent extensions of Markov processes. Given an excursion measure n we will say that the associated extension of the minimal process leaves 0 continuously a.s. if $n(X_{0+} > 0) = 0$ or, equivalently, in terms of its entrance law, $\lim_{s \to 0} n_s(B^c) = 0$ for every neighborhood B of 0, see e.g. [7]; if n is such that $n(X_{0+} = 0) = 0$, we will say that the extension leaves 0 by jumps a.s. The latter condition on n is equivalent to the existence of a jumping–in measure η, that is η is a σ–finite measure on $]0, \infty[$ such that the entrance law associated to n can be expressed as

$$n_s f = n(f(X_s), s < T_0) = \int_{]0,\infty[} \eta(dx) P_s f(x), s > 0,$$

for every $f \in C_b(\mathbb{R}^+)$, cf. Meyer [25].

Finally, observe that if n is a pseudo excursion measure that does not satisfy the condition (iii), we can still realize a Poisson point process of excursions on $(\mathbb{D}^+, \mathcal{G}_\infty)$ with characteristic measure n but we cannot form a process extending the minimal one by sticking together the excursions because the sum of lengths $\sum_{s \leq t} T_0(\Delta_s)$, is infinite $\mathbb{P}$-a.s. for every $t > 0$.

2.2 Some properties of excursion measures for self–similar Markov processes

Next, we deduce necessary and sufficient conditions that must be satisfied by an excursion measure in order that the associated recurrent extension of the minimal process is self–similar. For $c \in \mathbb{R}$, let H_c be the dilatation $H_c f(x) = f(cx)$.

Lemma 2. *Let n be an excursion measure and $\widetilde{X}$ the associated recurrent extension of the minimal process. The following are equivalent*

(i) The process $\widetilde{X}$ has the scaling property

(ii) There exists $\gamma \in]0,1[$ such that for any $c > 0$,

$$n(\int_0^{T_0} e^{-qs} f(X_s) ds) = c^{(1-\gamma)/\alpha} n(\int_0^{T_0} e^{-(qc^{1/\alpha} s)} H_c f(X_s) ds),$$

for $f \in C_b(\mathbb{R}^+)$.

(iii) There exists $\gamma \in]0,1[$ *such that for any* $c>0$,

$$n_s f = c^{-\gamma/\alpha} n_{s/c^{1/\alpha}} H_c f \quad \textit{for all} \quad s>0,$$

for $f \in C_b(\mathbb{R}^+)$.

Remark If one of the conditions (i–iii) in the preceding Lemma holds, then the subordinator σ which is the inverse local time of $\widetilde{X}$ is a stable subordinator of parameter γ, where γ is determined in the condition (ii) or (iii).

Proof. (ii) $\iff$ (iii) is straightforward.

(i) $\Rightarrow$ (ii). Suppose that there exists an excursion measure n such that the associated recurrent extension $\widetilde{X}$ has the scaling property (1). Let $\mathcal{M}$ be the random set of zeros of the process $\widetilde{X}$, i.e. $\mathcal{M}=\{t\geq 0 | \widetilde{X}(t)=0\}$. By construction $\mathcal{M}$ is the closed range of the subordinator $\sigma=(\sigma_t, t\geq 0)$, that is $\mathcal{M}$ is a regenerative set. The recurrence of $\widetilde{X}$ implies that $\mathcal{M}$ is unbounded a.s. By the scaling property for $\widetilde{X}$ we have that

$$\mathcal{M} \stackrel{d}{=} c\mathcal{M}, \qquad \text{for each} \qquad c>0,$$

that is $\mathcal{M}$ is self–similar. Thus the subordinator should have the scaling property and since the only Lévy processes that have the scaling property are the stable processes it follows that σ is a stable subordinator of parameter γ for some $\gamma \in]0,1[$ or, in terms of its Laplace exponent $\phi(\lambda) = n(1-e^{-\lambda T_0}) = \lambda^\gamma, \lambda>0$. Recall that the scaling property for the extension can be stated in terms of its resolvent by saying that for any $c>0$,

$$U_q f(x) = c^{1/\alpha} U_{qc^{1/\alpha}} H_c f(x/c), \quad \text{for all} \quad x\geq 0, \tag{5}$$

for $f\in C_b(\mathbb{R}^+)$. Using the compensation formula for Poisson point processes we get that

$$U_q f(0) = \frac{n(\int_0^{T_0} e^{-qs} f(X_s)ds)}{n(1-e^{-qT_0})}, \tag{6}$$

From equation (5) we have that the measure n should be such that

$$\frac{n(\int_0^{T_0} e^{-qs} f(X_s)ds)}{n(1-e^{-qT_0})} = c^{1/\alpha} \frac{n(\int_0^{T_0} e^{-qc^{1/\alpha}s} H_c f(X_s)ds)}{n(1-e^{-qc^{1/\alpha}T_0})},$$

and therefore we conclude that

$$n(\int_0^{T_0} e^{-qs} f(X_s)ds) = c^{(1-\gamma)/\alpha} n(\int_0^{T_0} e^{-(qc^{1/\alpha}s)} H_c f(X_s)ds).$$

(ii) $\Rightarrow$ (i). The scaling property of $\widetilde{X}$ is obtained by means of (5). In fact, the only thing that should be verified is that equation (5) holds for $x=0$, since we have the identity

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0}) U_q f(0), \quad x>0,$$

and the scaling property of the minimal process stated in terms of its resolvent V_q, i.e.

$$V_q f(x) = c^{1/\alpha} V_{qc^{1/\alpha}} H_c f(x/c), \quad x>0, c>0, q>0.$$

Indeed, by construction it follows that the formula (6) holds and the hypothesis (ii) implies that $n(1-e^{-qT_0}) = q^\gamma,\ q>0$; the conclusion is immediate. $\square$

In the following proposition we give a description of the sojourn measure of $\widetilde{X}$ and a necessary condition for the existence of a excursion measure n such that one of the conditions in Lemma 2 holds.

Lemma 3. *Let n be a normalized excursion measure and $\widetilde{X}$ the associated extension of the minimal process (X, T_0). Assume that one of the conditions (i–iii) in Lemma 2 holds. Then*

$$n\Big(\int_0^{T_0} 1_{\{X_s \in dy\}} ds\Big) = C_{\alpha,\gamma} y^{(1-\alpha-\gamma)/\alpha} dy, \quad y > 0,$$

with γ determined in (ii) of Lemma 2 and $C_{\alpha,\gamma} \in]0, \infty[$ a constant. As a consequence, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha\, \mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$, where I denotes the exponential functional (2).

Proof. Recall that the sojourn measure

$$n\Big(\int_0^{T_0} 1_{\{X_s \in dy\}} ds\Big) = \int_0^{\infty} n_s(dy) ds,$$

is a σ–finite measure on $]0, \infty[$ and is the unique excessive measure for the semigroup of the process $\widetilde{X}$, see e.g. Dellacherie et al. [12] XIX.46. Next, using the result (iii) in Lemma 2 and the Fubini's Theorem we obtain the following representation of the sojourn measure, for $f \geq 0$ measurable

$$\begin{aligned}
\int_0^{\infty} n_s f ds &= \int_0^{\infty} s^{-\gamma} n_1(H_{s^\alpha} f) ds \\
&= \int n_1(dz) \int_0^{\infty} s^{-\gamma} f(s^\alpha z) ds \\
&= C_{\alpha,\gamma} \int_0^{\infty} u^{(1-\alpha-\gamma)/\alpha} f(u) du,
\end{aligned}$$

with $0 < C_{\alpha,\gamma} = \alpha^{-1} \int n_1(dz) z^{-(1-\gamma)/\alpha} < \infty$. This proves the first part of the claimed result. We now prove that $\mathbf{E}(I^{-(1-\gamma)}) < \infty$. On the one hand, the function $\varphi(x) = \mathbb{E}_x(e^{-T_0})$ is integrable with respect to the sojourn measure. To see this, use the Markov property under n, to obtain

$$\begin{aligned}
n\Big(\int_0^{T_0} \varphi(X_s) ds\Big) &= \int_0^{\infty} n(\varphi(X_s), s < T_0) ds \\
&= \int_0^{\infty} n(e^{-T_0} \circ \theta_s, s < T_0) ds \\
&= \int_0^{\infty} n(e^{-(T_0 - s)}, s < T_0) ds \\
&= n(1 - e^{-T_0}) = 1.
\end{aligned}$$

On the other hand, using the representation of the sojourn measure, Fubini's Theorem and the scaling property we have that

$$\begin{aligned}
C_{\alpha,\gamma} \int_0^{\infty} \mathbb{E}_y(e^{-T_0}) y^{(1-\alpha-\gamma)/\alpha} dy &= C_{\alpha,\gamma} \int_0^{\infty} \mathbf{E}(e^{-y^{1/\alpha} I}) y^{(1-\alpha-\gamma)/\alpha} dy \\
&= C_{\alpha,\gamma} \alpha\, \mathbf{E}(I^{-(1-\gamma)}) \Gamma(1-\gamma).
\end{aligned}$$

Therefore, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha\, \mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$. □

We next study the extensions $\widetilde{X}$ that leave 0 a.s. by jumps. Using only the scaling property (1) it can be verified that the only possible jumping–in measures such that the associated excursion measure satisfies (ii) in Lemma 2 should be of the type

$$\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx, \quad x > 0,\ 0 < \alpha\beta < 1,$$

with a constant $b_{\alpha,\beta} > 0$, depending on α and β, cf. [34]. This being said we can state an elementary but satisfactory result on the existence of extensions of the minimal process that leaves 0 by jumps a.s.

Proposition 1. *Let $\beta \in]0, 1/\alpha[$. The following are equivalent*

(i) $\mathbf{E}(I^{\alpha\beta}) < \infty$;

(ii) The pseudo excursion measure $n^j = \mathbb{P}^\eta$, based on the jumping–in measure

$$\eta(dx) = x^{-(1+\beta)} dx, \qquad x > 0,$$

is an excursion measure;

(iii) The minimal process (X, T_0) admits an extension $\widetilde{X}$, that is a self–similar recurrent Markov process and leaves 0 by jumps a.s. according to the jumping–in measure $\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx$, with $b_{\alpha,\beta} = \beta / \mathbf{E}(I^{\alpha\beta})\Gamma(1 - \alpha\beta)$.

If one of these conditions holds then γ in (ii) in Lemma 2 is equal to $\alpha\beta$.

The condition (i) in Proposition 1 is easily verified under weak technical assumptions. Namely, if we assume the hypothesis (H2) the aforementioned condition is verified for every $\beta \in]0, (1/\alpha) \wedge \theta[$; this will be deduced from Lemma 4 below. On the other hand, the condition is verified in other settings, as can be seen in the following example.

Example 1 (Generalized self–similar saw tooth processes). Let $\alpha > 0$, ζ a subordinator such that $\mathbf{E}(\zeta_1) < \infty$, and X the α–self–similar process associated to the Lévy process $\xi = -\zeta$. Then ξ drifts to $-\infty$, X has a finite lifetime T_0 and X decreases from its starting point until the time T_0, when it is absorbed at 0. Furthermore, it was proved by Carmona et al. [10] that the Lévy exponential functional $I = \int_0^\infty \exp\{-\zeta_s/\alpha\} ds$, has finite integral moments of all orders. It follows that the condition (i) in Proposition 1 is satisfied by every $\beta \in]0, 1/\alpha[$. Thus for each $\beta \in]0, 1/\alpha[$ the α–self–similar extension $\widetilde{X}$ that leaves 0 by jumps according to the jumping–in measure in (iii) of Proposition 1, is a process having sample paths that looks like a saw with "rough" teeth. These are all the possible extensions of X, that is, it is impossible to construct an excursion measure such that its associated extension of (X, T_0) leaves 0 continuously a.s. since we know that the process X decreases to 0.

Proof of Proposition 1. Let $\eta(dx) = x^{-(1+\beta)} dx,\ x > 0$ and n^j be the pseudo excursion measure $n^j = \mathbb{P}^\eta$. By definition the entrance law associated to n^j is

$$n^j_s f = \int_0^\infty dx\, x^{-(1+\beta)} P_s f(x),\ s > 0.$$

Thus, for n^j to be an excursion measure, the only condition it needs to satisfy is $n^j(1-e^{-T_0})<\infty$. This follows from the elementary calculation

$$\begin{aligned}\int_0^\infty dx\, x^{-(1+\beta)}\,\mathbb{E}_x(1-e^{-T_0}) &= \int_0^\infty dx\, x^{-(1+\beta)}\,\mathbf{E}(1-e^{-x^{1/\alpha}I})\\ &= \alpha\,\mathbf{E}\Big(\int dy\; y^{-\alpha\beta-1}(1-e^{-yI})\Big)\\ &= \mathbf{E}(I^{\alpha\beta})\frac{\Gamma(1-\alpha\beta)}{\beta}.\end{aligned}$$

That is, $n^j(1-e^{-T_0})<\infty$ if and only if $\mathbf{E}(I^{\alpha\beta})<\infty$, which proves the equivalence between the assertions in (i) and (ii). If (ii) holds it follows from the results in [7] and Lemma 2 that associated to the normalized excursion measure $n^{j'}=b_{\alpha,\beta}\mathbb{P}^\eta$ there exists a unique extension of the minimal process (X,T_0) which is a self–similar Markov process and which leaves 0 by jumps according to the jumping–in measure $b_{\alpha,\beta}x^{-(1+\beta)}dx, x>0$, which establishes (iii). Conversely, if (iii) holds the Itô's excursion measure of $\widetilde{X}$ is $n^{j'}=b_{\alpha,\beta}\mathbb{P}^\eta$ and the statement in (ii) follows. □

2.3 The process $X^\natural$ analogous to the Bessel(3) process

Here we shall establish the existence of a self–similar Markov process $X^\natural$ that can be viewed as the self–similar Markov process (X,T_0) conditioned to never hit 0. In the case where (X,T_0) is a Brownian motion killed at 0, $X^\natural$ corresponds to the Bessel(3) process. To this end, we next recall some facts on Lévy processes and density transformations and deduce some consequences for self–similar Markov processes. We assume henceforth (H2).

The law of a Lévy process ξ, is characterized by a function $\Psi:\mathbb{R}\to\mathbb{C}$, defined by the relation

$$\mathbf{E}(e^{iu\xi_1})=\exp\{-\Psi(u)\},\qquad u\in\mathbb{R}\,.$$

The function Ψ is called the characteristic exponent of the Lévy process ξ and can be expressed thank to the Lévy–Khintchine formula as

$$\Psi(u)=iau+\frac{\sigma^2u^2}{2}+\int_{\mathbb{R}}(1-e^{iux}+iux1_{\{|x|<1\}})\Pi(dx),$$

where Π is a measure on $\mathbb{R}\setminus\{0\}$ such that $\int(|x|^2\wedge 1)\Pi(dx)<\infty$. The measure Π is called the Lévy measure, a the drift and σ^2 the Gaussian coefficient of ξ. Conditions (H2-b,c) imply that the Lévy exponent of ξ admits an analytic extension to the complex strip $\Im(z)\in[-\theta,0]$. Thus we can define a function $\psi:[0,\theta]\to\mathbb{R}$ by

$$\mathbf{E}(e^{\lambda\xi_1})=e^{\psi(\lambda)}\quad\text{and}\quad\psi(\lambda)=-\Psi(-i\lambda),\quad 0\le\lambda\le\theta.$$

Hölder's inequality implies that ψ is a convex function and that θ is the unique solution to the equation $\psi(\lambda)=0$ for $\lambda>0$. Furthermore, the function $h(x)=e^{\theta x}$ is invariant for the semigroup of ξ. Let $\mathbf{P}^\natural$ be the h–transform of $\mathbf{P}$ via the invariant function $h(x)=e^{\theta x}$. That is, the measure $\mathbf{P}^\natural$ is the unique measure on $(\mathbb{D},\mathcal{D})$ such that for every finite $\mathcal{D}_t$-stopping time T and each $A\in\mathcal{D}_T$

$$\mathbf{P}^\natural(A)=\mathbf{P}(e^{\theta\xi_T}A).$$

Under $\mathbf{P}^\natural$ the process $(\xi_t,t\ge 0)$ still is a Lévy process, with characteristic exponent

$$\Psi^\natural(u)=\Psi(u-i\theta),\qquad u\in\mathbb{R},$$

and drifts to ∞, more precisely,

$$0 < m^{\natural} := \mathbf{E}^{\natural}(\xi_1) = \psi'(\theta-) < \infty.$$

See e.g. Sato [32] § 33, for a proof of these facts and more about this change of measure.

Let $\mathbb{P}^{\natural}{}_x$ denote the law on $\mathbb{D}^+$ of the self–similar Markov process started at $x > 0$ associated to the Lévy process $\xi^{\natural}$ via Lamperti's transformation. In the sequel it will be implicit that the superscript $\natural$ refers to the measure $\mathbb{P}^{\natural}$ or $\mathbf{P}^{\natural}$. We now establish a relation between the probability measures $\mathbb{P}$ and $\mathbb{P}^{\natural}$ analogous to that between the law of a Brownian motion killed at 0 and the law of a Bessel(3) process, see e.g. McKean [23]. Informally, the law $\mathbb{P}^{\natural}{}_x$ can be interpreted as the law under $\mathbb{P}_x$ of X conditioned to never hit 0.

Proposition 2. *(i) Let $x > 0$ be arbitrary. Then we have that $\mathbb{P}^{\natural}{}_x$ is the unique measure such that for every $\mathcal{G}_t$–stopping time T we have*

$$\mathbb{P}^{\natural}{}_x(A) = x^{-\theta}\mathbb{P}_x(A\ X^{\theta}_T, T < T_0),$$

for any $A \in \mathcal{G}_T$. In particular, the function $h^* : [0,\infty[\to [0,\infty[$ defined by $h^*(x) = x^{\theta}$ is invariant for the semigroup P_t.

(ii) For every $x > 0$ and $t > 0$ we have

$$\mathbb{P}^{\natural}{}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A \mid T_0 > s),$$

for any $A \in \mathcal{G}_t$.

The proof of (i) in Proposition 2 is a straightforward consequence of the fact that $\mathbf{P}^{\natural}$ is the h–transform of $\mathbf{P}$ and that for every $\mathcal{G}_t$–stopping time T we have that $\tau(T)$ is a $\mathcal{F}_t$–stopping time. To prove (ii) in Proposition 2 we need the following lemma that provides us with a tail estimate for the law of the Lévy exponential functional I associated to ξ as defined in (2).

Lemma 4. *Under the conditions (H2) we have that*

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t) = C,$$

where

$$0 < C = \frac{\alpha}{m^{\natural}} \int t^{\alpha\theta-1}(\mathbf{P}(I > t) - \mathbf{P}(e^{\xi'_1} I > t))dt < \infty,$$

with $\xi'_1 =^d \xi_1$ and independent of I. If $0 < \alpha\theta < 1$, then

$$C = \frac{\alpha}{m^{\natural}}\,\mathbf{E}(I^{-(1-\alpha\theta)}).$$

Two proofs of this result have been given in a slightly restricted setting by Mejane [24]. However, one of these proofs can be extended to our case and in fact it is an easy consequence of a result on random equations originally due to Kesten [21], who in turn uses a difficult result on random matrices. A simpler proof of Kesten's result was given in Goldie [19].

Sketch of proof of Lemma 4. It is straightforward that the Lévy exponential functional I satisfies the equation in law

$$I =^d \int_0^1 e^{\xi_s/\alpha}ds + e^{\xi_1/\alpha} I' = Q + MI',$$

with I' the Lévy exponential functional associated to $\xi' = \{\xi'_t = \xi_{1+t} - \xi_1, t \geq 0\}$, a Lévy process independent of $\mathcal{F}_1$ and with the same distribution as ξ. Thus, according to [21] if the conditions (i–iv) below are satisfied then there exists a strictly positive constant C such that

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t) = C.$$

The hypotheses of Kesten's Theorem are

(i) M is not arithmetic

(ii) $\mathbf{E}(M^{\alpha\theta}) = 1$,

(iii) $\mathbf{E}(M^{\alpha\theta}\ln^+(M)) < \infty$,

(iv) $\mathbf{E}(Q^{\alpha\theta}) < \infty$.

Assuming the conditions (H2) the only thing that needs to be verified is that (iv) holds. Indeed,

$$\begin{aligned}\mathbf{E}(Q^{\alpha\theta}) &\leq \mathbf{E}\left(\sup\{e^{\theta\xi_s} : s \in [0,1]\}\right)\\ &\leq \frac{e}{e-1}\left(1 + \theta \sup\{\mathbf{E}(\xi^+_s e^{\theta\xi_s}) : s \in [0,1]\}\right) < \infty.\end{aligned}$$

The second inequality is obtained using the fact that $(e^{\theta\xi_t}, t \geq 0)$ is a positive martingale and a Doob's inequality. The first formula for the value of the limit, $C = \lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t)$ is a consequence of Lemma 2.2 and Theorem 4.1 in Goldie [19]. That the latter limit exists implies that $\mathbf{E}(I^a) < \infty$, for all $0 < a < \alpha\theta$. Now, to obtain the expression for C when $0 < \alpha\theta < 1$, we will use the following formula for the moments of I,

$$\mathbf{E}(I^a) = \frac{a}{-\psi(a/\alpha)}\,\mathbf{E}(I^{a-1}), \quad \text{for} \quad 0 < a < \alpha\theta, \tag{7}$$

which can be proved with arguments similar to that given by Bertoin and Yor [5] Proposition 2. We will also use the well known identity

$$\lambda^a = \frac{a}{\Gamma(1-a)}\int_0^\infty (1 - e^{-\lambda x})x^{-(1+a)}dx, \quad \lambda > 0, a \in]0,1[.$$

On the one hand, since $0 < \alpha\theta < 1$, Corollary 8.1.7 in Bingham et al. [6] implies

$$\lim_{s\to 0}\frac{\mathbf{E}(1 - e^{-sI})}{s^{\alpha\theta}} = C\Gamma(1-\alpha\theta).$$

On the other hand, by equation (7) we have

$$\begin{aligned}\mathbf{E}(I^{-(1-\alpha\theta)})\alpha\theta &= \lim_{a\uparrow\alpha\theta}\mathbf{E}(I^{a-1})a\\ &= \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\lim_{a\uparrow\alpha\theta}(-\psi(a/\alpha))\int_0^\infty s^{-(1+a)}\,\mathbf{E}\left(1 - e^{-sI}\right)ds\\ &= C\alpha\theta\lim_{a\uparrow\alpha\theta}\frac{-\psi(a/\alpha)}{\alpha\theta - a}\\ &= C\theta\psi'(\theta-).\end{aligned} \tag{8}$$

Indeed, write

$$\mathbf{E}(I^{a-1}) = \mathbf{E}(I^{a-1}1_{\{I\geq 1\}}) + \mathbf{E}(I^{a-1}1_{\{I<1\}}).$$

The first term tends to $\mathbf{E}(I^{\alpha\theta-1}1_{\{I\geq 1\}})$ as $a \uparrow \alpha\theta$, by dominated convergence. A consequence of equation (7) is that $\mathbf{E}(I^{a-1}) < \infty$ for every $0 < a < \alpha\theta$. Then by monotone convergence the second term tends to $\mathbf{E}(I^{\alpha\theta-1}1_{\{I<1\}})$. Then $\lim_{a\uparrow\alpha\theta} \mathbf{E}(I^{a-1}) = \mathbf{E}(I^{\alpha\theta-1})$. Next, using that the Stieltjes measure $q_{\alpha\theta-a}$ over $[0,\infty[$ defined by $q_{\alpha\theta-a}[0,s[= s^{\alpha\theta-a}$, $s > 0$ converges weakly to the Dirac mass at 0 as $a \uparrow \alpha\theta$ we obtain that

$$\lim_{a\uparrow\alpha\theta}(\alpha\theta - a)\int_0^\infty s^{-(1+a)}\,\mathbf{E}\left(1-e^{-sI}\right)ds = \lim_{a\uparrow\alpha\theta}\int_0^\infty \frac{\mathbf{E}\left(1-e^{-sI}\right)}{s^{\alpha\theta}}q_{\alpha\theta-a}(ds) = C\Gamma(1-\alpha\theta)$$

and the claim in equation (8) follows. □

The proof of Proposition 2 follows from standard arguments.

Proof of (ii) in Proposition 2. Recall that the law of T_0 under $\mathbb{P}_x$ is that of $x^{1/\alpha}I$ under $\mathbf{P}$. Thus we deduce from Lemma 4 that for every $x > 0$,

$$\lim_{s\to\infty} s^{\alpha\theta}\mathbb{P}_x(T_0 > s) = x^\theta C.$$

Using the Markov property and a dominated convergence argument, we obtain that

$$\begin{aligned}\mathbb{P}_x(A \mid T_0 > s) &= \mathbb{P}_x(A\ 1_{\{t<T_0\}}\mathbb{P}_{X_t}(T_0 > s-t)/\mathbb{P}_x(T_0 > s)) \\ &\xrightarrow[s\to\infty]{} x^{-\theta}\mathbb{P}_x(A\ X_t^\theta\ 1_{\{t<T_0\}}).\end{aligned}$$

□

By Proposition 2, the semigroup of X under ${\mathbb{P}^\natural}_x$ is given by

$$P_s^\natural f(x) := {\mathbb{E}^\natural}_x(f(X_s)) = x^{-\theta}\,\mathbb{E}_x(f(X_s)X_s^\theta 1_{\{s<T_0\}}), \qquad \text{for} \quad x > 0,$$

with f a positive or bounded measurable function. Let J be the Lévy exponential functional associated to the process $\xi^\natural$, i.e.

$$J = \int_0^\infty \exp\{-\xi_s^\natural/\alpha\}ds, \tag{9}$$

which is finite $\mathbf{P}^\natural$–a.s. since $\xi^\natural$ drifts to ∞. Now, since under $\mathbf{P}^\natural$ the process $(\xi_s^\natural, s \geq 0)$ is a non arithmetic Lévy process with $0 < m^\natural < \infty$, Theorem 1 in Bertoin and Yor [4] ensures that the measure ${\mathbb{P}^\natural}_x$ converges in the sense of finite dimensional distributions to a probability measure ${\mathbb{P}^\natural}_{0+}$ as $x \to 0$. Moreover, the law of X_s under ${\mathbb{P}^\natural}_{0+}$ is an entrance law for the semigroup $P_t^\natural$ and is related to the law of the Lévy exponential functional J under $\mathbf{P}^\natural$ by the formula

$${\mathbb{E}^\natural}_{0+}(f(X_s^{1/\alpha})) = \frac{\alpha}{m^\natural}\,\mathbf{E}^\natural(f(s/J)/J), \quad s > 0, \tag{10}$$

for f measurable and positive. Recall also that $m^\natural/\alpha = \mathbf{E}^\natural(1/J) < \infty$, cf. [4] for a proof of these facts.

The next result states that under the hypotheses (H2) the conditions (H1) hold and gives a first description of the entrance law $(\mathbf{n}_s, s > 0)$.

Proposition 3. *Assume the hypotheses (H2).*

(i) If $0 < \alpha\theta < 1$, then the hypotheses (H1) hold for $\kappa = \theta$. Furthermore, the q–potential of the entrance law $(\mathbf{n}_s, s > 0)$, admits the representation

$$\int_0^\infty dse^{-qs}\,\mathbf{n}_s\, f = \gamma_{\alpha,\theta}\int_0^\infty f(y)\,\mathbf{E}^\natural(\exp\{-qy^{1/\alpha}J\})y^{(1-\alpha-\alpha\theta)/\alpha}dy,$$

where

$$\gamma_{\alpha,\theta} = \Big(\alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)\Big)^{-1},$$

for every $f \in C_b(\mathbb{R}^+)$.

(ii) If $\alpha\theta \geq 1$, then either the hypothesis (H1-a) or (H1-b) fails to hold.

Proof. (i) That the hypothesis (H1-a) holds is easily proved. Indeed, since $0 < \alpha\theta < 1$ the Corollary 8.1.7. in Bingham et al. [6] implies that the result in Lemma 4 is equivalent to

$$\lim_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^\theta} = \lim_{x\to 0}\frac{\mathbf{E}(1-e^{-x^{1/\alpha}I})}{x^\theta} = \Gamma(1-\alpha\theta)\frac{\alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})}{m^\natural}. \tag{11}$$

To prove (H1-b) we recall the identity,

$$\frac{V_q f(x)}{x^\theta} = V_q^\natural(f/h^*)(x),$$

where $V_q^\natural$ is the resolvent of the semigroup $P_t^\natural$ and $h^*(x) = x^\theta, x > 0$. As was already pointed out, the results in [4] are applicable in our setting to the self–similar process $X^\natural$. In particular, formula (4) op. cit. states that

$$\lim_{x\to 0} V_q^\natural g(x) = \frac{\alpha}{m^\natural}\int_0^\infty g(y^\alpha)\,\mathbf{E}^\natural(e^{-qyJ})dy,$$

for every function $g \in C_b(\mathbb{R}^+)$. Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{V_q f(x)}{x^\theta} &= \lim_{x\to 0} V_q^\natural(f/h^*)(x)\\ &= \frac{\alpha}{m^\natural}\int_0^\infty f(y^\alpha)y^{-\alpha\theta}\,\mathbf{E}^\natural(e^{-qyJ})dy,\\ &= \frac{1}{m^\natural}\int_0^\infty f(y)\,\mathbf{E}^\natural(e^{-qy^{1/\alpha}J})y^{(1-\alpha-\alpha\theta)/\alpha}dy\end{aligned} \tag{12}$$

for every $f \in C_K]0,\infty[$. Thus we have verified the hypotheses (H1) and the expression of the q–resolvent of the entrance law $(\mathbf{n}_s, s > 0)$ follows from the identity (3) using the calculations in equation (11) and (12).

(ii) If $\alpha\theta \geq 1$, the Fatou's lemma and the scaling property imply

$$\liminf_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^\theta} \geq \int_0^\infty e^{-s}s^{-\alpha\theta}\left(\liminf_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t)\right)ds = \infty.$$

But from the proof of (i) we know that the limit

$$\lim_{x\to 0}\frac{V_q f(x)}{x^\theta}, \quad q > 0,$$

still exists and is not 0 for every non–negative function $f \in C_K]0,\infty[$ and, indeed, $f > 0$ in a set of positive Lebesgue measure. As a consequence, even if there exists $\kappa < \theta$, such that the limit $\lim_{x\to 0} x^{-\kappa}\,\mathbb{E}_x(1-e^{-T_0})$, exists and is positive, the limit $\lim_{x\to 0} x^{-\kappa}V_q f(x)$ is equal to zero for every function continuous f with bounded support on $]0,\infty[$. □

Proposition 3 proves that the hypotheses (H2) and $0<\alpha\theta<1$ imply the hypotheses (H1). In the next Proposition we establish a partial converse.

Proposition 4. *Assume that there exists a $\kappa>0$ such that the hypothesis (H1) hold. Then*

(i) $0<\alpha\kappa<1$,

(ii) the hypotheses (H2-b) and (H2-c) are satisfied with $\theta=\kappa$.

Proof. To prove (i) we recall that under the hypotheses (H1) Theorem 2.1 in [34] states that the q–resolvent of the entrance law $(\mathbf{n}_s, s>0)$ is characterized by the equation (3). Next, it is easily verified using the self–similarity of the minimal process (X, T_0), that for every $q>0$, $c>0$

$$\lim_{x\to 0}\frac{V_q f(x)}{\mathbb{E}_x(1-e^{-T_0})}=c^{(1-\alpha\kappa)/\alpha}\lim_{x\to 0}\frac{V_{qc^{1/\alpha}}H_c f(x)}{\mathbb{E}_x(1-e^{-T_0})}.$$

Then the excursion measure $\mathbf{n}$ is such that for every $c>0$

$$\mathbf{n}(\int_0^{T_0}e^{-qs}f(X_s)ds)=c^{(1-\alpha\kappa)/\alpha}\,\mathbf{n}(\int_0^{T_0}e^{-qc^{1/\alpha}s}H_c f(X_s)ds).$$

The latter fact implies that (ii) in Lemma 2 is satisfied with $\gamma=\alpha\kappa$ and $0<\alpha\kappa<1$. Next we prove (ii). We first prove that under the hypothesis (H1) the process $(X^{\kappa}_t, t>0)$ is a martingale for $\mathbb{P}_x$, which implies Cramér's condition (H2-b). Indeed, since the hypothesis (H1-a) holds we have that

$$\lim_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^{\kappa}}=B\in]0,\infty[,$$

and, given that $0<\alpha\kappa<1$, the existence of this limit is equivalent to the existence of the limit

$$\lim_{s\to\infty}s^{\alpha\kappa}\mathbb{P}_x(T_0>s)=x^{\kappa}B/\Gamma(1-\alpha\kappa).$$

This fact suffices to prove that for every $x>0$ and $t>0$

$$\lim_{s\to\infty}\mathbb{P}_x(A|T_0>s)=x^{-\kappa}\mathbb{P}_x(X^{\kappa}_t, A\cap\{t<T_0\}),$$

for any $A\in\mathcal{G}_t$. To see this just repeat the arguments in the proof of (ii) in Proposition 2. In particular, we have that for every $x>0$ and $t>0$, $x^{\kappa}=\mathbb{E}_x(X^{\kappa}_t, t<T_0)$. Using the Markov property we obtain that for every $x>0$, under $\mathbb{P}_x$ the process X^{κ} is a martingale and as a consequence Cramér's condition follows. Moreover, the Lévy process ξ associated to X via Lamperti's transformation has a characteristic exponent Ψ which admits an analytic extension to the complex strip $\Im(z)\in[-\kappa,0[$ defined by $\psi(z)=-\Psi(-iz)$, see the survey at the beginning of this subsection. Now to prove that the hypothesis (H2-c) is satisfied, we recall that under the hypotheses (H1) we have that

$$\lim_{s\to\infty}s^{\alpha\kappa}\,\mathbf{P}(I>s)=x^{-\kappa}\lim_{s\to\infty}s^{\alpha\kappa}\mathbb{P}_x(T_0>s)=B/\Gamma(1-\alpha\kappa),$$

and that $\mathbf{E}(I^{-(1-\alpha\kappa)})<\infty$, the latter being a consequence of Lemma 3. Repeating the arguments in the calculation of the constant in the proof of Lemma 4 we obtain that

$$\mathbf{E}(I^{-(1-\alpha\kappa)})=B\psi'(\theta-)/\Gamma(1-\alpha\kappa)<\infty,$$

that is the exponent ψ of ξ has a left derivative at κ which is equivalent to

$$\mathbf{E}(\xi_1 e^{\kappa\xi_1}) < \infty.$$

Using the elementary relation

$$0 \le (\xi_1 \exp\{\kappa\xi_1\})^- = \xi_1^- \exp\{\kappa\xi_1\} = \xi_1^- \exp\{-\kappa\xi_1^-\} \le \kappa^{-1}$$

with $a^- = (-a) \vee 0$, we obtain that $0 \le \mathbf{E}((\xi_1 e^{\kappa\xi_1})^-) < 1/\kappa$. Therefore, $\mathbf{E}(\xi_1 e^{\kappa\xi_1}) < \infty$ if and only if $\mathbf{E}(\xi_1^+ e^{\kappa\xi_1}) < \infty$, which ends the proof. □

Remark

1. If $0 < \alpha\theta < 1$ we have the following equality

$$\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)}).$$

This can be seen by making elementary calculations to obtain that

$$\int e^{-s}\, \mathbf{n}_s\, 1 ds = \gamma_{\alpha,\theta} \int_0^\infty \mathbf{E}^\natural(e^{-y^{1/\alpha}J}) y^{(1-\alpha-\alpha\theta)} dy = \frac{\mathbf{E}^\natural(J^{-(1-\alpha\theta)})}{\mathbf{E}(I^{-(1-\alpha\theta)})},$$

and comparing this with the fact that $\int e^{-s}\, \mathbf{n}_s\, 1 ds = 1$ gives the equality

2. A consequence of Lemma (4) is that

$$\mathbf{E}(I^{\beta\alpha}) < \infty \quad \text{for every} \quad 0 < \beta < \theta$$

and that $\mathbf{E}(I^{\alpha\theta}) = \infty$. Then under the hypotheses (H2) any extension which leaves 0 by jumps a.s. has a jumping–in measure $\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx$, $x > 0$, with $0 < \beta < \theta \wedge 1/\alpha$ and $b_{\alpha,\beta}$ as defined in Proposition 1.

3 Existence of recurrent extensions that leaves 0 continuously

We next study the excursion measure such that the related extension leaves 0 continuously. To this end, we suppose throughout the rest of this section that the hypotheses (H2) holds.

Theorem 1. *There exists a pseudo excursion measure* $\mathbf{n}'$ *such that* $\mathbf{n}'(X_{0+} > 0) = 0$. *Its associated entrance law* $(\mathbf{n}'_s, s > 0)$ *is given by*

$$\mathbf{n}'_s f = \mathbb{E}^\natural{}_{0+}(f(X_s) X_s^{-\theta}), \qquad s > 0.$$

We have that $\mathbf{n}'$ *is an excursion measure if and only if* $0 < \alpha\theta < 1$. *Assume that this condition holds and let*

$$a_{\alpha,\theta} = \alpha\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)}) \Gamma(1-\alpha\theta)/m^\natural.$$

Then the measure $(a_{\alpha,\theta})^{-1}\, \mathbf{n}'$, *is the normalized excursion measure* $\mathbf{n}$.

Proof. We know from Proposition 2 that the function $h(x) = x^{-\theta}$ is excessive for the semigroup $P_t^\natural$ and that the corresponding h–transform is P_t. Let $\mathbf{n}'$ be the h–transform of $\mathbb{E}^\natural{}_{0+}$ by means of $h(x) = x^{-\theta}$. That is, $\mathbf{n}'$ is the unique measure in $\mathbb{D}^+$ that is carried by $\{T_0 > 0\}$, such that under $\mathbf{n}'$ the coordinate process is Markovian with semigroup P_t and for every $\mathcal{G}_t$–stopping time T and any $A_T \in \mathcal{G}_T$

$$\mathbf{n}'(A_T, T < T_0) = \mathbb{E}^\natural{}_{0+}(A_T, X_T^{-\theta}).$$

Therefore, $\mathbf{n}'$ is a pseudo excursion measure such that $\mathbf{n}'(X_{0+} > 0) = 0$ and the entrance law associated to $\mathbf{n}'$ is defined by

$$\mathbf{n}'_s f := \mathbf{n}'(f(X_s), s < T_0) = \mathbb{E}^\natural{}_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0, \tag{13}$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable. This proves the existence of a pseudo excursion measure such that $\mathbf{n}'(X_{0+} > 0) = 0$. To determine when $\mathbf{n}'$ is in fact an excursion measure we have to specify when $\mathbf{n}'(1 - e^{-T_0})$ is finite. Using standard arguments we obtain that

$$\begin{aligned}
\mathbf{n}'(1 - e^{-T_0}) &= \int_0^\infty dse^{-s}\, \mathbf{n}'(T_0 > s) \\
&= \int_0^\infty dse^{-s}\, \mathbb{E}^\natural{}_{0+}(X_s^{-\theta}) \\
&= \begin{cases} \alpha\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^\natural & \text{if} \quad \alpha\theta < 1 \\ \infty & \text{if} \quad \alpha\theta \geq 1, \end{cases}
\end{aligned}$$

the third equality is obtained from (10). If $0 < \alpha\theta < 1$, then $\mathbf{E}^\natural(J^{-(1-\alpha\theta)}) < \infty$ since $\mathbf{E}^\natural(J^{-1}) < \infty$. As a consequence $\mathbf{n}'(1 - e^{-T_0}) < \infty$, if and only if $0 < \alpha\theta < 1$. If we assume that $0 < \alpha\theta < 1$, it follows that the measure $a^{-1}_{\alpha,\theta}\, \mathbf{n}'$ is a normalized excursion measure compatible with the semigroup P_t. Furthermore, it is straightforward to check that $a^{-1}_{\alpha,\theta}\, \mathbf{n}'$ satisfies the condition (ii) in Lemma 2 for $\gamma = \alpha\theta$. The normalized excursion measure $a^{-1}_{\alpha,\theta}\, \mathbf{n}'$ is equal to the measure $\mathbf{n}$ since this is the unique normalized excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. □

In the following theorem we give a simple criterion to determine, in terms of the Lévy process ξ, whether there exists a self–similar recurrent extension of (X, T_0) that leaves 0 continuously. Furthermore, with this result we give a complete solution to the problem posed by Lamperti since we have already established the existence of self–similar recurrent extensions of the minimal process that leave 0 by jumps.

Theorem 2. *(i) Assume* $0 < \alpha\theta < 1$. *The minimal process admits a unique self–similar recurrent extension* $\widetilde{X} = (\widetilde{X}_t, t \geq 0)$ *that leaves* 0 *continuously a.s. The resolvent of* $\widetilde{X}$ *is determined by*

$$U_q f(0) = \frac{\gamma_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}^\natural(e^{-qy^{1/\alpha}J}) y^{(1-\alpha-\alpha\theta)/\alpha} dy,$$

with $\gamma_{\alpha,\theta}$ *as defined in Proposition 3 and*

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0}) U_q f(0), \qquad x > 0,$$

for $f \in C_b(\mathbb{R}^+)$. *The resolvent* U_q *is Fellerian.*

(ii) If $\alpha\theta \geq 1$, *there does not exist a self-similar recurrent extension that leaves* 0 *continuously.*

Proof. To obtain (i) we use the Lemma 1. This enables us to apply the results in Blumenthal [7] to ensure that associated to the excursion measure $\mathbf{n}$ described in Theorem 1 there exists a Markov process $\widetilde{X}$ having a Feller resolvent that is an extension of the minimal process. The self–similarity of $\widetilde{X}$ follows from Lemma 2. The only thing that needs a justification is the expression for the q–resolvent of the extension. Using the compensation formula for Poisson point processes we obtain that

$$U_q f(0) = \mathbf{n}\left(\int_0^{T_0} e^{-qs} f(X_s) ds\right) / \, \mathbf{n}(1 - e^{-qT_0}),$$

for every $f \in C_b(\mathbb{R}^+)$. From Lemma 2 we deduce that $\mathbf{n}(1 - e^{-qT_0}) = q^{\alpha\theta}$. The expression of $U_q f(0)$ is then obtained from Proposition 3. The proof of (ii) is a straightforward consequence of Lemma 5 below. $\square$

The next lemma states that if $\alpha\theta \geq 1$, the only excursion measures compatible with (X, T_0) which satisfy (ii) in Lemma 2 are those associated to a jumping–in measure as in (ii) in Proposition 1.

Lemma 5. *Assume that $\alpha\theta \geq 1$. If there exists a normalized excursion measure $\mathbf{m}$ compatible with the minimal process such that conditions (ii) and (iii) in Lemma 2 are satisfied, then $\mathbf{m}(X_{0+} = 0) = 0$.*

Sketch of Proof. We recall from the proof of Proposition 3 that if $\alpha\theta \geq 1$ we have that

$$\liminf_{x \to 0} \frac{\mathbb{E}_x(1 - e^{-T_0})}{x^\theta} = \infty,$$

and that

$$\lim_{x \to 0} \frac{V_q f(x)}{x^\theta}, \quad q > 0,$$

exists in $\mathbb{R}$ for every function $f \in C_K]0, \infty[$. Therefore,

$$\lim_{x \to 0} \frac{V_q f(x)}{\mathbb{E}_x\left(1 - e^{-T_0}\right)} = 0,$$

for every function $f \in C_K]0, \infty[$. Now, we may simply repeat the arguments in the proof of Lemma 1.1 in [34] to prove that for $q > 0$

$$\mathbf{m}(\int_0^{T_0} e^{-qs} f(X_s) ds) = b \int_0^\infty V_q f(x) x^{-(1+\beta)} dx,$$

for some $\beta \in]0, 1/\alpha[$ and a constant $b \in]0, \infty[$. The result follows. $\square$

Corollary 1. *Assume $0 < \alpha\theta < 1$.*

(i) The law of T_0 under $\mathbf{n}$ is

$$\mathbf{n}(T_0 \in ds) = \frac{\alpha\theta}{\Gamma(1 - \alpha\theta)} \, s^{-(1+\alpha\theta)} ds.$$

(ii) Under $\mathbf{n}$ the law of the height of the excursion, say $H := \sup_{0 \leq t \leq T_0} X_s$, is given by

$$\mathbf{n}(H > z) = p_{\alpha,\theta} z^{-\theta}, \qquad z > 0.$$

with $p_{\alpha,\theta} = p\left(\alpha\theta \, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1 - \alpha\theta)\right)^{-1}$, and $p \in]0, 1]$ a constant that depends on the law of ξ.

Proof. The result in (i) follows from the fact that the subordinator σ which is the inverse local time of $\widetilde{X}$ is a stable subordinator of parameter $\alpha\theta$, cf. Lemma 2. The main ingredient in the proof of (ii) is that the tail distribution of the random variable $S_\infty = \sup_{r>0} \xi_r$ is such that

$$\lim_{s\to\infty} e^{\theta s}\,\mathbf{P}(S_\infty > s) = p/m^\natural\theta,$$

for a constant $p \in]0,1]$, cf. Bertoin and Doney [3] for a proof of this fact and an expression of the constant p. We deduce from this a tail estimate for the behavior of the supremum of the minimal process (X, T_0) as the initial point tends to 0. More precisely, defining $S^X_\infty := \sup_{0\le r\le T_0} X_r$, we have

$$\lim_{x\to 0} x^{-\theta}\mathbb{P}_x(S^X_\infty > z) = z^{-\theta}(p/m^\natural\theta), \quad z>0.$$

Let $H_t = \sup_{t\le s\le T_0} X_s,\ t>0$. Besides, we have that for any $z>0$

$$\lim_{t\to 0+} \mathbf{n}(H_t > z, t < T_0) = \mathbf{n}(H > z),$$

and that for any $\epsilon, \delta > 0$, there exists a $t_0 > 0$ such that

$$\mathbf{n}(X_t \in (\epsilon,\infty), t<T_0) \le \delta, \quad \forall t < t_0.$$

Therefore,

$$\mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t < T_0) \le \mathbf{n}(H_t > z, t<T_0) \le \delta + \mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t<T_0),$$

and by the Markov property under $\mathbf{n}$, we get that

$$\begin{aligned}\mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t<T_0) &= (a_{\alpha,\theta})^{-1}\,\mathbb{E}^\natural{}_{0+}(X_t \in]0,\epsilon[, X_t^{-\theta}\,\mathbb{E}_{X_t}(S^X_\infty > z)) \\ &\sim p_{\alpha,\theta} z^{-\theta}\,\mathbb{E}^\natural{}_{0+}(X_t\in]0,\epsilon[) \\ &\sim p_{\alpha,\theta} z^{-\theta},\end{aligned}$$

for t small enough. Thus,

$$p_{\alpha,\theta} z^{-\theta} \le \mathbf{n}(H>z) \le \delta + p_{\alpha,\theta} z^{-\theta},$$

and the result follows by letting $\delta \to 0$. □

If $0 < \alpha\theta < 1$, it was shown by Vuolle-Apiala that given an excursion measure, the extension $\widetilde{X}$ associated to this excursion measure either leaves 0 continuously or by jumps. This fact is natural when we observe that the excursions that leave 0 continuously have different duration that those leaving 0 by jumps. Indeed, the duration of the former has distribution

$$\mathbf{n}(T_0 > t) = t^{-\alpha\theta}(\Gamma(1-\alpha\theta))^{-1},$$

and for the latter

$$n^j(T_0 > t) = t^{-\alpha\beta}(\Gamma(1-\alpha\beta))^{-1}, \qquad 0<\beta<\theta.$$

In the case when the Lévy process ξ is a Brownian motion with a negative drift, the criterion in Theorem 2 coincides with the classification from Feller's diffusion theory for 0 to be a regular or an exit boundary point, as is explained in Example 2 below. By analogy, one can say that 0 is a regular boundary point for $\widetilde{X}$ if $0<\alpha\theta<1$ and an exit boundary point if $1\le\alpha\theta$. Even in the case $\alpha\theta<0$, which is not considered in this chapter, it is easy to see that if $\theta<0$ in Cramér's condition then the Lévy process ξ drifts to ∞. The only way to extend a self–similar Markov process X associated to a Lévy process that drifts to ∞ is by making 0 an entrance boundary point. This possibility is considered by Bertoin and Caballero [2], Bertoin and Yor [4, 5] and Caballero and Chaumont [9].

4 Excursions conditioned by their durations

It is well known that the excursion measure for the Brownian motion can be described using the law of the excursion process conditioned to return to 0 at time 1, i.e. the law of a Bessel(3) bridge of length 1, see e.g. McKean [23] or Revuz and Yor [27] §XII.4. In this section we follow this idea to describe the law under the excursion measure $\mathbf{n}$ defined in Theorem 1 of the excursion process conditioned to return to zero at a given time. We then give an alternative description of the excursion measure $\mathbf{n}$. To that end, we will make the additional hypotheses

(H2-d) $\mathbf{E}(\xi_1) > -\infty$ and the distribution of the Lévy exponential functional I has a continuous density on $[0,\infty[$, say ρ, with respect to Lebesgue measure.

The condition that the law of the exponential functional I has a continuous density is satisfied by a wide variety of Lévy processes, cf. Carmona et al. [10] Proposition 2.1.

We next introduce another self–similar process. Denote by $\widehat{\xi} = (-\xi_s, s > 0)$ the dual Lévy process and by $\widehat{\mathbf{P}}$, and $\widehat{\mathbf{E}}$, its probability and expectation. Then define $(\widehat{\mathbb{P}}_x, x > 0)$ to be the distribution on $\mathbb{D}^+$ of the α–self–similar process associated to the Lévy process with law $\widehat{\mathbf{P}}$. The process $\widehat{X}$ is usually called the dual α–self–similar process; the term dual is justified by the relation

$$\int_0^\infty g(x)V_q f(x)x^{(1-\alpha)/\alpha}dx = \int_0^\infty f(x)\widehat{V}_q g(x)x^{(1-\alpha)/\alpha}dx, \tag{14}$$

for every $f, g :]0,\infty[\to \mathbb{R}^+$ measurable, see e.g. Lemma 2 in [4]. By hypothesis (H2-d) we have that $0 < m := |\psi'(0^+)| = \widehat{\mathbf{E}}(\xi_1) < \infty$. Let $\widehat{\mathbb{P}}_{0+}$ be the limit in the sense of finite dimensional marginals of $\widehat{\mathbb{P}}_x$ as $x \to 0$, whose existence is ensured by Theorem 1 in [4]. The latter theorem also establishes that for every $t > 0$ and for $f : \mathbb{R}^+ \to \mathbb{R}^+$, measurable we have

$$\widehat{\mathbb{E}}_{0+}(f(X_t)) = \frac{\alpha}{m}\mathbf{E}(f((t/I)^\alpha)/I), \tag{15}$$

where I is defined in (2). Hypothesis (H2-d) implies that for any $t > 0$ the law of X_t under $\widehat{\mathbb{P}}_{0+}$ has a density with respect to the measure $\upsilon(dy) = y^{(1-\alpha)/\alpha}dy, y > 0$, given by the formula

$$\frac{\widehat{\mathbb{P}}_{0+}(X_t \in dy)}{\upsilon(dy)} = m^{-1}y^{-1/\alpha}\rho(ty^{-1/\alpha}) := \widehat{p}_t(y), \qquad y > 0.$$

Let $(\mu_s(dy) = \widehat{\mathbb{P}}_{0+}(X_s \in dy), s > 0)$. A consequence of the duality relation (14) is that the relation $\mu_s\widehat{P}_{t-s} = \mu_t$ for $s < t$ can be shifted to the semigroup of the minimal process P_t as $\widehat{p}_t = P_s\widehat{p}_{t-s}$ υ–a.s. It was proved in Rivero [28] section 4, that these densities can be used to construct a regular version of the family of probability measures $(\mathbb{P}_x(\cdot|T_0 = r), r > 0)$ when the underlying Lévy process is a subordinator. Morever, the same argument applies to any Lévy process assuming only (H2-d). Here the densities $(\widehat{p}_t, t \geq 0)$ will be used to construct a bridge for the coordinate process under $\mathbb{E}^{\natural}{}_{0+}$; the techniques here used are reminiscent of those in Fitzsimmons et al. [15].

Recall that the semigroup $(P^{\natural}_t, t \geq 0)$ is the h–transformation of the semigroup $(P_t, t \geq 0)$ via the invariant function $h(x) = x^\theta, x > 0$. Using the fact that for every $t > s > 0$, the equality $\widehat{p}_t = P_s\widehat{p}_{t-s}$ υ–a.s. holds, we obtain that for $r > 0$ arbitrary, the function

$$h^{\natural r}(s, x) = \widehat{p}_{r-s}(x)x^{-\theta}1_{\{s<r\}}, \quad x > 0, s > 0,$$

is excessive for the semigroup $(\pi_t \otimes P^{\natural}_t, t \geq 0)$ of the space-time process. Let $\overline{\Lambda}^r$ be the h–transform of the measure $\mathbb{E}^{\natural}{}_{0+}$ by means of the space–time excessive function $h^{\natural r}(s,x)$. Then under $\overline{\Lambda}^r$ the space process $(X_t, t > 0)$ is an inhomogeneous Markov process with entrance law

$$\overline{\Lambda}^r_s f = \mathbb{E}^{\natural}{}_{0+}(f(X_s)\widehat{p}_{r-s}(X_s)X_s^{-\theta}), \quad 0 < s < r,$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable, and inhomogeneous semigroup

$$K^r_{t,t+s}(x,dy) = \frac{P^{\natural}_s(x,dy)h^{\natural r}(t+s,y)}{h^{\natural r}(t,x)} = \frac{P_s(x,dy)\widehat{p}_{r-(t+s)}(y)}{\widehat{p}_{r-t}(x)}, \quad y > 0; \quad t, t+s < r.$$

Observe that the inhomogeneous semigroup $K^r_{t,t+s}$ is that of X conditioned to die at 0 at time r, cf. [28] Lemma 7. Moreover, using the fact that $\overline{\Lambda}^r$ is a h–transform of the measure $\mathbb{E}^{\natural}{}_{0+}$ it is easily verified that the measure $\overline{\Lambda}^r$ has the property

$$\overline{\Lambda}^r(F(X_s, 0 \leq s < r)) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(F(r^{\alpha}X_s, 0 \leq s < 1)),$$

for every positive measurable F. In particular, the total mass of $\overline{\Lambda}^r$ is determined by

$$b_r := \overline{\Lambda}^r(1) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(1),$$

and it will be shown below that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\, \mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}m} < \infty. \tag{16}$$

Therefore, assuming the hypotheses (H2-a,b,c,d) and $\overline{\Lambda}^1(1) < \infty$, we can define a probability measure on $\mathcal{G}_\infty$ by $\Lambda^r = b_r^{-1}\overline{\Lambda}^r$. The distribution under Λ^r of the lifetime T_0 is the Dirac distribution at r i.e. $\Lambda^r(T_0 = r) = 1$, cf. [28] Lemma 7. We can now state the main result of this section.

Proposition 5 (Itô's description of the measure n). *Assume hypotheses (H2-a,b,c,d) holds and $0 < \alpha\theta < 1$. Then $\overline{\Lambda}^1(1) < \infty$. Let $\mathbf{n}$ be the unique normalized excursion measure such that $\mathbf{n}(X_{0+} > 0) = 0$. For $F \in \mathcal{G}_\infty$,*

$$\mathbf{n}(F) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\int_0^\infty \Lambda^r(F \cap \{T_0 = r\})\frac{dr}{r^{1+\alpha\theta}}.$$

The proof of this proposition is similar to that given in [27] Theorem XII.4.2 for the analogous result for Brownian excursion measure.

Proof. We first show that

$$\mathbf{n}(F) = \frac{m}{a_{\alpha,\theta}}\int_0^\infty \overline{\Lambda}^r(F \cap \{T_0 = r\})dr, \tag{17}$$

with $a_{\alpha,\theta}$ as defined in Theorem 1. We deduce from this that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\, \mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}m}.$$

Indeed, by the monotone class theorem it is enough to prove the assertion for sets F of the form

$$F = \bigcap_1^n \{X(t_i) \in B_i\},$$

with $0 < t_1 < t_2 < \cdots < t_n$ and Borel sets $B_i \subset]0,\infty[, i \in \{1,\dots,n\}$. On the one hand, according to Theorem 1 we have

$$\mathbf{n}(F) = \int_{B_1} \mathbf{n}_{t_1}(dx_1) \int_{B_2} P_{t_2-t_1}(x_1,dx_2)\cdots \int_{B_n} P_{t_n-t_{n-1}}(x_{n-1},dx_n),$$

On the other hand, using that $F \cap \{T_0 < t_n\} = \emptyset$ we have that the right hand term in (17) can be written as

$$\frac{m}{a_{\alpha,\theta}} \int_{t_n}^{\infty} dr \int_{B_1} \overline{\Lambda}^r_{t_1}(dx_1) \int_{B_2} K_{t_1,t_2}(x_1,dx_2)\cdots \int_{B_n} K_{t_{n-1},t_n}(x_{n-1},dx_n). \tag{18}$$

Recall from Theorem 1 that

$$\overline{\Lambda}^r_{t_1}(dx_1) = \mathbb{P}^{\natural}{}_{0^+}(X_{t_1} \in dx_1)\widehat{p}_{r-t_1}(x_1)x_1^{-\theta} = a_{\alpha,\theta}\,\mathbf{n}_{t_1}(dx_1)\widehat{p}_{r-t_1}(x_1).$$

Using this identity and the expression of the transition probabilities $K_{t_i,t_{i+1}}$ we get that (18) is equal to

$$m\int_{t_n}^{\infty} dr \int_{B_1} \mathbf{n}_{t_1}(dx_1) \int_{B_2} P_{t_2-t_1}(x_1,dx_2)\cdots \int_{B_n} P_{t_n-t_{n-1}}(x_{n-1},dx_n)\widehat{p}_{r-t_n}(x_n).$$

Finally, using

$$m\int_s^{\infty} \widehat{p}_{r-s}(x)dr = \int_s^{\infty} \rho((r-s)x^{-1/\alpha})\frac{dr}{x^{1/\alpha}} = 1,$$

for all $x > 0$, we conclude that both expressions in (17) for $\mathbf{n}(F)$ coincide. In particular, if $F = 1-e^{-T_0}$ we have that

$$1 = \mathbf{n}(1-e^{-T_0}) = \frac{m}{a_{\alpha,\theta}} \int_0^{\infty} \overline{\Lambda}^r(1)(1-e^{-r})dr = \frac{\overline{\Lambda}^1(1)m}{a_{\alpha,\theta}}\left(\frac{\Gamma(1-\alpha\theta)}{\alpha\theta}\right).$$

The value of $\overline{\Lambda}^1(1)$ in (16) is obtained by using the expression for $a_{\alpha,\theta}$ and we derive from (17) that

$$\mathbf{n}(F) = \frac{m\overline{\Lambda}^1(1)}{a_{\alpha,\theta}} \int_0^{\infty} \Lambda^r(F \cap \{T_0 = r\})\frac{dr}{r^{1+\alpha\theta}},$$

and the result follows. □

Remark A result equivalent to that in Proposition 5 can be obtained for the excursion measure n^j obtained via the jumping-in measure $\eta(dx) = b_{\alpha,\beta}x^{-(1+\beta)}dx$. The method is similar and we leave the details to the interested reader.

5 Duality

In this section we will construct a self–similar Markov process which is in weak duality with the process $\widetilde{X}$ and whose excursion measure is the image under time reversal of $\mathbf{n}$. This will be given under the hypotheses (H2) and

(H2-e) $\mathbf{E}(\xi_1^-) < \infty$, with $a^- = (-a)\vee 0$.

Next we introduce some notation. Let $\xi^{\natural}$ be a Lévy process with law $\mathbf{P}^{\natural}$ and $\widehat{\xi^{\natural}}$ its dual, i.e. $\widehat{\xi^{\natural}} = -\xi^{\natural}$. Denote by $\widehat{\mathbf{P}^{\natural}}$ and $\widehat{\mathbf{E}^{\natural}}$ the probability and expectation for $\widehat{\xi^{\natural}}$. The process $\widehat{\xi^{\natural}}$ drifts to $-\infty$ and satisfies the hypotheses (H2-a,b,c). Indeed, that (H2-b) holds follows from

$$\widehat{\mathbf{E}^{\natural}}(e^{\theta\xi_1}) = \mathbf{E}^{\natural}(e^{-\theta\xi_1}) = \mathbf{E}(e^{-\theta\xi_1}e^{\theta\xi_1}) = 1,$$

in the same way is verified that (H2-c) holds,

$$\widehat{\mathbf{E}^{\natural}}(\xi_1^+ e^{\theta\xi_1}) = \mathbf{E}^{\natural}((-\xi_1)^+ e^{-\theta\xi_1}) = \mathbf{E}(\xi_1^-) < \infty.$$

Let $(\widehat{\mathbb{P}^{\natural}}_x, x \geq 0)$ be the law on $\mathbb{D}^+$ of the α–self–similar process $\widehat{X}^{\natural} = (\widehat{X}^{\natural}_t, t \geq 0)$ associated by Lamperti's transformation to the Lévy process with law $\widehat{\mathbf{P}^{\natural}}$. The process $\widehat{X}^{\natural}$ has a lifetime $\widehat{T}_0 = \inf\{t > 0 : \widehat{X}^{\natural}_t = 0\}$ which is finite $\widehat{\mathbb{P}^{\natural}}_x$–a.s. for all $x \geq 0$. Denote by $(\widehat{P}^{\natural}_t, t \geq 0)$ and $(\widehat{V}^{\natural}_q, q > 0)$ the semigroup and resolvent of the minimal process for $\widehat{X}^{\natural}$, i.e.

$$\widehat{P}^{\natural}_t f(x) = \widehat{\mathbb{P}^{\natural}}_x(f(X_t), t < T_0), \quad t \geq 0,$$

and

$$\widehat{V}^{\natural}_q f(x) = \int e^{-qt}\widehat{P}^{\natural}_t f(x)dt, \quad q > 0.$$

By the duality relation (14), the resolvents $V^{\natural}_q$ and $\widehat{V}^{\natural}_q$ are in weak duality with respect to the measure $\upsilon(dx) = x^{(1-\alpha)/\alpha}dx,\ x > 0$. Furthermore, it follows that the resolvents V_q and $\widehat{V}^{\natural}_q$, are in weak duality with respect to the measure $\zeta(dx) = x^{(1-\alpha-\alpha\theta)/\alpha}dx,\ x > 0$.

We assume henceforth that $0 < \alpha\theta < 1$. The results in section 3 can be applied to the minimal process $(\widehat{X}^{\natural}, \widehat{T}_0)$ to ensure that there exists a unique normalized excursion measure $\widehat{\mathbf{n}}$, compatible with the semigroup $(\widehat{P}^{\natural}_t, t \geq 0)$ and its associated entrance law admits the representation

$$\widehat{\mathbf{n}}_s f = (\widehat{a}_{\alpha,\theta})^{-1}\widehat{\mathbb{E}}_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0,$$

where $\widehat{a}_{\alpha,\theta} = \alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m$, for f continuous and bounded. To see this it should be verified that the measure $\widehat{\mathbf{P}^{\natural}}^{\natural}$, obtained by h–transformation of the law $\widehat{\mathbf{P}^{\natural}}$ by means of the function $h(x) = e^{\theta x}$ is $\widehat{P}$. To that end, it suffices to prove that both probability measures have the same 1–dimensional marginals. Indeed,

$$\widehat{\mathbf{P}^{\natural}}^{\natural}(f(\xi_s)) = \widehat{\mathbf{P}^{\natural}}(f(\xi_s)e^{\theta\xi_s}) = \mathbf{P}^{\natural}(f(-\xi_s)e^{-\theta\xi_s}) = \mathbf{P}(f(-\xi_s)) = \widehat{\mathbf{P}}(f(\xi_s)),$$

for every f continuous and bounded. Then the α–self–similar Markov process associated to the Lévy process with law $\widehat{\mathbf{P}^{\natural}}^{\natural}$ is equivalent to that associated to the Lévy process with law $\widehat{\mathbf{P}}$. Remark that the law of J under $\widehat{\mathbf{P}^{\natural}}^{\natural}$ is the same as that of I under $\mathbf{P}$.

Then the minimal process $(\widehat{X}^{\natural}, \widehat{T}_0)$ admits a unique extension $(\widetilde{Z}_t, t \geq 0)$, that leaves 0 continuously a.s. Let $(\widehat{U}_q, q > 0)$ denote the resolvent of the process $\widetilde{Z}$. Because of the weak duality relation between the resolvents V_q, and $\widehat{V}^{\natural}_q$ it is natural to ask if this property is inherited by the resolvents U_q and $\widehat{U}_q$. That is the content of the following result.

Lemma 6. *The resolvents $(U_q, q > 0)$ and $\widehat{U}_q$ are in weak duality with respect to the measure $\zeta(dx) = x^{(1-\alpha-\alpha\theta)/\alpha}dx, x > 0$.*

Proof. From Proposition 3 we have that the resolvent at 0 of $\widetilde{Z}$ is determined by the expression

$$\widehat{U}_q f(0) = \frac{\widehat{\gamma}_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}(e^{-qy^{1/\alpha} I}) y^{(1-\alpha-\alpha\theta)/\alpha} dy,$$

with $\widehat{\gamma}_{\alpha,\theta} = (\widehat{a}_{\alpha,\theta} m)^{-1}$. Recall that the resolvent at 0 of $\widetilde{X}$ is given by

$$U_q f(0) = \frac{\gamma_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}^\natural(e^{-qy^{1/\alpha} J}) y^{(1-\alpha-\alpha\theta)/\alpha} dy.$$

On the other hand, for any $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ we have

$$\begin{aligned}
\int_0^\infty \zeta(dy) g(y) U_q f(y) &= \int_0^\infty \zeta(dy) g(y) V_q f(y) + U_q f(0) \int_0^\infty \zeta(dy) g(y)\, \mathbb{E}_y(e^{-qT_0}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{V}_q^\natural g(y) + U_q f(0) \int_0^\infty \zeta(dy) g(y)\, \mathbf{E}(e^{-qy^{1/\alpha} I}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{V}_q^\natural g(y) \\
&\quad + \frac{\widehat{a}_{\alpha,\theta} m}{a_{\alpha,\theta} m^\natural} \widehat{U}_q g(0) \int_0^\infty \zeta(dx) f(x)\, \mathbf{E}^\natural(e^{-qx^{1/\alpha} J}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{U}_q g(y),
\end{aligned}$$

where the last equality follows from the fact that the constants $\gamma_{\alpha,\theta}$ and $\widehat{\gamma}_{\alpha,\theta}$ are equal. To see this recall that $\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)})$, as remarked after Proposition 3. □

Some results on time reversal can be derived from the preceding facts. To give a precise statement we introduce some notation. Let ϱ denote the operator of time reversal at time T_0, that is

$$(\varrho X(\omega))(t) = \begin{cases} X_{(T_0-t)^-}(\omega) & \text{if} \quad 0 \le t < T_0 < \infty \\ 0 & \text{otherwise} \end{cases}$$

and let $\varrho\, \mathbf{n}$ denote the image under time reversal at time T_0 of $\mathbf{n}$. Recall that L is a return time if

$$L \circ \theta_t = (L-t)^+, \quad \text{a.s. for all} \quad t \ge 0.$$

The first part of the following result is an extension for self–similar process of the celebrated result on time reversal of Williams [36]: a three dimensional Bessel process starting from 0 and reversed at its last exit time from $x > 0$, is identical in law to a Brownian motion killed at its first hitting time of 0. In the second part we determine $\varrho\, \mathbf{n}$.

Proposition 6. *(i) If L is a finite return time then under* $\mathbb{E}^\natural{}_{0+}$ *the reversed process* $(X_{(L-t)-}, 0 \le t < L)$ *is Markovian and has semigroup* $(\widehat{P}_t^\natural, t \ge 0)$.

(ii) We have that $\varrho\, \mathbf{n} = \widehat{\mathbf{n}}$.

Proof. (i) The potential of the measure $\mathbb{E}^\natural{}_{0+}$ is determined by

$$\begin{aligned}
\mathbb{E}^\natural{}_{0+}\Big(\int_0^\infty ds f(X_s)\Big) &= a_{\alpha,\theta} \int_0^\infty ds\, \mathbf{n}_s(f h^*) \\
&= a_{\alpha,\theta} \int f(y) y^{(1-\alpha)/\alpha} dy,
\end{aligned}$$

with the notation of Sections 2.3 and 3. Because of the weak duality between the resolvents $V^{\natural}_{\lambda}$ and $\widehat{V}^{\natural}_{\lambda}$ with respect to the measure $y^{(1-\alpha)/\alpha}dy, y>0$, the statement in (i) is a direct consequence of a result of Nagasawa on time reversal. A general version of Nagasawa's result can be found in Dellacherie et al. [12]§ XVIII.46.
(ii) Assuming that the excursion of $\widetilde{X}$ from 0 starts and ends at 0 and using the weak duality in Lemma 6 it follows from a result due to Mitro [26] § 4 that $\varrho\,\mathbf{n}=\widehat{\mathbf{n}}$. To see that under our hypotheses the excursions of the self–similar process $\widetilde{X}$ from 0 starts and ends at 0, it should be verified that $\widetilde{X}_{g_t}=0$ and $\widetilde{X}_{D^-_{g_t}}=0$ for all t a.s. with $g_t=\sup\{s\leq t:\widetilde{X}_s=0\}$ and $D_t=\inf\{t\leq s:\widetilde{X}_s=0\}$, see e.g. Getoor and Sharpe [18] § 9. In fact, since we already know that $\mathbf{n}(X_{0+}>0)=0$, it suffices to verify that $\mathbf{n}(X_{T_0-}>0)=0$. The latter is a straightforward consequence of the Markov property and that $\mathbb{P}_x(X_{T_0-}>0)=0$ for all $x>0$, since X is a self–similar Markov process associated to a Lévy process that drifts to $-\infty$, see e.g. [22] Theorem 4.1. □

6 Examples

Example 2 (Self–similar diffusions). Here we consider the case when the Lévy process is a Brownian motion with negative drift. Let $(\xi_t=\varepsilon B_t-\mu t, t\geq 0)$ with $(B_t, t\geq 0)$ a Brownian motion and $\varepsilon,\mu>0$. The hypotheses (H2) are satisfied with $\theta=2\mu/\varepsilon^2$ and under $\mathbf{P}^{\natural}$ the law of $\xi^{\natural}$ is that of $\varepsilon B_t+\mu t$. Then the α–self–similar Markov process X associated to ξ has continuous paths and has an infinitesimal generator of the form

$$Lf(x)=(\varepsilon^2/2-\mu)x^{1-1/\alpha}f'(x)+\varepsilon^2/2x^{2-1/\alpha}f''(x),\quad x>0.$$

Then for $\alpha>0$ we have that $0<\alpha\theta<1$ if and only if $0<\mu<\varepsilon^2/2\alpha$. This corresponds to the case when the point 0 is a regular boundary point for the self–similar diffusion associated to the infinitesimal generator L just described; in the case $1\leq\alpha\theta$, or equivalently $\varepsilon^2/2\alpha\leq\mu$, 0 is an exit boundary point, see e.g. Lamperti [22] Theorem 5.1 and Vuolle-Apiala [34] Theorem 3.1 for a related discussion. If $0<\mu<\varepsilon^2/2\alpha$ holds, the process X admits a unique extension that is continuous and is characterized by Theorem 2. Furthermore, using the fact that the law of J under $\mathbf{E}^{\natural}$ is that of $2\alpha^2/(\varepsilon^2 Z_{\alpha\theta})$, with $Z_{\alpha\theta}$ a random variable of law gamma of parameter $\alpha\theta$, (see e.g. Dufresne [13]), we deduce that the entrance law in Theorem 1 has a density w.r.t. Lebesgue measure

$$\frac{\mathbf{n}_s(dy)}{dy}=c_{\alpha\theta}s^{-2(1-\alpha\theta)-1}y^{2(1-\alpha\theta)/\alpha-1}\exp(-y^{1/\alpha}s^{-1}d_{\varepsilon,\alpha})\qquad y>0,$$

with

$$c_{\alpha\theta}=\frac{(1-\alpha\theta)\alpha}{\Gamma(1-\alpha\theta)\mu^2}\left(\frac{\varepsilon^2}{2\alpha^2}\right)^{\alpha\theta}\quad\text{and}\quad d_{\varepsilon,\alpha}=\frac{2\alpha^2}{\varepsilon^2}.$$

Example 3 (Reflected stable processes). Let Y be a stable process of parameter $a\in]0,2[$ and $(\underline{\mathbb{P}}_x, x\geq 0)$ its law. Assume that Y has no negative jumps and $|Y|$ is not a subordinator. Define $\rho=\underline{\mathbb{P}}(Y_1>0)$ and

$$X'_t=\begin{cases}Y_t-\inf_{0\leq s\leq t}Y_s & \text{if } t\geq T_{]-\infty,0]}\\ Y_t & \text{if } t<T_{]-\infty,0]}\end{cases}$$

with $T_{]-\infty,0]}$ the first hitting time of $]-\infty,0]$ by Y. Then $\rho\in]0,1[$ and 0 is a regular recurrent state for X'. (We refer to Bertoin [1] § VIII and Chaumont [11] for background on stable processes and its excursion theory.) We denote by (X, T_0) the process X' killed at $T_{]-\infty,0]}$; this process is $1/a$–self–similar. The hypotheses on Y imply that

$$\underline{\mathbb{P}}_x(T_0 < \infty, X_{T_0-} = 0) = 1, \quad x > 0.$$

Let ξ be the Lévy process associated to (X, T_0) via Lamperti's transformation (see Caballero and Chaumont [9] for a precise description of ξ). We claim that the hypothesis (H2) are satisfied for $\theta = a(1-\rho)$. This can be viewed either by barehand calculations using the results in [9] or by the following arguments.

It is known that the function $h(x) = x^{a(1-\rho)}, x > 0$ is, up to a multiplicative constant, the only invariant function for the semigroup of the process (X, T_0). Then Cramér's condition (H2-b) for ξ, is satisfied with $\theta = a(1-\rho)$. A consequence of this fact and Proposition 3.1 in[24] is that the Lévy exponential functional $I = \int_0^\infty \exp\{a\xi_s\}ds$, has finite moments

$$\mathbf{E}(I^{\beta/a}) < \infty \quad \text{for every} \quad 0 < \beta < a(1-\rho).$$

The excursion measure for X' away from 0, say $\underline{\mathbf{n}}$, is an excursion measure compatible with the minimal process (X, T_0) such that its entrance law satisfies (iii) in Lemma 2 with $\gamma = 1-\rho$, and $\underline{\mathbf{n}}(X_{0+} > 0) = 0$ (see [11] and the reference therein). Thus $\mathbf{E}(I^{-\rho}) < \infty$, by Lemma 3. Therefore, it is easily verified by repeating the arguments in the proof of Proposition 4 that the condition (H2-c) is satisfied.

Finally, the excursion measure $\mathbf{n}$ defined in Theorem 1 is equal to $\underline{\mathbf{n}}$ and the recurrent extension in Theorem 2 associated to $\mathbf{n}$ is equivalent to X'.

Example 4. Let ξ be a non–arithmetic Lévy process with no positive jumps such that ξ derives to $-\infty$. We assume that ξ is neither the negative of a subordinator nor a deterministic drift. The case of the negative of a subordinator was discussed in example 1 and the case of a deterministic drift can be treated in the same way. From the theory of Lévy processes with no positive jumps we know that $\mathbf{E}(e^{\lambda\xi_1}) < \infty$, for all $\lambda > 0$. Then the convex function $\psi(\lambda) : \mathbb{R}^+ \to \mathbb{R}$, defined by $\mathbf{E}(e^{\lambda\xi_1}) = e^{\psi(\lambda)}$, is such that $\psi(0) = 0$, and $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Since ξ drifts to $-\infty$ there exists a unique $\theta > 0$, such that $\psi(\theta) = 0$. It follows that ξ satisfies the conditions (H2). Let $0 < \alpha < 1/\theta$, and let (X, T_0) be the α–self–similar minimal process associated to ξ. Owing to the absence of positive jumps, we have that $X_{T_{[z,\infty[}} = z$ whenever $T_{[z,\infty[} < T_0$, with $T_{[z,\infty[} = \inf\{t > 0 : X_t \geq z\}$. The excursion measure $\mathbf{n}$ compatible with the process (X, T_0) defined in Theorem 1 has the property:

> Under the probability measure on $\mathbb{D}^+$, $\mathbf{n}\,|(T_{[z,\infty[} < T_0)$, the processes $(X_t, t \leq T_{[z,\infty[})$ and $(X_{T_z+t}, t \leq T_0 - T_{[z,\infty[})$, are independent. The law of the former is $\mathbb{E}^{\natural}{}_{0+}$ killed at $T_{[z,\infty[}$ and of the latter is that of (X, T_0) started at z.

Here $\mathbf{n}\,|(T_{[z,\infty[} < T_0)$ means $\mathbf{n}(A \cap \{T_{[z,\infty[} < T_0\})/\,\mathbf{n}(\{T_{[z,\infty[} < T_0\})$ for $A \in \mathcal{G}_\infty$. This claim is easily verified using the fact that the measure $\mathbf{n}$ is a multiple of the h–transform of $\mathbb{E}^{\natural}{}_{0+}$ via the excessive function $h^*(x) = x^{-\theta}, x > 0$. Moreover, the law of the Lévy exponential functional $I = \int_0^\infty \exp\{\xi_s/\alpha\}ds$, associated to ξ is self–decomposable and as a consequence the law of I has a continuous density, cf. [28] Proposition 4. Therefore, to apply the results in Sections 4 & 5, the only hypothesis that should be made on ξ is that $\mathbb{E}(\xi_1) > -\infty$.

A On dual extensions

This section is motivated by Section 5, where we proved that given two minimal process X and $\widehat{X}$ which are self–similar and that are in weak duality, there exist Markov processes $\widetilde{X}$ and $\widetilde{Z}$ extending

(X, T_0) and $(\widehat{X}, \widehat{T}_0)$ respectively, which still are in weak duality. The purpose of this section is to give a generalization of this fact under the hypotheses of Blumenthal. The result given here is of independent interest and to make the section self–contained, we next introduce some notation. Let $(Y_t, t \geq 0)$ and $(\widehat{Y}_t, t \geq 0)$ be Markov processes having 0 as a trap. Denote by P, E, (resp. $\widehat{\mathrm{P}}, \widehat{\mathrm{E}}$) the probabilities and expectation for Y, (resp. $\widehat{Y}$) and by T_0 (resp. $\widehat{T}_0$) the first hitting time of 0 for Y (resp. for $\widehat{Y}$), i.e. $T_0 = \inf\{t > 0 : Y_t = 0\}$. Assume $\mathrm{P}_x(T_0 < \infty) = \widehat{\mathrm{P}}_x(T_0 < \infty) = 1$ for any $x > 0$. Let Q^0_t, and W^0_λ, (resp. $\widehat{Q}^0_t$, $\widehat{W}^0_\lambda$) denote the semigroup and λ–resolvent for Y killed at 0, (resp. $\widehat{Y}$). For $\lambda > 0$, define the functions $\varphi_\lambda, \widehat{\varphi}_\lambda : \mathbb{R}^+ \to [0, 1]$, by

$$\varphi_\lambda(x) = \mathrm{E}_x(e^{-\lambda T_0}); \qquad \widehat{\varphi}_\lambda = \widehat{\mathrm{E}}_x(e^{-\lambda T_0}), \qquad x > 0.$$

The main assumptions of this section are

(H3-a) $Y, \widehat{Y}$, both satisfy the basic hypotheses in [7];

(H3-b) the resolvents W^0_λ and $\widehat{W}^0_\lambda$ are in weak duality with respect to a σ–finite measure $\zeta(dx)$ on $]0, \infty[$;

(H3-c) We have

$$\int_{]0,\infty[} \zeta(dx)\varphi_\lambda(x) < \infty; \quad \int_{]0,\infty[} \zeta(dx)\widehat{\varphi}_\lambda(x) < \infty, \quad \text{for all} \quad \lambda > 0.$$

Theorem 3. *Assume hypotheses (H3). Then there exist excursion measures m and $\widehat{m}$ compatible with the semigroups $(Q^0_t, t \geq 0)$ and $(\widehat{Q}^0_t, t \geq 0)$ respectively. The Laplace transforms of the entrance laws $(m_s, s > 0)$ and $(\widehat{m}_s, s > 0)$ associated to m and $\widehat{m}$ respectively, are determined by*

$$\int_0^\infty e^{-\lambda s} m_s f ds = \int_{]0,\infty[} \zeta(dx) f(x) \widehat{\varphi}_\lambda(x); \quad \int_0^\infty e^{-\lambda s} \widehat{m}_s f ds = \int_{]0,\infty[} \zeta(dx) f(x) \varphi_\lambda(x),$$

for $\lambda > 0$, and f continuous and bounded. Furthermore, associated to these excursion measures there exist Markov processes Y^* and $\widehat{Y}^*$ which are extensions for Y and $\widehat{Y}$ respectively and which are still in weak duality with respect to the measure $\zeta(dx)$.

The proof of this theorem will be given via three lemmas. The first of them ensures the existence of the excursion measures.

Lemma 7. *The family of finite measures $M_\lambda f = \int_{]0,\infty[} \zeta(dx) f(x) \widehat{\varphi}_\lambda(x)$, $\lambda > 0$, is such that*

(i) $\lim_{\lambda \to \infty} M_\lambda 1 = 0$

(ii) For $\mu, \lambda > 0$, $\mu \neq \lambda$

$$(\mu - \lambda) M_\lambda W^0_\mu f = M_\lambda f - M_\mu f,$$

for f continuous and bounded.

Proof. That $M_\lambda \longrightarrow 0$, as $\lambda \to \infty$, follows from the monotone convergence theorem. Using the weak duality for the resolvents W^0_λ and $\widehat{W}^0_\lambda$, we get

$$\begin{aligned} M_\lambda W^0_\mu f &= \int_{]0,\infty[} \zeta(dx) W^0_\mu f(x) \widehat{\varphi}_\lambda(x) \\ &= \int_{]0,\infty[} \zeta(dx) f(x) \widehat{W}^0_\mu \widehat{\varphi}_\lambda(x). \end{aligned}$$

The result is then obtained from the elementary identity

$$\widehat{W}^0_\mu \widehat{\varphi}_\lambda(x) = \frac{\widehat{\mathrm{E}}_x(e^{-\lambda T_0} - e^{-\mu T_0})}{\mu - \lambda}.$$

□

From Lemma 7 and Theorem 6.9 of Getoor and Sharpe [17], there exists a unique entrance law $(m_t, t > 0)$, for the semigroup $(Q_t, t \geq 0)$, such that for each $\lambda > 0$

$$M_\lambda f = \int_0^\infty e^{-\lambda t} m_t f dt,$$

for f measurable and bounded, and

$$\int_0^1 m_t 1 dt < \infty.$$

According to Blumenthal [7], for an entrance law $(m_s, s > 0)$ there exists a unique excursion measure m, such that its entrance law is $(m_s, s > 0)$. The same method ensures the existence of an excursion measure $\widehat{m}$ and an entrance law $(\widehat{m}_t, t > 0)$, for the semigroup $(\widehat{Q}_t, t \geq 0)$.

Using the results in [7] we obtain that associated to the excursion measure m (resp. to $\widehat{m}$) there exists a unique Markov process Y^* extending Y (resp. $\widehat{Y}^*$ extends $\widehat{Y}$) and the λ–resolvent of Y^* is determined by

$$W_\lambda f(0) = \frac{M_\lambda f}{\lambda M_\lambda 1}; \quad W_\lambda f(x) = W^0_\lambda f(x) + \varphi_\lambda(x) W_\lambda f(0), \quad x > 0,$$

for f measurable and bounded; the λ–resolvent for $\widehat{Y}^*$, say $\widehat{W}_\lambda$, is defined in a similar way. To establish weak duality with respect to the σ–finite measure $\zeta(dx)$ for the resolvents W_λ and $\widehat{W}_\lambda$ we will need the following technical result.

Lemma 8. *For every $\lambda > 0$, we have that $\lambda M_\lambda 1 = \lambda \widehat{M}_\lambda 1$.*

Proof. This result is a consequence of the following identity, for $\lambda, \mu > 0$

$$\lambda M_\lambda 1 - \mu M_\mu 1 = \lambda \widehat{M}_\lambda 1 - \mu \widehat{M}_\mu 1;$$

and the fact that

$$\lim_{\mu \to \infty} \mu M_\mu 1 = 0,$$

since $m(1 - e^{-\mu T_0}) = \mu M_\mu 1$, with m the excursion measure associated to the entrance law $(m_s, s > 0)$. Thus, to end the proof we just have to prove the former identity. Indeed, this follows from the fact that

$$M_\lambda \varphi_\mu = \int_{]0,\infty[} \zeta(dx) \widehat{\varphi}_\lambda(x) \varphi_\mu(x) = \widehat{M}_\mu \widehat{\varphi}_\lambda,$$

and the following elementary identities: for $\lambda, \mu > 0$

$$(\lambda - \mu) M_\lambda \varphi_\mu = \lambda M_\lambda 1 - \mu M_\mu 1, \quad \text{and} \quad (\lambda - \mu) \widehat{M}_\lambda \widehat{\varphi}_\mu = \lambda \widehat{M}_\lambda 1 - \mu \widehat{M}_\mu 1.$$

□

Finally, the following lemma establishes weak duality for the resolvents W_λ and $\widehat{W}_\lambda$.

Lemma 9. *For every $\lambda > 0$ and every measurable functions $f, g : [0,\infty[\to \mathbb{R}^+$, we have*

$$\int_{]0,\infty[} \zeta(dy) g(y) W_\lambda f(y) = \int_{]0,\infty[} \zeta(dy) f(y) \widehat{W}_\lambda g(y).$$

The proof of this lemma is a straightforward consequence of Lemma 8 and the construction of W_λ and $\widehat{W}_\lambda$; see the proof of Lemma 6.

Remarks

1. Observe that
$$\lim_{\lambda\to 0} \int_0^\infty ds e^{-\lambda s} m_s f = \int_0^\infty ds m_s f = \int_{]0,\infty[} \zeta(dy) f(y).$$
By the weak duality relation in Lemma 9 we have that $\zeta(dy)$ is invariant for the semigroup of Y^* and, since 0 is a recurrent state for Y^*, $\zeta(dy)$ is in fact the unique (up to a multiplicative constant) excessive measure for this semigroup, see e.g. Dellacherie et al. [12] XIX.46.

2. We have not considered here the possibility of a *stickiness* parameter in the construction of the processes Y^* and $\widehat{Y}^*$; that is constructing Y^* and $\widehat{Y}^*$ via the subordinators
$$\sigma_t = \mathrm{d}t + \sum_{s\leq t} T_0(\Delta_s); \quad \widehat{\sigma}_t = \widehat{\mathrm{d}}t + \sum_{s\leq t} \widehat{T}_0(\Delta_s), \quad t > 0,$$
for some $\mathrm{d}, \widehat{\mathrm{d}} > 0$ (see section 2.1 for the notation or Blumenthal [8] § 5 for an account). In such a case, the λ–resolvent for Y^* (resp. $\widehat{Y}^*$) at 0 is given by
$$W_\lambda f(0) = \frac{\mathrm{d}f(0) + M_\lambda f}{\lambda \mathrm{d} + \lambda M_\lambda 1}; \quad W_\lambda f(0) = \frac{\widehat{\mathrm{d}} f(0) + \widehat{M}_\lambda f}{\lambda \widehat{\mathrm{d}} + \lambda \widehat{M}_\lambda 1},$$
for f continuous and bounded, and, if $\mathrm{d} = \widehat{\mathrm{d}}$, then the resolvents W_λ and $\widehat{W}_\lambda$ are still in weak duality but this time with respect to the measure $\zeta^d(dx) = d\delta_0(dx) + \zeta(dx)$.

3. Assume moreover that for every $x > 0$, $\widehat{\mathrm{P}}_x(T_0 \in dt)$ is absolutely continuous with respect to Lebesgue measure, having a density
$$a(x,t) = \frac{\widehat{\mathrm{P}}_x(T_0 \in dt)}{dt}, \quad x, t > 0,$$
which is jointly Borel measurable. Then for $\lambda > 0$,
$$\int_0^\infty ds e^{-\lambda s} m_s f = \int_{]0,\infty[} \zeta(dx) \widehat{\varphi}_\lambda(x) f(x) = \int_0^\infty ds e^{-\lambda s} \int_{]0,\infty[} \zeta(dx) a(x,s) f(x),$$
for f continuous and bounded. The second equality is a consequence of Fubini's theorem. By inverting the Laplace transform we obtain that for $s > 0$,
$$m_s f = \int_{]0,\infty[} \zeta(dx) a(x,s) f(x).$$
A similar result was obtained by Getoor in [16] Proposition 10.10 in a different setting.

Bibliography

[1] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[2] J. Bertoin and M.-E. Caballero. Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli*, 8(2):195–205, 2002.

[3] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994.

[4] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002.

[5] J. Bertoin and M. Yor. On the entire moments of self-similar Markov process and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse Math.*, 11(1):19–32, 2002.

[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[7] R. M. Blumenthal. On construction of Markov processes. *Z. Wahrsch. Verw. Gebiete*, 63(4):433–444, 1983.

[8] R. M. Blumenthal. *Excursions of Markov processes*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992.

[9] M. Caballero and L. Chaumont. Weak convergence of positive self–similar Markov processes when the initial state tends to 0. In preparation, 2003.

[10] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997.

[11] L. Chaumont. Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bull. Sci. Math.*, 121(5):377–403, 1997.

[12] C. Dellacherie, B. Maisonneuve, and P. A. Meyer. *Probabilités et potentiel: Processus de Markov (fin). Compléments du calcul stochastique*, volume V. Hermann, Paris, 1992.

[13] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990.

[14] P. Embrechts and M. Maejima. *Self-similar processes*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2002.

[15] P. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 101–134. Birkhäuser Boston, Boston, MA, 1993.

[16] R. K. Getoor. Excursions of a Markov process. *Ann. Probab.*, 7(2):244–266, 1979.

[17] R. K. Getoor and M. J. Sharpe. Last exit times and additive functionals. *Ann. Probability*, 1:550–569, 1973.

[18] R. K. Getoor and M. J. Sharpe. Excursions of dual processes. *Adv. in Math.*, 45(3):259–309, 1982.

[19] C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.

[20] J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, 21(3):500–510, 1984.

[21] H. Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, 131:207–248, 1973.

[22] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[23] H. P. McKean, Jr. Excursions of a non-singular diffusion. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 1:230–239, 1962/1963.

[24] O. Mejane. Cramer's estimate for the exponential functional of a Lévy process. Unpublished, Laboratoire de Statistiques et Probabilités, Université Paul Sabatier, 2002. Available via http://front.math.ucdavis.edu/math.PR/0211409.

[25] P. A. Meyer. Processus de Poisson ponctuels, d'après K. Ito. In *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970)*, pages 177–190. Lecture Notes in Math., Vol. 191. Springer, Berlin, 1971.

[26] J. B. Mitro. Exit systems for dual Markov processes. *Z. Wahrsch. Verw. Gebiete*, 66(2):259–267, 1984.

[27] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[28] V. M. Rivero. A law of iterated logarithm for increasing self–similar Markov process. *Stochastics and Stochastics Reports*, 75(6):443–472, 2003.

[29] L. C. G. Rogers. Itô excursion theory via resolvents. *Z. Wahrsch. Verw. Gebiete*, 63(2):237–255, 1983.

[30] T. S. Salisbury. Construction of right processes from excursions. *Probab. Theory Related Fields*, 73(3):351–367, 1986.

[31] T. S. Salisbury. On the Itô excursion process. *Probab. Theory Related Fields*, 73(3):319–350, 1986.

[32] K.-I. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[33] M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[34] J. Vuolle-Apiala. Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, 22(2):546–565, 1994.

[35] J. Walsh. The cofine topology revisited. In *Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976)*, pages 131–151. Amer. Math. Soc., Providence, R. I., 1977.

[36] D. Williams. Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974.