Chapitre IV

Recurrent extensions of self–similar Markov processes and Cramér's condition II

Abstract

This chapter is a continuation of Chapter III. We indicate how the methods used there can be extended to study the recurrent extensions of a positive self–similar Markov process that makes a jump to 0. The unique excursion measure $\overline{\mathbf{n}}$ under which the excursion process leaves 0 continuously is constructed as well as its associated self–similar recurrent extension. The image under time reversal of $\overline{\mathbf{n}}$ is determined and we construct a dual self–similar recurrent Markov process associated to it. We make explicit the law of the meander process and that of the excursion process conditioned to have a given length. We construct a self–similar Markov process conditioned to hit 0 continuously.

Key words. Self–similar Markov process, description of excursion measures, weak duality, Lévy processes.
A.M.S. Classification. 60 J 25 (60 G 18).

1 Introduction

Let $(\mathbb{Q}_x, x \geq 0)$ be the law of a $\mathbb{R}^+$–valued self-similar Markov process Y started at $x \geq 0$. Assume that Y hits 0 at some finite time and then dies. We will refer to $(Y, \mathbb{Q})$ as the minimal process. This chapter is the companion of Chapter III. There we studied the excursion measures and the recurrent extensions of a self–similar Markov process Y that hits 0 continuously, i.e.

$$\mathbb{Q}_x(T_0 < \infty, Y_{T_0-} = 0) = 1 \qquad \text{for all } x > 0,$$

where $T_0 = \inf\{t > 0 : Y_{t-} = 0 \text{ or } Y_t = 0\}$. Here we are interested in the same problem but for a self–similar Markov process that hits 0 by a jump a.s.

$$\mathbb{Q}_x(T_0 < \infty, Y_{T_0-} > 0) = 1 \qquad \text{for all } x > 0. \tag{1}$$

As was proved by Lamperti [21], the former corresponds to a self–similar Markov process associated to a Lévy process ξ with an infinite lifetime and which drifts to $-\infty$, $\lim_{t\to\infty} \xi_s = -\infty$ a.s., while the latter corresponds to one associated to a Lévy process killed at an independent exponential time (i.e. jumps to $-\infty$ with some strictly positive rate).

In this Chapter, instead of using Brownian motion as a thread for introducing our main results, as we did in Chapter III, we prefer to use stable processes with negative jumps, that is Lévy processes which are self–similar. This choice is more appropiate to the present framework since it is well known that stable processes with negative jumps hit $]-\infty,0[$ a.s. by a jump. As a consequence a stable process killed at its first hitting time of $]-\infty,0[$ is a positive self–similar Markov process that satisfies the property (1). With this in mind we next briefly recall some known results on stable processes. We refer to Chaumont [8, 10] for an account of stable processes and their excursion theory.

Let (X,P) be an a–stable Lévy process for $a\in]0,2[$, i.e. a real–valued Lévy process that is $1/a$–self–similar, and we assume that X has negative jumps and that $|X|$ is not a subordinator. We denote by P^0 the law of the process X killed at its first entrance into $]-\infty,0[$ and take 0 as a cemetery point. Since X has negative jumps we have that X hits $]-\infty,0]$ by a jump and

$$P^0_x(X_{T_0-}>0,\ T_0<\infty)=1,\qquad \forall x>0,$$

where $T_0=\inf\{t\geq 0: X^0_t=0\}$. We denote $(\xi,\mathbf{Q})$ the Lévy process associated to (X^0,P^0) via Lamperti's [21] transformation. According to Lamperti, under our assumptions the real–valued Lévy process $(\xi,\mathbf{Q})$ is a Lévy process killed at an independent exponential time. A consequence of the results of Silverstein [25] is that the function

$$h_\rho(x)=x^{a(1-\rho)}, x\geq 0,\qquad \rho=P(X_1\geq 0),$$

is, up to a multiplicative constant, the unique invariant function for P^0, i.e. for any $t>0$

$$P^0_x(h_\rho(X_t))=h_\rho(x),\qquad \text{for all } x\geq 0.$$

It follows that the function $h(x)=e^{a(1-\rho)x}, x\in\mathbb{R}$, is an invariant function for the process $(\xi,\mathbf{Q})$. Next, let $P^\natural$ be the h–transform of P^0 via the invariant function h_ρ. The probability measure $P^\natural$ is the law of a positive $1/a$–self–similar Markov process such that

$$P^\natural_x(\lim_{t\to\infty}X_t=\infty, T_0=\infty)=1\qquad x\geq 0.$$

It is not hard to see that the Lévy process associated to $(X^\natural,P^\natural)$ via Lamperti's transformation is in fact the process $(\xi,\mathbf{Q})$ h–transformed via the function $h(x)=e^{a(1-\rho)x}, x\in\mathbb{R}$, and can be interpreted as $(\xi,\mathbf{Q})$ conditioned to drift to ∞. Furthermore, Chaumont [8, 10] showed that the measures P and $P^\natural$ are related in the same way as the law of a Brownian motion killed at 0 is related to that of a Bessel(3) process, see e.g. [22]. Using this fact Chaumont obtains a description of the unique excursion measure n compatible with the law of (X^0,P^0) such that $n(X_{0+}>0)=0$ and $n(1-e^{-T_0})=1$, which is reminiscent of Imhof's [18] description of Itô's excursion measure for the Brownian motion using the law of a Bessel(3) process. The measure n is the Itô's excursion measure of X reflected at its infimum, that is $((X_t-\inf_{s\leq t}X_s, t\geq 0),P)$. In section 2 we obtain, under some hypotheses, results that are analogous to those above and then are used to construct the unique excursion measure, say $\overline{\mathbf{n}}$, compatible with $(Y,\mathbb{Q})$ and such that $\overline{\mathbf{n}}(Y_{0+}>0)=0$ and $\overline{\mathbf{n}}(1-e^{-T_0})=1$. Associated to this excursion measure there is a unique self–similar recurrent extension of the process $(Y,\mathbb{Q})$, say $(\widetilde{Y},\widetilde{\mathbb{Q}})$, which, in the case of the stable process corresponds to the stable process reflected at its infimum.

We noted above that (X^0,P^0) hits 0 by a jump and, by the Markov property, it follows that $n(X_{T_0-}=0)=0$, i.e. the excursions end by a jump a.s. Chaumont [8] Corollaire 1 proved that conditionally on the value of X_{T_0-} the image under time reversal of n is equal to the law, say $P^{*\downarrow}$, of the dual stable process, $(X^*, P^*) = (-X, P)$, killed at its first hitting time of $]-\infty, 0]$ and conditioned to hit 0 continuously. In part (ii) of Theorem 2 we determine the image under time reversal of $\overline{\mathbf{n}}$ and we deduce therefrom that a similar property for the image under time reversal of $\overline{\mathbf{n}}$ conditioned on the value of Y_{T_0-} still holds. In part(iii) of Theorem 2 we construct a process Z_θ whose excursion measure from 0 is the image under time reversal of $\overline{\mathbf{n}}$ and which is in weak duality with the process $\widetilde{Y}$. Then we prove that the process Z_θ started at 0 is equal in law to the process obtained by time reversing one by one the excursions from 0 of the process $\widetilde{Y}$ started at 0. The latter result is reminiscent of Theorem 4.8 of Getoor and Sharpe [16]. In the stable process setting one can use the result of Doney [12] to interpret the process Z_θ as the process (X^*, P^*) conditioned to stay positive and reflected at its future infimum. Doney gives a pathwise construction of a Lévy process conditioned to stay positive by using Tanaka's [26] method.

Section 4 is devoted to the construction of the law under $\overline{\mathbf{n}}$ of the excursion process conditioned by its length and to establishing an absolute continuity relation between this law and that of the meander process. This relation between the law of the excursion process conditioned by its length and that of the meander process was established by Chaumont [10] Théorème 2 for stable processes with negative jumps.

In addition to $(X^\natural, P^\natural)$ there is another process, say $(X^\downarrow, P^\downarrow)$, associated to (X^0, P^0) which plays an important rôle in the understanding of n. This is process can be thought of as (X^0, P^0) conditioned to hit 0 continuously. More precisely, Silverstein's [25] results imply that the function $h'_\rho(x) = x^{a(1-\rho)-1}, x \geq 0$ is excessive for (X^0, P^0). Using this, Chaumont [8] Section 1.3 constructs a process $(X^\downarrow, P^\downarrow)$ as a h–transform of (X^0, P^0) via the function h'_ρ and shows that this is a self–similar Markov process that hits 0 continuously. Actually, the function $h^\downarrow(x) = \exp\{(a(1-\rho)-1)x\}, x \in \mathbb{R}$, is invariant for the Lévy process $(\xi, \mathbf{Q})$ and the corresponding h–transform can be thought of as $(\xi, \mathbf{Q})$ conditioned to tend to $-\infty$ as the time tends to ∞. The purpose of Section 5 is, under supplementary hypotheses, to provide a construction of a self–similar Markov process $Y^\downarrow$ that can be thought ofas $(Y, \mathbb{Q})$ conditioned to hit 0 continuously. The results of Section III.3 can be applied to this process to ensure the existence of an excursion measure $\mathbf{n}^\downarrow$, such that $\mathbf{n}^\downarrow(X_{0+} > 0) = 0$ and $\mathbf{n}^\downarrow(X_{T_0-} > 0) = 0$. Furthermore this excursion measure is absolutely continuous w.r.t. the excursion measure $\overline{\mathbf{n}}$.

In Section 6.1 we verify that stable processes with negative jumps satisfy our hypotheses and we go into more detail about the results recalled above. Moreover, with the aim of establishing further connections with the results in Chapter III in Section 6.2, we work in the framework of Section III.5 to determine the weak dual of a self–similar Markov process that leaves 0 by a jump and hits 0 continuously.

2 Settings and first results

Our first purpose is to establish the analogues of Propositions III.2 and III.3 and Theorems III.1 and III.2 for the class of self–similar Markov processes that hit 0 by a jump. With this aim we recall that the techniques used in the proofs of those results are based essentially on two facts which are deduced from the hypothesis that the underlying Lévy process satisfies Cramér's condition. Under this assumption we can ensure that there exists a $\theta > 0$ such that the function $h(x) = x^\theta, x \geq 0$ is invariant for the semi-group of the process Y, and that the law ${\mathbb{Q}^\natural}_x$, which is the h–transform of $\mathbb{Q}_x$ via $h(x) = x^\theta$, has a limit ${\mathbb{Q}^\natural}_{0+}$ as x goes to 0 in the sense of finite dimensional laws. The probability measure ${\mathbb{Q}^\natural}_x$ can be viewed as the law of the process Y conditioned to never hit 0. Therefore, in order to establish the main results of sections III.2 and III.3 in the present case, we just have to ensure that the latter facts still hold and the same proofs will still be valid. We devote this section to this task.

Let $\mathbf{Q}'$ be a measure on the space $(\mathbb{D}, \mathcal{D})$, of càdlàg trajectories with values in $\mathbb{R}$ endowed with the σ–algebra generated by the coordinate maps and $(\mathcal{D}'_t, t \geq 0)$ the natural filtration. Assume that under $\mathbf{Q}'$ the canonical process is a Lévy process and that the convex set

$$C = \{\lambda \in \mathbb{R} : \mathbf{Q}'(e^{\lambda \xi_1}) < \infty\},$$

contains a point different from 0, $C \setminus \{0\} \neq \emptyset$. Then the characteristic exponent of ξ, i.e. $\Psi : \mathbb{R} \to \mathbb{C}$, defined by

$$\mathbf{Q}'(e^{i\lambda \xi_1}) = e^{-t\Psi(\lambda)} \quad \lambda \in \mathbb{R},$$

admits an analytic extension to the complex strip $-\Im(z) \in C$. Thus we can define the Laplace exponent $\psi : C \to \mathbb{R}$ of $\mathbf{Q}'$ by

$$\mathbf{Q}'(e^{\lambda \xi_1}) = e^{\psi(\lambda)}, \quad \text{with} \quad \psi(\lambda) = -\Psi(-i\lambda), \quad \lambda \in C.$$

Hölder's inequality implies that ψ is a convex function on C. Let $\mathbf{Q}$ be the law of the Lévy process ξ which is obtained by killing ξ' at a rate $\boldsymbol{k}$, that is ξ' is killed at an independent exponential random variable of parameter $\boldsymbol{k} > 0$. Then the Laplace exponent $\psi_{\boldsymbol{k}}$ of ξ under $\mathbf{Q}$ is

$$\mathbf{Q}(e^{\lambda \xi_1}) = e^{\psi_{\boldsymbol{k}}(\lambda)}, \quad \psi_{\boldsymbol{k}}(\lambda) = \psi(\lambda) - \boldsymbol{k}, \quad \lambda \in C.$$

We will denote by ζ the lifetime of ξ, by $(\mathcal{D}_t, t \geq 0)$ the filtration of the killed process, by Δ the cemetery point for ξ and, as usual we extend the functions $f : \mathbb{R} \to \mathbb{R}$ to $\mathbb{R} \cup \Delta$ by $f(\Delta) = 0$.

We assume henceforth

(HI–a) ξ is not arithmetic, i.e. the state space is not a subgroup of $c\mathbb{Z}$ for any real c;

(HI–b) there exists $\theta > 0$ such that $\mathbf{Q}(e^{\theta \xi_1}, 1 < \zeta) = 1$;

(HI–c) $\mathbf{Q}(\xi_1^+ e^{\theta \xi_1}, 1 < \zeta) < \infty$.

We will refer to (HI–b) as Cramér's condition by analogy with Chapter III. Condition (HI-b) holds if and only if we kill ξ' at an independent exponential time $\boldsymbol{k} = \psi(\theta)$ for some θ in $C \cap]0, \infty[$. A sufficient condition for (HI-c) is that θ belongs to the interior of C. Cramér's condition implies that the function $h(x) = e^{\theta x}, x \in \mathbb{R}$, is invariant for the semi-group of ξ under $\mathbf{Q}$. Let $\mathbf{Q}^\natural$ be the h–transform of $\mathbf{Q}$ via the function $h(x) = e^{\theta x}$. That is $\mathbf{Q}^\natural$ is the unique measure on the space of càdlàg trajectories with lifetime such that

$$\mathbf{Q}^\natural(F_T) = \mathbf{Q}(F_T e^{\theta \xi_T}, T < \zeta) \qquad \text{for every stopping time } T \text{ of } \mathcal{D}_t.$$

Moreover, under $\mathbf{Q}^\natural$ the canonical process still is a Lévy process but with infinite lifetime and finite mean $m^\natural = \psi'(\theta) > 0$, owing to (HI-c) and the convexity of $\psi_{\boldsymbol{k}}$. Thus $\xi^\natural$ drifts to ∞, $\lim_{s \to \infty} \xi_s = \infty$ $\mathbf{Q}^\natural$–a.s. The characteristic exponent of $\xi^\natural$ is given by $\Psi^\natural(\lambda) = \Psi(\lambda - i\theta) + \boldsymbol{k}$ for $\lambda \in \mathbb{R}$.

Hereafter we take an arbitrary fixed $\alpha > 0$. Next, let $(\mathbb{Q}_x, x > 0)$ be the law of the α–self–similar Markov process Y associated to $(\xi, \mathbf{Q})$ via Lamperti's transformation. That is, let

$$A_t = \int_0^t \exp\{(1/\alpha)\xi_s\} ds \qquad t \geq 0$$

and let $\tau(t)$ be its inverse,

$$\tau(t) = \inf\{s > 0 : A_s > t\},$$

with the convention $\inf\{\emptyset\} = \infty$. For $x > 0$, let $\mathbb{Q}_x$ be the law of the process

$$Y_t = x \exp\{\xi_{\tau(tx^{-1/\alpha})}\}, \qquad t > 0,$$

with the convention that the above quantity is 0 if $\tau(tx^{-1/\alpha}) = \infty$. The Volkonskii theorem ensures that the process Y is a strong Markov process in the filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)}, t \geq 0)$. Furthermore, by construction the process Y has the scaling property: for every $c > 0$ the law of the process $(cY_{tc^{-1/\alpha}}, t \geq 0)$ under $\mathbb{Q}_x$ is $\mathbb{Q}_{cx}$. It follows that Y has a finite lifetime $T_0 = \inf\{t > 0 : Y_t = 0\}$ and that it has the same law under $\mathbb{Q}_x$ as $x^{1/\alpha} A_{\boldsymbol{e}}$ under $\mathbf{Q}'$ with

$$A_{\boldsymbol{e}} = \int_0^{\boldsymbol{e}} \exp\{(1/\alpha)\xi'_s\} ds, \tag{2}$$

with $\boldsymbol{e}$ an exponential random variable of parameter $\boldsymbol{k}$ independent of ξ'. Since ξ has a finite lifetime, Y hits 0 by a jump in finite time and then dies. We denote (Y, T_0) the process killed at 0 and by $(P_t, t \geq 0)$ and $(V_q, q > 0)$ its semi-group and resolvent respectively. Observe that the results of Section III 2.3 are still valid under the assumptions of this chapter since their proofs only use the property that the self–similar Markov process hits 0 in a finite time a.s.

Remark 1. The process Y is obtained by applying first an operation of killing and then a time change to the Lévy process. If the order of this construction is inverted, first time change and then killing according to a multiplicative functional, we obtain an equivalent self–similar Markov process. More precisely, given a Lévy process with law $\mathbf{Q}'$ and infinite lifetime, we construct a self–similar Markov process $(Y', \mathbb{Q}'_x, x \geq 0)$ via Lamperti's transformation of ξ'. This process either hits 0 continuously or never hits 0 a.s. Next we kill the process Y' according to the multiplicative functional

$$M_t = \exp\{-\boldsymbol{k}\varphi(t)\}, \quad \varphi(t) = \int_0^t (Y'_s)^{-1/\alpha} ds, \quad t < T'_0 = \inf\{r > 0 : Y'_r = 0\},$$

to obtain a self–similar Markov process Y''. See Lamperti [21] for a detailed study of the additive functional φ. The Feymann-Kac formula allows us to determine the infinitesimal generator of Y'', which is equal to that of Y. Thus the processes Y and Y'' are equivalent.

After this slight digression on the construction of Y we continue with our program. Let $({\mathbb{Q}^{\natural}}_x, x > 0)$ be the law of the α–self–similar Markov process $Y^{\natural}$ associated to the Lévy process $\xi^{\natural}$ with law $\mathbf{Q}^{\natural}$ via Lamperti's transformation. Since $\xi^{\natural}$ drifts to ∞ we have that $Y^{\natural}$ never hits 0 and $\lim_{t\to\infty} Y^{\natural}_t = \infty$, ${\mathbb{Q}^{\natural}}_x$–a.s. for all $x > 0$. As in Section III.3, the process $Y^{\natural}$ can be thought of as the process Y conditioned never to hit 0, thanks to the following statements which are the analogues of Proposition III.2

(i) *Let $x > 0$ be arbitrary. We have that ${\mathbb{Q}^{\natural}}_x$ is the unique measure such that for every $\mathcal{G}_t$ stopping time, T, we have*

$${\mathbb{Q}^{\natural}}_x(A) = x^{-\theta}\, \mathbb{Q}_x(A\ Y_T^{\theta}, T < T_0),$$

for any $A \in \mathcal{G}_T$. In particular, the function $h^* : [0,\infty[\to [0,\infty[$ defined by $h^*(x) = x^{\theta}$ is invariant for the semi-group P_t.

(ii) *For every $x > 0$ and $t > 0$ we have*

$${\mathbb{Q}^{\natural}}_x(A) = \lim_{s\to\infty} \mathbb{Q}_x(A \mid T_0 > s),$$

for any $A \in \mathcal{G}_t$.

The proof of (i) is the same as (i) in Proposition III.2; the proof of (ii) needs a lemma just as in the proof of (ii) in Proposition III.2

Lemma 1. *Under the hypothesis (HI) we have that there exists a constant* $C \in]0,\infty[$ *such that*

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{Q}'(A_{\boldsymbol{e}} > t) = C.$$

Moreover, if $0 < \alpha\theta < 1$ *then*

$$C = \frac{\alpha}{m^{\natural}}\,\mathbf{Q}'(A_{\boldsymbol{e}}^{-(1-\alpha\theta)}).$$

Proof. This proof, like that of the analogous result in Chapter III, is based on a result of Kesten [20] and Goldie [17] on random equations. We claim that $A_{\boldsymbol{e}}$ has the same law as $D + MA'_{\boldsymbol{e}'}$ with $D = \int_0^1 \exp\{\xi'_s\}1_{\{s<\boldsymbol{e}\}}ds$, $M = e^{(1/\alpha)\xi'_1}1_{\{1<\boldsymbol{e}\}}$ and $A'_{\boldsymbol{e}'}$ with the same law as $A_{\boldsymbol{e}}$ and independent of (D, M). Furthermore, $\mathbf{Q}'(D^{\alpha\theta}) < \infty$. These two facts enable us to apply the results of Kesten and Goldie to prove that

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{Q}'(A_{\boldsymbol{e}} > t) = C,$$

for some $C \in]0,\infty[$ whose expression can be found in [17]. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable and bounded function and put $\widetilde{D} = \int_0^1 \exp\{(1/\alpha)\xi'_s\}ds$ and $\widetilde{M} = \exp\{(1/\alpha)\xi'_1\}$. Indeed, using the lack of memory of the exponential law we obtain

$$\begin{aligned}
\mathbf{Q}'(f(A_{\boldsymbol{e}})) &= \mathbf{Q}'(f(D)1_{\{\boldsymbol{e}<1\}}) + \mathbf{Q}'(1_{\{\boldsymbol{e}>1\}}f(\widetilde{D} + \widetilde{M}\int_0^{\boldsymbol{e}-1}\exp\{(1/\alpha)(\xi_{1+s}-\xi_1)\}ds)) \\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}<1\}}) + e^{-\boldsymbol{k}}\,\mathbf{Q}'(f(\widetilde{D} + \widetilde{M}A'_{\boldsymbol{e}'})) \\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}<1\}}) + \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}>1\}}) \\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})),
\end{aligned}\tag{3}$$

where we observe that in the second equality the random variable $A'_{\boldsymbol{e}'}$ is independent of $\sigma(\xi'_s, s \le 1)$ and $\boldsymbol{e}$. We next prove that $\mathbf{Q}'(D^{\alpha\theta}) < \infty$.

$$\begin{aligned}
\mathbf{Q}'(D^{\alpha\theta}) &\le \mathbf{Q}'(\sup\left\{e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}}; s \le 1\right\}) \\
&\le \frac{e}{e-1}\left(1 + \sup_{\{0\le s\le 1\}}\mathbf{Q}'(e^{\theta\xi_s}\log^+(e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}})1_{\{s<\boldsymbol{e}\}})\right) \\
&= \frac{e}{e-1}\left(1 + \theta\sup_{\{0\le s\le 1\}}\mathbf{Q}'(e^{\theta\xi_s}\xi_s^+1_{\{s<\boldsymbol{e}\}})\right) < \infty,
\end{aligned}\tag{4}$$

where the second inequality is due to the fact that the process $e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}}$ is a positive martingale for $\mathbf{Q}'$ and a so we can apply a Doob's inequality, with the convention $0\log^+(0) = 0$. The last right–hand term is finite due to assumption (HI–c).

In the case $0 < \alpha\theta < 1$, the value of the constant C is determined as in the proof of Lemma III.4 using the identity

$$\mathbf{Q}'(A_{\boldsymbol{e}}^{\alpha\beta}) = \frac{\alpha\beta}{-\psi_{\boldsymbol{k}}(\beta)}\,\mathbf{Q}'(A_{\boldsymbol{e}}^{\alpha\beta-1}), \quad \beta < \theta,$$

whose proof can be found in Carmona, Petit & Yor [7] Proposition 3.1.(i) $\square$

Corollary 1. *For each $\beta \in]0, \theta \wedge (1/\alpha)[$ there exists a self–similar recurrent extension of (Y, T_0) that leaves 0 a.s. by a jump according to the jump-in measure $\eta^{\beta}(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx, x > 0$, with $b_{\alpha,\theta} = \beta / \mathbf{Q}'(A_{\mathbf{e}}^{\alpha\beta})\Gamma(1-\alpha\beta)$.*

The proof of Corollary 1 is a straightforward consequence of Lemma 1 and Proposition III.1; see the remarks at the end of section III.2.

Furthermore, since the Lévy process $\xi^{\natural}$ has a strictly positive finite mean $\mathbf{Q}^{\natural}(\xi_1) = m^{\natural}$ we know from [1] that there exists a measure $\mathbb{Q}^{\natural}{}_{0+}$ which is the limit in the sense of finite dimensional laws of $\mathbb{Q}^{\natural}{}_x$ as $x \to 0+$. Under $\mathbb{Q}^{\natural}{}_{0+}$ the law of Y_s is an entrance law for the semi-group of $Y^{\natural}$ and is related to the law of the Lévy exponential functional $J = \int_0^{\infty} \exp\{-(1/\alpha)\xi^{\natural}_s\} ds$ by the formula

$$\mathbb{Q}^{\natural}{}_{0+}(f(Y_s^{1/\alpha})) = \frac{\alpha}{m^{\natural}} \mathbf{Q}^{\natural}(f(s/J)J^{-1}), \qquad s > 0, \tag{5}$$

for f measurable and positive, see [1]. Assume $0 < \alpha\theta < 1$. Then to construct an excursion measure $\overline{\mathbf{n}}$ compatible with the minimal process (Y, T_0) such that $\overline{\mathbf{n}}(Y_{0+} > 0) = 0$ and $\overline{\mathbf{n}}(1 - e^{-T_0}) = 1$, we can argue as in the proof of Theorem III.1. Indeed, this is an h–transform of $\mathbb{Q}^{\natural}{}_{0+}$ via the excessive function $x^{-\theta}, x > 0$. Furthermore, the proof of Proposition III.3 can also be extended to the present case to ensure that the measure $\overline{\mathbf{n}}$ is the unique excursion measure with these properties, that is compatible with the minimal process (Y, T_0). We have the following results.

Theorem 1. *Assume $0 < \alpha\theta < 1$.*

(i) The excursion measure $\overline{\mathbf{n}}$ is such that for every $\mathcal{G}_t$–stopping time T

$$\overline{\mathbf{n}}(A_T, T < T_0) = (a_{\alpha,\theta})^{-1}\, \mathbb{Q}^{\natural}{}_{0+}(A_T Y_T^{-\theta}), \qquad A_T \in \mathcal{G}_T,$$

with $a_{\alpha,\theta} = \alpha\, \mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^{\natural}$.

(ii) The q–potential of the entrance law $(\overline{\mathbf{n}}_s, s > 0)$, associated to $\overline{\mathbf{n}}$, admits the representation

$$\int_0^{\infty} e^{-qs}\, \overline{\mathbf{n}}_s\, f ds = (m^{\natural} a_{\alpha,\theta})^{-1} \int_0^{\infty} f(y)\, \mathbf{Q}^{\natural}(e^{-y^{-1/\alpha}J}) y^{1/\alpha - 1 - \theta} dy,$$

for $f \in C_b(\mathbb{R}^+)$.

(iii) The minimal process (Y, T_0) admits a unique self–similar recurrent extension $\widetilde{Y}$ that leaves 0 continuously a.s. The resolvent of $\widetilde{Y}$ is given by

$$U_q f(0) = \frac{1}{(m^{\natural} a_{\alpha,\theta}) q^{\alpha\theta}} \int_0^{\infty} f(y)\, \mathbf{Q}^{\natural}(e^{-y^{-1/\alpha}J}) y^{1/\alpha - 1 - \theta} dy$$

and $U_q f(x) = V_q f(x) + \mathbb{Q}_x(e^{-qT_0}) U_q f(0)$, for $x > 0$ and $f \in C_b(\mathbb{R}^+)$. The resolvent U_q is Fellerian.

The proof of (i) in Theorem 1 is the same as that of Theorem III.1.(i); (ii) in Theorem 1 is proved as Proposition III.3.(i); last, the proof of Theorem III.2.(i) applies to prove (iii) in Theorem 1.

Remark 2. We can deduce as in the proof of Proposition III.3 that

$$\mathbf{Q}'(A_{\mathbf{e}}^{-(1-\alpha\theta)}) = \mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)}).$$

Remark 3. If $\alpha\theta \geq 1$, the arguments given in Theorem III.2 show that there does not exist an excursion measure compatible with the semigroup of Y such that the excursion process leaves 0 continuously.

3 Time reversed excursions

In this section we are interested in determining the image under time reversal of the unique excursion measure $\overline{\mathbf{n}}$ compatible with Y such that $\overline{\mathbf{n}}(Y_{0+} > 0) = 0$. Furthermore, we would like to determine whether the self–similar recurrent extension $\widetilde{Y}$ of Y admits a weak dual process and, if so, to identify it.

To this end, we recall that the process $Y^\natural$ has a weak dual that we denote by $\widehat{Y}^\natural$. The latter is the self–similar Markov process associated to $-\xi^\natural$, the dual of $\xi^\natural$. More precisely, let $V_q^\natural, \widehat{V}_q^\natural$ be the q–resolvents of $Y^\natural$ and $\widehat{Y}^\natural$ respectively. Then

$$\int_0^\infty dx x^{1/\alpha-1} f(x) V_q^\natural g(x) = \int_0^\infty dx x^{1/\alpha-1} g(x) \widehat{V}_q^\natural f(x),$$

for all measurable functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$. See Bertoin and Yor [1]. Next, since the process Y is the h-transform of $Y^\natural$ via the excessive function $h(x) = x^{-\theta}, x > 0$, we have that the q–resolvent V_q of Y is in weak duality with $\widehat{V}_q^\natural$ with respect to the measure $x^{1/\alpha-1-\theta}dx, x > 0$.

Since the process $-\xi^\natural$ drifts to $-\infty$ it follows that

$$\widehat{\mathbb{Q}}_x^\natural(Y_{T_0-} = 0, T_0 < \infty) = 1 \quad \text{for all } x > 0.$$

Now, that Y hits 0 by a jump implies that the excursions of $\widetilde{Y}$ away from 0 terminate by a jump a.s., i.e. $\overline{\mathbf{n}}(Y_{T_0-} = 0) = 0$, and by the self–similarity it is easy to prove that $\overline{\mathbf{n}}(Y_{T_0-} \in dx) = x^{-(1+\gamma)}dx, x > 0$ for some $\gamma > 0$. These two statements allow us to guess that the candidate for a weak dual of $\widetilde{Y}$ should be a recurrent extension of $\widehat{Y}^\natural$ that leaves 0 by a jump a.s. We formalize this statement in the following theorem. Let $\varrho : \mathbb{D}^+ \to \mathbb{D}^+$ be the operator of time–reversal at time T_0,

$$\varrho Y(t) = \begin{cases} Y_{(T_0-t)-} & \text{if} \quad 0 \leq t < T_0 < \infty \\ 0 & \text{otherwise,} \end{cases}$$

and $\varrho\, \overline{\mathbf{n}}$ the image under time reversal at T_0 of $\overline{\mathbf{n}}$.

Theorem 2. *(i) For each $\beta \in]0, \theta]$ the process $\widehat{Y}^\natural$ admits a self–similar recurrent extension $Z_\beta = (Z_{\beta,t}, t \geq 0)$ that leaves 0 by a jump according to the jumping-in measure*

$$\eta_\beta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx, x > 0,$$

with $b_{\alpha,\beta} = \beta/\Gamma(1-\alpha\beta)\widehat{\mathbf{Q}}^\natural(I^{\alpha\beta})$, and $I = \int_0^\infty \exp\{(1/\alpha)\widehat{\xi}_s^\natural\}ds$. The resolvent of Z_β is given by

$$\mathcal{U}_q f(0) = b_{\alpha,\beta} q^{-\alpha\beta} \int_0^\infty y^{-(1+\beta)} \widehat{V^\natural}_q f(y) dy; \quad \mathcal{U}_q f(x) = \widehat{V^\natural}_q f(x) + \widehat{\mathbb{Q}^\natural}_x(e^{-qT_0})\mathcal{U}_q f(0),$$

for $x > 0$.

(ii) The image under time reversal of $\overline{\mathbf{n}}$, is given by

$$\varrho\, \overline{\mathbf{n}}(\cdot) = b_{\alpha,\theta} \int_0^\infty dx x^{-(1+\theta)} \widehat{\mathbb{Q}^\natural}_x(\cdot).$$

In particular, $\overline{\mathbf{n}}(Y_{T_0-} \in dx) = b_{\alpha,\theta} x^{-(1+\theta)} dx, x > 0$ and $\varrho\, \overline{\mathbf{n}}(\cdot | Y_{T_0-} = x) = \widehat{\mathbb{Q}^\natural}_x(\cdot)$.

(iii) The process Z_θ is in weak duality with $\widetilde{Y}$ w.r.t. $x^{1/\alpha-1-\theta}dx, x>0$.

We have noted that in the stable processes setting, the self–similar process $\widetilde{Y}$ corresponds to a stable process reflected at its infimum and Z_θ is, as we will see later, the dual stable process conditioned to stay positive and reflected at its future infimum. Thus, in this case, (iii) in Theorem 2 establishes that these processes are in weak duality. We have said in the Introduction that Z_θ has the same law started at 0 as the process obtained by time reversing one by one the excursions from 0 of $\widetilde{Y}$ started from 0. This result still holds in a greater generality. To give a precise statement, in the sequel, we denote $\widetilde{\mathbb{Q}}_\cdot$ and $\widetilde{\mathbb{Q}}_\cdot^{\wedge}$ the law of the processes $\widetilde{Y}$ and Z_θ, respectively. We have the following corollary which is reminiscent of Theorem 4.8 of Getoor & Sharpe [16].

Corollary 2. *For any $t>0$, let $g_t=\sup\{s<t:\widetilde{Y}_s=0\}$, $d_t=\inf\{s>t:\widetilde{Y}_s=0\}$ and*

$$\overleftarrow{Y}_t=\begin{cases}Y_{(d_t-(t-g_t))-} & \text{if}\quad 0<g_t<d_t<\infty\\ Y_t & \text{otherwise.}\end{cases}$$

Then the process $\overleftarrow{Y}=(\overleftarrow{Y}_t, t\geq 0)$ has the same law under $\widetilde{\mathbb{Q}}_0$ as Z_θ under $\widetilde{\mathbb{Q}}_0^{\wedge}$.

We postpone the proof of Corollary 2 until subsection 3.1.

Proof of Theorem 2. (i) According to Proposition III.1 all that we have to verify in order to prove (i) is that $\widehat{\mathbf{Q}^\natural}(I^{\alpha\beta})<\infty$ for every $\beta\in]0,\theta]$. Indeed, due to (HI–c) we have that $-\widehat{\mathbf{Q}^\natural}(\xi_1)=m^\natural\in]0,\infty[$, and by the identity (5) that $\widehat{\mathbf{Q}^\natural}(I^{-1})=m^\natural/\alpha<\infty$ (observe that I under $\widehat{\mathbf{Q}^\natural}$ is equal to J under $\mathbf{Q}^\natural$). Therefore, for every $0<\alpha\beta\leq\alpha\theta<1$ we have that $\widehat{\mathbf{Q}^\natural}(I^{\alpha\beta-1})<\infty$. The claim follows using the identity

$$\widehat{\mathbf{Q}^\natural}(I^{\alpha\beta})=\frac{\alpha\beta}{-\boldsymbol{\psi}(\beta)}\widehat{\mathbf{Q}^\natural}(I^{\alpha\beta-1})\quad\text{for } 0<\beta\leq\theta, \tag{6}$$

with $\boldsymbol{\psi}:[0,\theta]\to\mathbb{R}$ defined by

$$\widehat{\mathbf{Q}^\natural}(e^{\lambda\xi_1})=e^{\boldsymbol{\psi}(\lambda)},\qquad 0\leq\lambda\leq\theta.$$

The identity (6) can be proved with arguments similar to those given by Bertoin & Yor [2]. Note that $\boldsymbol{\psi}(\lambda)=\psi_{\boldsymbol{k}}(\theta-\lambda)$, for every $0\leq\lambda\leq\theta$.

(ii) We first note that an application of Lemma III.3 proves that the entrance laws $(\overline{\mathbf{n}}_s(dy), s>0)$ and

$$N^\theta_s f=b_{\alpha,\theta}\int_0^\infty dx\, x^{-(1+\theta)}\widehat{P}^\natural_s f(x),\quad s>0,$$

for the semi-groups $(P_t, t\geq 0)$ and $(\widehat{P^\natural_s}, s\geq 0)$ respectively have the same potential

$$\int_0^\infty ds\,\overline{\mathbf{n}}_s\,f=C_{\alpha,\alpha\theta}\int_0^\infty f(x)x^{1/\alpha-1-\theta}dx=\int_0^\infty ds N^\theta_s f,$$

with $C_{\alpha,\alpha\theta}=(m^\natural a_{\alpha,\theta})^{-1}$. This enable us to use a result on time reversal of Kusnetzov measures established in Dellacherie, Maisonneuve & Meyer [11] XIX.33 to verify the claimed result.

(iii) We should prove that for any $q>0$ and f,g measurable positive functions

$$\int_0^\infty dx x^{1/\alpha-1-\theta}f(x)U_q g(x)=\int_0^\infty dx x^{1/\alpha-1-\theta}g(x)\mathcal{U}_q f(x),$$

with U_q the resolvent of $\widetilde{Y}$ defined in Theorem 1. Indeed, this is an elementary consequence of the identity (7) established in Lemma 2 below. Specifically,

$$\begin{aligned}
&\int_0^\infty y^{1/\alpha-1-\theta} f(y) U_q g(y) dy \\
&= \int_0^\infty y^{1/\alpha-1-\theta} f(y) V_q g(y) dy + U_q g(0) \int_0^\infty y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{V^\natural}_q f(y) dy \\
&\quad + \left(\int_0^\infty x^{1/\alpha-1-\theta} g(x) \widehat{\mathbb{Q}^\natural}_x(e^{-qT_0}) dx\right) \left(b_{\alpha,\theta} q^{-\alpha\theta} \int_0^\infty y^{-(1+\theta)} \widehat{V^\natural}_q f(y) dy\right) \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{V^\natural}_q f(y) dy + \mathcal{U}_q f(0) \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{\mathbb{Q}^\natural}_y(e^{-qT_0}) dy \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \mathcal{U}_q g(y).
\end{aligned}$$

□

Lemma 2. *For every $q > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable*

$$b_{\alpha,\theta} \int_0^\infty y^{-(1+\theta)} \widehat{V^\natural}_q f(y) dy = C_{\alpha,\alpha\theta} \int_0^\infty y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) dy, \tag{7}$$

with $C_{\alpha,\alpha\theta} = (m^\natural a_{\alpha,\theta})^{-1}$

We can prove Lemma 2 either by bare hands calculations or by using the following result proved by Carmona, Petit and Yor [6] Proposition 2.3.

Lemma 3. *The random variable $A_{\boldsymbol{e}}$ has a density $\rho(t) = \boldsymbol{k}\, {\mathbb{Q}^\natural}_1(Y_t^{-(1/\alpha)-\theta})$ for $t > 0$.*

Proof of Lemma 2. Let $W(x) = x^{-1/\alpha-\theta}, x > 0$. Using the fact that under $\mathbb{Q}_y$ the law of T_0 is that of $y^{1/\alpha} A_{\boldsymbol{e}}$ under $\mathbf{Q}'$, the self–similarity and the weak duality between the resolvents $V^\natural$ and $\widehat{V}^\natural$, we get

$$\begin{aligned}
&C_{\alpha,\alpha\theta} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y) \int_0^\infty dt\, {\mathbb{Q}^\natural}_1(Y_t^{-1/\alpha-\theta}) e^{-qy^{1/\alpha}t} \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y) \int_0^\infty ds y^{-1/\alpha} y^{1/\alpha+\theta}\, {\mathbb{Q}^\natural}_y(Y_s^{-1/\alpha-\theta}) e^{-qs} \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1} f(y) V_q^\natural W(y) \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1} W(y) \widehat{V}_q^\natural f(y).
\end{aligned}$$

The claim follows since $b_{\alpha,\theta}/\boldsymbol{k} = C_{\alpha,\alpha\theta}$, due to identity (6), and $\boldsymbol{k} = \psi(\theta)$. □

Furthermore, Lemma 3 allows us to obtain a tail estimate for the law of T_0.

Lemma 4. *For any $x > 0$,*

$$\lim_{\epsilon \to 0+} \frac{\mathbb{Q}_x(T_0 \le \epsilon)}{\epsilon} = x^{-1/\alpha} \boldsymbol{k}.$$

Proof. First we prove that the limit exists. To this end we note that the function $s \mapsto f_s(\cdot) = \mathbb{Q}^{\natural}_{\cdot}(Y_s^{-(1/\alpha)-\theta}), s > 0$ is an exit law for the semigroup $(P^{\natural}_t, t \geq 0)$, i.e. for every $s > 0, t \geq 0$, $P^{\natural}_t f_s(x) = f_{t+s}(x), x > 0$. Thus the function $C_t(\cdot) = \int_0^t f_s(\cdot)ds$ is, in the terminology of potential theory, an "additive process"

$$C_{t+s}(\cdot) = C_t(\cdot) + P^{\natural}_t C_s(\cdot), \quad t, s \geq 0.$$

An ergodic local theorem due to Feyel [14], ensures that the limit $\lim_{t\to 0} C_t/t$, exists. In particular, the following limit exists

$$\lim_{\epsilon\to 0+} \frac{\mathbb{Q}_1(T_0 \leq \epsilon)}{\epsilon} = \lim_{\epsilon\to 0+} \frac{\mathbf{Q}'(A_{\boldsymbol{e}} \leq \epsilon)}{\epsilon} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^{\epsilon} \boldsymbol{k}\, \mathbb{Q}^{\natural}{}_1(Y_s^{-(1/\alpha)-\theta})ds := a.$$

Using the self–similarity we have that under $\mathbb{Q}_x$ the law of T_0 is the same as that of $x^{1/\alpha}T_0$ under $\mathbb{Q}_1$; thus

$$\lim_{\epsilon\to 0+} \frac{\mathbb{Q}_x(T_0 \leq \epsilon)}{\epsilon} = x^{-1/\alpha}a.$$

We next prove that $a = \boldsymbol{k}$. On the one hand, we use Fatou's lemma twice to see that $a \geq \boldsymbol{k}$,

$$\begin{aligned} a &= \liminf_{\epsilon\to 0} \frac{\boldsymbol{k}}{\epsilon}\int_0^{\epsilon} \mathbb{Q}^{\natural}{}_1(Y_s^{-(1/\alpha)-\theta})ds \\ &\geq \boldsymbol{k}\int_0^1 \liminf_{\epsilon\to 0} \mathbb{Q}^{\natural}{}_1(Y_{u\epsilon}^{-(1/\alpha)-\theta})du \\ &\geq \boldsymbol{k}\int_0^1 \mathbb{Q}^{\natural}{}_1(\liminf_{\epsilon\to 0}(Y_{u\epsilon}^{-(1/\alpha)-\theta}))du = \boldsymbol{k}. \end{aligned}$$

On the other hand, Theorem 1 and Fatou's lemma imply that

$$\begin{aligned} 1 &= \liminf_{\epsilon\to 0} \frac{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)}{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)} \\ &= \liminf_{\epsilon\to 0} \frac{\epsilon}{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)}\, \overline{\mathbf{n}}(\epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon), 1 < T_0) \\ &= (cst) \liminf_{\epsilon\to 0+} \mathbb{Q}^{\natural}{}_{0+}(\epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon)Y_1^{-\theta}) \\ &\geq (cst)\, \mathbb{Q}^{\natural}{}_{0+}(\liminf_{\epsilon\to 0+} \epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon)Y_1^{-\theta})) \\ &\geq (cst)a\, \mathbb{Q}^{\natural}{}_{0+}(Y_1^{-(1+\alpha\theta)/\alpha)}), \end{aligned}$$

where $cst = (\Gamma(1-\alpha\theta)/(\alpha\theta a_{\alpha,\theta}))$ and $a_{\alpha,\theta}$ is defined in Theorem 1. The rightmost hand term in the last inequality is equal to $(a/\boldsymbol{k})$, which proves $\boldsymbol{k} \geq a$. To see this we recall that $\mathbb{Q}^{\natural}{}_{0+}(Y_1^{-(1+\alpha\theta)/\alpha)}) = \frac{\alpha}{m^{\natural}}\, \mathbf{Q}^{\natural}(J^{\alpha\theta})$ by identity (5) and using (6) we get

$$(cst)\frac{\alpha}{m^{\natural}}\, \mathbf{Q}^{\natural}(J^{\alpha\theta}) = \frac{\mathbf{Q}^{\natural}(J^{\alpha\theta})}{\alpha\theta\, \mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})} = 1/\boldsymbol{k}.$$

□

Remark 4. It is interesting to observe that the preceding tail estimate is equivalent to

$$\lim_{\epsilon\to 0+} \frac{\mathbf{Q}'(A_{\boldsymbol{e}} \leq \epsilon)}{\mathbf{Q}'(\boldsymbol{e} \leq \epsilon)} = 1.$$

This a natural fact if the Lévy process ξ' does not drift to $-\infty$, $\limsup_{t\to\infty} \xi'_t = \infty$ $\mathbf{Q}$–p.s., since in this case $A_\infty = \infty$, $\mathbf{Q}'$–a.s. and therefore the small values of $A_{\boldsymbol{e}}$ should depend just on those of $\boldsymbol{e}$. Whereas, if ξ' drifts to $-\infty$ then $A_\infty < \infty$, $\mathbf{Q}'$–a.s. and it is easily deduced from Lemma 4 that $\mathbf{Q}'(A_\infty \leq \epsilon) = o(\epsilon)$.

3.1 Proof of Corollary 2

Getoor & Sharpe [16] Theorem 4.8 proved an analogous result for any Markov processes X and $\widehat{X}$ which are in duality, whose semi–groups have dual densities w.r.t. an invariant measure ζ and such that X leaves and hits continuously a recurrent regular state b. The proof of Getoor and Sharpe's result relies mainly on the fact (which they prove) that the excursion measure $\widehat{n}$ is the image under time reversal of n, with n and $\widehat{n}$ the excursion measures of X and $\widehat{X}$ from b, respectively. This relation between $\widehat{n}$ and n was proved by Mitro [23] assuming only that X and $\widehat{X}$ are weak duals and that the excursions from b start and end continuously. It follows that Theorem 4.8 in [16] is still true under these weaker hypotheses. Next, Kaspi [19] § 4 mentions that his results provide a tool to prove this result in a greater generality, namely when X does not enter or leave b continuously. However, for the sake of completeness we provide a sketch of the proof of Corollary 2.

First, we observe that versions of the processes Z_θ and $\widetilde{Y}$ can be constructed simultaneously using the same P.P.P. of excursions. More precisely, take a Poisson point process $\Delta = (\Delta_s, s \geq 0)$ with values in $\mathbb{D}^+$ and characteristic measure $\overline{\mathbf{n}}$. Thus each atom is a path and $T_0(\Delta_s)$ denotes its lifetime. We set $\sigma_t = \sum_{s\leq t} T_0(\Delta_s)$, for $t > 0$. This defines a subordinator with Laplace exponent $\phi(\lambda) = \overline{\mathbf{n}}(1 - e^{-\lambda T_0}), \lambda > 0$. Let L_t be the inverse of σ. On the one hand, the process $\widetilde{Y}$ is constructed, following [4], using this P.P.P. as we did in Chapter III.2. On the other hand, define a process $\underset{\leftarrow}{Y}$ as follows. For $t \geq 0$, let $s = L_t$, thus $\sigma_{s-} \leq t \leq \sigma_s$, and

$$\underset{\leftarrow}{Y}(t) = \begin{cases} \Delta_s((\sigma_s - t)-) & \text{if} \quad \sigma_{s-} < \sigma_s \\ 0 & \text{if} \quad \sigma_{s-} = \sigma_s \text{ or } s = 0. \end{cases}$$

Lemma 5. *The process $\underset{\leftarrow}{Y}$ is a self–similar recurrent extension of $\widehat{Y}^\natural$ and has the same law as Z_θ.*

Proof. Recall that ϱ is the function that time-reverses the trajectories at their lifetimes. The image under ϱ of Δ, say $\varrho\Delta$, still is a P.P.P. of excursions with characteristic measure $\varrho\,\overline{\mathbf{n}}$. We have that the subordinator $\widehat{\sigma}$ constructed as σ, but this time using $\varrho\Delta$, is equal to σ and

$$\underset{\leftarrow}{Y}(t) = \begin{cases} \varrho\Delta_s(t - \widehat{\sigma}_{s-}) & \text{if} \quad \widehat{\sigma}_{s-} < \widehat{\sigma}_s \\ 0 & \text{if} \quad \widehat{\sigma}_{s-} = \widehat{\sigma}_s \text{ or } s = 0. \end{cases}$$

Since $\varrho\,\overline{\mathbf{n}}$ is an excursion measure compatible with the law of $\widetilde{Y}^\natural$ we have from results in Blumenthal [4] that $\underset{\leftarrow}{Y}$ is the unique self–similar recurrent extension of $\widetilde{Y}^\natural$ whose excursion measure from 0 is $\varrho\,\overline{\mathbf{n}}$. $\square$

Moreover, we have the equality between random sets

$$\{t > 0 : \widetilde{Y}(t)\} = \{t > 0 : \overset{\leftarrow}{Y}(t)\} = \{t > 0 : \underset{\leftarrow}{Y}(t)\},$$

and by construction it is easily seen that the processes $\overset{\leftarrow}{Y}$ and $\underset{\leftarrow}{Y}$ both started at 0, are identical. This ends the proof of Corollary 2.

4 Normalized excursion and meander for $\widetilde{Y}$

Motivated by the description of Itô's excursion measure for Brownian motion using the law of a Bessel(3) bridge, in Section III.4 we obtained a description of the excursion measure $\mathbf{n}$ of Theorem III.1

in terms the law of the excursion process conditioned to have a given length. The purpose of this section is to obtain an analogous result for the excursion measure of Theorem 1.

With the aim of giving a handy description of the excursion measure conditioned by its length, in the following proposition we construct a version of the conditional law $\overline{\mathbf{n}}(\,\cdot\,|T_0 = r)$. For any $r > 0$, define the function $h^{\natural r} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h^{\natural r}(s,x) = \mathbb{Q}^{\natural}{}_x(Y_{r-s}^{-(1/\alpha)-\theta})1_{\{s<r\}}, \quad x > 0, s \geq 0,$$

and $h^{\natural r}(s,0) = 0,\ s \geq 0$. Let b_r be the constant given by

$$b_r := \frac{\alpha^2\theta\,\mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}\boldsymbol{k}} r^{-(1+\alpha\theta)} = \frac{\alpha\theta}{\boldsymbol{k}\Gamma(1-\alpha\theta)} a_{\alpha,\theta} r^{-(1+\alpha\theta)}, \tag{8}$$

with $a_{\alpha,\theta}$ defined in Theorem 1.

Proposition 1. *(i) For any $r > 0$, the function $h^{\natural r}$ is excessive for the semi–group of the space–time Markov process $((t, Y_t^{\natural}), t \geq 0)$.*

(ii) For any $r > 0$, the probability measure $\mathbf{\Lambda}^r$ over $\mathcal{G}_{r-}$ defined by

$$\mathbf{\Lambda}^r(F) = (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(Fh^{\natural r}(t, Y_t)), \qquad F \in \mathcal{G}_t,\ t < r,$$

is such that for every $H \in \mathcal{G}$

$$\overline{\mathbf{n}}(H) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)} \int_0^{\infty} \mathbf{\Lambda}^r(H) \frac{dr}{r^{1+\alpha\theta}}.$$

Proof. The proof of (i) is a straightforward consequence of the Markov property.

For any $r > 0$, let $\overline{\mathbf{\Lambda}}^r$ be the h transform of the space-time process over $Y^{\natural}$ with law $\mathbb{Q}^{\natural}{}_{0+}$, via the excessive function $h^{\natural r}$. Then under $\overline{\mathbf{\Lambda}}^r$ the space process $Y^{\natural}$ is an inhomogeneous Markov process with entrance law

$$\overline{\mathbf{\Lambda}}^r_s(f) = \mathbb{Q}^{\natural}{}_{0+}(f(Y_s)h^{\natural r}(s, Y_s)), \qquad s > 0,$$

and for $s, t \geq 0$ its transition probabilities are given by

$$K^r_{t,t+s}(x, dy) = \frac{P^{\natural}_s(x, dy)h^{\natural r}(t+s, y)}{h^{\natural r}(t, x)}, \quad x > 0, y > 0,$$

where the quotient is taken to be 0 if the denominator is 0. The measure $\overline{\mathbf{\Lambda}}^r$ is a finite measure with total mass

$$\begin{aligned}
\overline{\mathbf{\Lambda}}^r(1) &= \lim_{s\to 0} \overline{\mathbf{\Lambda}}^r_s(1) \\
&= \lim_{s\to 0} \mathbb{Q}^{\natural}{}_{0+}(h^{\natural r}(s, Y_s)) \\
&= \mathbb{Q}^{\natural}{}_{0+}(Y_r^{-(1/\alpha)-\theta}) \\
&= b_r < \infty,
\end{aligned}$$

where the third equality is a consequence of the Markov property and the fourth follows from (5). To finish the proof we just have to prove that the probability measures $\mathbf{\Lambda}^r := (b_r)^{-1}\overline{\mathbf{\Lambda}}^r$ satisfy the identity in (ii) of Proposition 1. To that end it suffices to show the identity for any F_t of the form

$F_t = F \cap \{t < T_0\}, F \in \mathcal{G}_t, t > 0$. Indeed, recall from Theorem 1 that for every positive and $\mathcal{G}_t$–measurable H_t we have

$$\overline{\mathbf{n}}(H_t, t < T_0) = (a_{\alpha,\theta})^{-1}\, \mathbb{Q}^{\natural}{}_{0+}(H_t Y_t^{-\theta}),$$

and the expression for b_1 in (8). Therefore, using Fubini's theorem and that the law of T_0 under $\mathbb{Q}_x$ for $x > 0$ has a density

$$\mathbb{Q}_x(T_0 \in ds)/ds = \boldsymbol{k} x^{\theta}\, \mathbb{Q}^{\natural}{}_x(Y_s^{-(1/\alpha)-\theta}), \qquad x > 0$$

and $\mathbb{Q}_0(T_0 \in ds) = \delta_0(ds)$, we get that

$$\begin{aligned}
\overline{\mathbf{n}}(F \cap \{t < T_0\}) &= \overline{\mathbf{n}}(1_F \int_t^{\infty} \boldsymbol{k} Y_t^{\theta}\, \mathbb{Q}^{\natural}{}_{Y_t}(Y_{r-t}^{-(1/\alpha)-\theta}) dr) \\
&= \boldsymbol{k} \int_0^{\infty} \overline{\mathbf{n}} \left(1_F Y_t^{\theta}\, \mathbb{Q}^{\natural}{}_{Y_t}(Y_{r-t}^{-(1/\alpha)-\theta}) 1_{\{t<r\}}\right) dr \\
&= \frac{\alpha\theta}{\Gamma(1-\alpha\theta)} \int_0^{\infty} \frac{dr}{r^{1+\alpha\theta}} (b_r)^{-1}\, \mathbb{Q}^{\natural}{}_{0+}(1_F h^{\natural r}(t, Y_t)) \\
&= \int_0^{\infty} \frac{dr}{r^{1+\alpha\theta}}\, \boldsymbol{\Lambda}^r(F \cap \{t < T_0\}),
\end{aligned}$$

where the last equality holds due to the fact that $\boldsymbol{\Lambda}^r$ is an h–transform of $\mathbb{Q}^{\natural}{}_{0+}$. □

By an argument similar to that given in the previous proof it is proved that for any $x > 0$, $t > 0$ and positive measurable g,

$$\mathbb{Q}_x(F_t \cap \{t < T_0\} g(T_0)) = \int_0^{\infty} g(r) \boldsymbol{k} x^{\theta}\, \mathbb{Q}^{\natural}{}_x(Y_r^{-(1/\alpha)-\theta}) \frac{\mathbb{Q}^{\natural}{}_x(F_t h^{\natural r}(t, X_t))}{h^{\natural r}(0, x)} dr, \qquad F_t \in \mathcal{G}_t.$$

That is, the h–transform of the spac–time process $((t, Y_t^{\natural}), t \geq 0)$ started at $(0, x)$ via the excessive function $h^{\natural r}$ is a version of the conditional law

$$\mathbb{Q}_x(\cdot \mid T_0 = r).$$

As a consequence, the transition probabilities $K^r_{t,t+s}$ defined in the proof of Proposition 1 are those of Y conditioned to hit 0 at time r.

When the process $Y = X^0$ is a stable process X killed at its first hitting time of the set $]-\infty, 0]$, Chaumont [10] proved that the law of the excursion process conditioned to have a given length is absolutely continuous w.r.t. the law of the stable meander process. An analogous result still holds in our setting. To give a precise statement we next recall the definition of the law of the meander process. For any $r > 0$, the probability measure M^r defined over $\mathbb{D}^+([0, r])$ by

$$M^r(\cdot) := \overline{\mathbf{n}}(\cdot \circ k_r, T_0 > r)/\, \overline{\mathbf{n}}(T_0 > r),$$

with k_r the killing operator at time $r > 0$, is called the law of the meander process. This corresponds to the law of the process $(\widetilde{Y}_{g_t+s}, 0 \leq s \leq t - g_t)$ conditioned by $t - g_t = r$ for some $t > r$ and g_t the last hitting time of 0 before t, $g_t = \sup\{s \leq t : \widetilde{Y}_s = 0\}$, cf. Getoor [15].

We can now state a corollary to Proposition 1 which is the analogue of Theorem 3 in [10]:

Corollary 3. *For any $r > 0$, $t < r$ and $F \in \mathcal{G}_t$ we have that*

$$\boldsymbol{\Lambda}^r(F) = \frac{r\boldsymbol{k}}{\alpha\theta} M^r(F Y_r^{-1/\alpha}).$$

Proof. On the one hand, by the very definition of the law of the meander and Theorem 1 we have that

$$M^r(F) = \frac{r^{\alpha\theta}\Gamma(1-\alpha\theta)}{a_{\alpha,\theta}}\,\mathbb{Q}^{\natural}{}_{0+}(F\ Y_r^{-\theta}).$$

On the other hand, by Proposition 1 and the Markov property we have that

$$\Lambda^r(F) = (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(Fh^{\natural r}(t,Y_t)) = (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(F\ Y_r^{-(1/\alpha)-\theta}).$$

The result follows by identifying the constants. □

The law of the excursion process conditioned by its length Λ^r constructed in Chapter III.4 can be thought of as the law of a bridge for the process with law $\mathbb{E}^{\natural}_{0+}$ because the excursion hits 0 continuously. In fact, it can be proved that for every $t < r$ and $F \in \mathcal{G}_t$,

$$\Lambda^r(F) = \lim_{\epsilon\to 0}\mathbb{E}^{\natural}_{0+}(F|X_r \le \epsilon).$$

The arguments used to prove a such result are similar to those given in [10] Lemme 2 and we omit them. An analogue result does not have meaning for the law $\mathbf{\Lambda}^r$ since the excursions are ended by a jump to 0 a.s. However, the following identity holds for any $r > 0$,

$$\mathbf{\Lambda}^r(\cdot) = \lim_{\epsilon\to 0}\overline{\mathbf{n}}(\cdot|r < T_0 \le r+\epsilon). \tag{9}$$

This can be proved as in [10] or using the tail estimation in Lemma 4. Indeed, using the Markov property and a dominated convergence argument we have that for any $r > 0$, $t < r$ and $F \in \mathcal{G}_t$

$$\begin{aligned}\overline{\mathbf{n}}(F|r < T_0 \le r+\epsilon) &= \frac{\epsilon}{\overline{\mathbf{n}}(r < T_0 \le r+\epsilon)}\,\overline{\mathbf{n}}(F\cap\{r<T_0\}\left[\mathbb{Q}_{Y_r}(T_0\le\epsilon)/\epsilon\right])\\ &\sim (\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)\,\overline{\mathbf{n}}(FY_r^{-1/\alpha}),\end{aligned}$$

as $\epsilon \longrightarrow 0$. By the Markov property and Proposition 1

$$(\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)\,\overline{\mathbf{n}}(FY_r^{-1/\alpha}) = (cst)\,\mathbb{Q}^{\natural}{}_{0+}(Fh^{\natural r}(t,Y_t)) = \mathbf{\Lambda}^r(F),$$

with $cst = (\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)(a_{\alpha,\theta})^{-1} = (b_r)^{-1}$.

Remark 5. The law of the excursion process conditioned by its length can be defined following Chaumont [10] since most of his arguments are easily generalized to any self–similar Markov process. The resulting measure is equal to the law $\mathbf{\Lambda}^r$ constructed here. We omit the details.

5 The process conditioned to hit 0 continuously

For the moment we leave aside hypotheses (HI-b,c) of Section 2 and work instead under hypotheses

(HI-d) there exists $\gamma < 0$ for which $\mathbf{Q}(e^{\gamma\xi_1}1_{\{1<\zeta\}}) = 1$.

(HI-e) $\mathbf{Q}(\xi_1^- e^{\gamma\xi_1}1_{\{1<\zeta\}}) < \infty$.

Under these hypotheses we will prove the existence of a self–similar Markov process $\mathrm{Y}^{\downarrow}$ that can be thought of as Y conditioned to hit 0 continuously.

The hypothesis (HI-d) implies that under $\mathbf{Q}$ the function $h^{\downarrow}(x) = e^{\gamma x}, x \in \mathbb{R}$ is an invariant function for the semigroup of the Lévy process with law $\mathbf{Q}$. Let $\mathbf{Q}^{\downarrow}$ be the h–transform of $\mathbf{Q}$ via the invariant function $h^{\downarrow}$. Under $\mathbf{Q}^{\downarrow}$ the canonical process is still a Lévy process with infinite lifetime that drifts to $-\infty$. Furthermore, by hypothesis (HI-e), we have that $m^{\downarrow} = \mathbf{Q}^{\downarrow}(\xi_1) \in]-\infty, 0[$. We will be interested in the self–similar Markov process $\mathrm{Y}^{\downarrow}$ of law $({\mathbb{Q}^{\downarrow}}_x, x \geq 0)$, which is the Markov process associated to the Lévy process with law $\mathbf{Q}^{\downarrow}$ via Lamperti's transformation. Since the Lévy process $\xi^{\downarrow}$ drifts to $-\infty$ we have that $\mathrm{Y}^{\downarrow}$ hits 0 continuously at some finite time ${\mathbb{Q}^{\downarrow}}_x$ a.s. for every $x > 0$. As a consequence of the following result we will refer to $\mathrm{Y}^{\downarrow}$ as the process Y conditioned to hit 0 continuously.

Proposition 2. *(i) For any $x > 0$, we have that ${\mathbb{Q}^{\downarrow}}_x$ is the unique measure such that for every $\mathcal{G}_t$–stopping time T we have*

$$ {\mathbb{Q}^{\downarrow}}_x(F_T, T < T_0) = x^{-\gamma}\, \mathbb{Q}_x(F_T Y_T^{\gamma}, T < T_0), $$

for every $F_T \in \mathcal{G}_T$.

(ii) For every $x > 0, t > 0$ we have

$$ \lim_{\epsilon \to 0} \mathbb{Q}_x(F_t \cap \{t < T_0\} | Y_{T_0-} \leq \epsilon) = {\mathbb{Q}^{\downarrow}}_x(F_t \cap \{t < T_0\}), \qquad F_t \in \mathcal{G}_t. $$

The proof of (i) in Proposition 2 is an immediate consequence of the fact that $\mathbf{Q}^{\downarrow}$ is an h–transform. To prove (ii) we will need the following Lemma in which we determine the tail distribution of a Lévy process at a given exponential time.

Lemma 6. *Let σ be a Lévy process of law P. Assume that σ is non–arithmetic and that there exists $\vartheta > 0$ for which $1 < E(e^{\vartheta \sigma_1}) < \infty$, and $E(\sigma_1^{+} e^{\vartheta \sigma_1}) < \infty$. Let T_λ be a exponential random variable of parameter $\lambda = \log E(e^{\vartheta \sigma_1})$ and independent of σ. We have that*

$$ \lim_{x \to \infty} e^{\vartheta x} P(\sigma_{T_\lambda} \geq x) = \frac{1 - e^{-\lambda} + \lambda}{\mu^{\natural} \vartheta}, $$

with $\mu^{\natural} = \mathbf{Q}(\sigma_1 e^{\vartheta \sigma_1})$.

Lemma 6 is a consequence of the renewal theorem for real–valued random variables and an application of Cramér's method as explained by Feller [13] §XI.6.

Proof. The following three claims enable us to put Lemma 6 in a context similar to that of [13] XI.6. First, the function $Z(x) = P(\sigma_{T_\lambda} < x)$, satisfies a renewal equation. More precisely, for $z(x) = \int_0^1 dt \lambda e^{-\lambda t} P(\sigma_t < x)$ and $L(dy) = e^{-\lambda} P(\sigma_1 \in dy)$ we have that

$$ Z(x) = z(x) + \int_{-\infty}^{\infty} L(dy) Z(x - y). $$

This is an elementary consequence of the fact that the process $(\sigma'_s = \sigma_{1+s} - \sigma_1, s \geq 0)$ is a Lévy process independent of $(\sigma_r, r \leq 1)$ with the same law as σ. Second, the measure L is a defective law, $L(\mathbb{R}) < 1$, such that

$$ \int_{-\infty}^{\infty} e^{\vartheta y} L(dy) = e^{-\lambda} E(e^{\vartheta \sigma_1}) = 1; \quad \text{and} \quad \int_{-\infty}^{\infty} y e^{\vartheta y} L(dy) < \infty, $$

by hypothesis. Third, the function $z^\natural(x) = e^{\vartheta x}(z(\infty) - z(x))$ is directly Riemann integrable; with

$$z(\infty) = \lim_{x\to\infty} z(x) = \int_0^1 dt\lambda e^{-\lambda t} P(\sigma_t < \infty) = 1 - e^{-\lambda}.$$

The latter follows using the fact that $z^\natural(x) = e^{\vartheta x}\int_0^1 dt\lambda e^{-\lambda t}P(\sigma_t \geq x)$, is the product of an exponential function and a decreasing one and that $z^\natural$ is integrable. To see that $z^\natural$ is integrable, use Fubini's theorem to establish

$$\begin{aligned}\int_{-\infty}^{\infty} z^\natural(x)dx &= \int_0^1 dt\lambda e^{-\lambda t} E\left(\int_{-\infty}^{\infty} dx e^{\vartheta x} 1_{\{\sigma_t \geq x\}}\right)\\ &= \frac{1}{\vartheta}\int_0^1 dt\lambda e^{-\lambda t} E(e^{\vartheta\sigma_t})\\ &= \frac{\lambda}{\vartheta} < \infty.\end{aligned}$$

Therefore, we can repeat the arguments given in the proof of Theorem XI.6.2 in [13] but this time using the renewal theorem for real–valued random variables to prove that

$$\begin{aligned}\lim_{x\to\infty} e^{\vartheta x}P(\sigma_{T_\lambda} \geq x) &= \lim_{x\to\infty} e^{\vartheta x}(Z(\infty) - Z(x))\\ &= \frac{z(\infty)}{\mu^\natural\vartheta} + \frac{\int_{-\infty}^{\infty} z^\natural(x)dx}{\mu^\natural} = \frac{1 - e^{-\lambda} + \lambda}{\mu^\natural\vartheta}.\end{aligned}$$

□

Proof of Proposition 2 (ii). Observe that under $\mathbb{Q}_x$ the random variable Y_{T_0-} has the same law as $x\exp(\xi'_{\boldsymbol{e}})$ under $\mathbf{Q}'$, with $\boldsymbol{e}$ an exponential random variable of parameter $\boldsymbol{k} = \psi(\theta) = \psi(\gamma) > 0$, and independent of ξ'. Moreover, applying Lemma 6 to $-\xi'$ under $\mathbf{Q}'$ we obtain by hypotheses (HI-d) that

$$\lim_{y\to\infty} e^{-\gamma y}\,\mathbf{Q}'(\xi_{\boldsymbol{e}} \leq -y) = \frac{1 - e^{-\boldsymbol{k}} + \boldsymbol{k}}{\gamma\mu^\downarrow} := \mathrm{d}_{\boldsymbol{k}},$$

with $\mu^\downarrow = \mathbf{Q}'(\xi_1 e^{\gamma\xi_1}) \in]-\infty, 0[$, which is finite by hypothesis (HI-e). Thus, we have the following estimate of the left tail distribution of Y_{T_0-}:

$$\lim_{\epsilon\to 0} \epsilon^\gamma\,\mathbb{Q}_x(Y_{T_0-} \leq \epsilon) = x^\gamma \mathrm{d}_{\boldsymbol{k}}. \tag{10}$$

The proof of (ii) in Proposition 2 now follows by a standard application of the Markov property, estimate (10) and a dominated convergence argument. □

In the sequel, we will assume in addition that the hypotheses (HI-b,c) are satisfied. This implies in turn that the hypotheses (H2) of Chapter III are satisfied. Indeed, for $\hat\theta = \theta - \gamma$ we have that

$$\mathbf{Q}^\downarrow(e^{\hat\theta\xi_1}) = \mathbf{Q}(e^{\theta\xi_1}1_{\{1<\zeta\}}) = 1,$$

and

$$\mathbf{Q}^\downarrow(\xi_1^+ e^{\hat\theta\xi_1}) = \mathbf{Q}(\xi_1^+ e^{\theta\xi_1}1_{\{1<\zeta\}}) < \infty,$$

by hypotheses (HI-b) and (HI-c), respectively. By hypothesis (HI-e) we have that

$$\mathbf{Q}^\downarrow(\xi_1) = m^\downarrow \in]-\infty, 0[.$$

Since the measure $\mathbf{Q}^{\downarrow}$ satisfies the hypotheses (H2) of Chapter III we know from Theorem III.1 that if $0 < \alpha(\theta-\gamma) < 1$ then there exists a unique excursion measure $\mathbf{n}^{\downarrow}$ compatible with the semigroup of $(\mathrm{Y}^{\downarrow}, T_0)$, and associated to it a self–similar recurrent extension of $(\mathrm{Y}^{\downarrow}, T_0)$, say $\widetilde{\mathrm{Y}^{\downarrow}}$. The absolutely continuity relations in part (i)of Proposition 2 are inherited by the excursion measures $\overline{\mathbf{n}}$ and $\mathbf{n}^{\downarrow}$. More precisely, for every $\mathcal{G}_t$ stopping time, T, we have that

$$\mathbf{n}^{\downarrow}(F_T, T < T_0) = c_\gamma\, \overline{\mathbf{n}}(F_T Y_T^{\gamma}, T < T_0), \qquad F_T \in \mathcal{G}_T, \tag{11}$$

with $\overline{\mathbf{n}}$ the excursion measure of Theorem 1 and

$$c_\gamma = \frac{\mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)}{\mathbf{Q}^{\natural}(J^{-(1-\alpha\theta+\alpha\gamma)})\Gamma(1-\alpha\theta+\alpha\gamma)}.$$

To see this we just have to note that the measure $\mathbf{Q}^{\downarrow\natural}$ obtained by h-transforming $\mathbf{Q}^{\downarrow}$ via the invariant function $h_{\theta-\gamma}(x) = e^{(\theta-\gamma)x}, x \in \mathbb{R}$, is identical to the measure $\mathbf{Q}^{\natural}$ constructed in Section 2.

Furthermore, it is natural to hope that the conditioning on hitting 0 continuously should act just at the end of the excursions. This let us guess that the meander processes associated to $\widetilde{Y}$ and $\widetilde{\mathrm{Y}^{\downarrow}}$ should be related. This is indeed the case; a standard calculation shows that for every $r > 0$ the meander processes of length r, $(\widetilde{Y}, M^r)$ and $(\widetilde{\mathrm{Y}^{\downarrow}}, M^{\downarrow,r})$ are identical in law conditionally on their values at time r,

$$M^r(\cdot|Y_r = x) = M^{\downarrow,r}(\cdot|Y_r = x), \qquad x > 0,$$

in the obvious notation.

Regardless of the value of $\alpha(\theta-\gamma)$, we can always construct a pseudo–excursion measure $\mathbf{n}^{\downarrow}$ as an h–transform of $\overline{\mathbf{n}}$ via the excessive function $x^{\gamma}, x > 0$, and this pseudo–excursion measure is still compatible with the minimal process $(\mathrm{Y}^{\downarrow}, T_0)$. For us a pseudo–excursion measure has the same properties as an excursion measure except that it is possible that it does not integrate $1 - e^{-T_0}$. The latter holds if and only if $1 \leq \alpha(\theta-\gamma)$.

Remark 6. Observe that another consequence of Lemma 6 is that

$$\lim_{y\to\infty} y^{\theta}\, \mathbb{Q}_1(Y_{T_0-} \geq y) = \frac{1 - e^{-\boldsymbol{k}} + \boldsymbol{k}}{\theta m^{\natural}}.$$

6 Examples

6.1 Further details for stable processes

In the Introduction we noted (X, P) a real valued a–stable process with negative jumps and we assume that X is not the negative of a subordinator. Since X is a Lévy process its law is determined by its characteristic exponent which in turn can be described as

$$E(e^{i\lambda X_1}) = \exp\{-c|\lambda|(1 - i\beta \mathrm{sgn}(\lambda)\tan(a\pi/2))\} \qquad \lambda \in \mathbb{R}, c > 0, \beta \in [-1, 1[$$

(the case $\beta = 1$ is excluded since we assume that X has some negative jumps). The case where X does not have negative jumps enters in the setting considered in Chapter III. The Lévy measure of X has the form

$$\mathbf{\Pi}(dx) = C_+ x^{-1-a} 1_{\{x>0\}} + C_- |x|^{-1-a} 1_{\{x<0\}} dx,$$

for some constants $C_+, C_- \geq 0$ such that $\beta = C_+ - C_-/C_+ + C_-$. In a recent work, Caballero and Chaumont [5] determined explicitly the characteristics of the Lévy process $(\xi, \mathbf{Q})$ associated via Lamperti's transformation to the positive $1/a$–self–similar Markov process (X^0, P^0). The process $(\xi, \mathbf{Q})$ is a Lévy process whose characteristic exponent is given by

$$\Psi(\lambda) = \boldsymbol{k} + id\lambda + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda x} - 1 - ix\lambda 1_{\{|x|<1\}})\Pi(dx), \lambda \in \mathbb{R},$$

where $\boldsymbol{k} = \lim_{s\to 0} s^{-1} P(T_{]-\infty,0[} \leq s)$ (this limit was calculated by Bingham [3]), $d \in \mathbb{R}$ is a drift coefficient whose value is not important for us here and

$$\Pi(dx) = C_+(e^x(e^x - 1)^{-1-a})1_{\{x>0\}} + C_-(e^x|e^x - 1|^{-1-a})1_{\{x<0\}}dx,$$

see [5] for the details. We have to verify that this Lévy process satisfies the conditions (HI) in order to apply our results to stable processes. Recall that to pass from the process X^0 to the process ξ we have to make the transformation

$$\xi_t = \log(X_{\varphi^{-1}(t)}), \quad \text{with } \varphi^{-1}(t) \text{ the inverse of } \varphi(t) = \int_0^t (X_s^0)^{-a} ds, \quad t < T_0.$$

Indeed, ξ is not arithmetic since the stable process is not. To verify that (HI-b) holds, we recall that the function $h_\rho(x) = x^{a(1-\rho)} x \geq 0$ is invariant for (X^0, P^0), see Silverstein [25] or Chaumont [10]. Since the measure $P^\natural$ is the h–transform of P^0 via the invariant function h_ρ we have that for every stopping time T in the filtration of X^0 we have

$$P_x^\natural(T < \infty) = x^{-a(1-\rho)} P_x^0(X_T^{a(1-\rho)} 1_{\{T<T_0\}}).$$

In particular, for $T = \varphi^{-1}(t)$ with $t > 0$, which is a stopping time for X^0, we have

$$P_x^\natural(1_{\{\varphi^{-1}(t)<\infty\}}) = x^{-a(1-\rho)} P_x^0(X_{\varphi^{-1}(t)}^{a(1-\rho)} 1_{\{\varphi^{-1}(t)<T_0\}}) = \mathbf{Q}(e^{a(1-\rho)\xi_t} 1_{\{t<\zeta\}}) = \mathbf{Q}(e^{a(1-\rho)\xi_t}),$$

and the leftmost term is equal to 1 since Lamperti [21] Lemma 3.1 proved that whenever the self–similar Markov process never hits 0 we have $\varphi(\infty) = \infty$ a.s. independently of the starting point, which is indeed the case under $P^\natural$. According to Sato [24] Theorem 25.3, the condition (HI-c) is equivalent to

$$\int_{\{x>1\}} xe^{a(1-\rho)x} \frac{e^x}{(e^x - 1)^{1+a}} dx < \infty,$$

and that the latter holds is straightforward. We have thus proved that the conditions (HI) are satisfied by the Lévy process associated to a stable process killed at $]-\infty, 0[$ with $\theta = a(1-\rho)$, and since the self–similarity index is $\alpha = 1/a$ we have that $0 < \alpha\theta = 1 - \rho < 1$. In this particular case most of the results in Section 2 are well known, see [10]. The recurrent extension of X^0 is exactly the process X reflected at its infimum $(X - \underline{X}, P)$, since it is a strong Markov process that leaves 0 continuously and its excursion measure n is the unique excursion measure compatible with the law P^0 such that $n(X_{0+}^0 > 0) = 0$ and $n(1 - e^{-T_0}) < \infty$.

We will denote by (X^*, P^*) the dual stable process $(X^*, P^*) = (-X, P)$, by $(X^{*,0}, P^{*,0})$ the dual stable process killed at $]-\infty, 0[$ and by $\overline{X}_t^* = \sup_{s\leq t} X^*, t \geq 0$. One can construct the dual stable process conditioned to stay positive $(X^{*,\uparrow}, P^{*,\uparrow})$ analytically, as an h–transform of $P^{*,0}$ via the invariant function $x^{a\rho}, x \geq 0$, or pathwise, by using Tanaka's method [26]; that is $X^{*,\uparrow}$ is obtained by time–reversing one by one the excursions from 0 of the process X reflected at its supremum $(\overline{X}^* - X^*, P^*)$.

For details on the latter construction see the recent work of Doney [12]. From Doney's construction it is easily deduced that the process

$$R_t = \begin{cases} (\overline{X}^* - X^*)_{(d_t-(t-g_t))-} & \text{if } 0 < g_t \leq d_t < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $g_t = \sup\{s < t : (\overline{X}^* - X^*)_s = 0\}$ and $d_t = \inf\{s > t : (\overline{X}^* - X^*_s) = 0\}$, has the same distribution under P^*_0 as the process $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ under $P^{*,\uparrow}_{0+}$, where $\underline{\underline{X}}^{*,\uparrow}_t = \inf\{X^{*,\uparrow}_s, s \geq t\}$, and $P^{*,\uparrow}_{0+}$ is the limit in the Skorohod sense of $P^{*,\uparrow}_x$ as $x \to 0+$, see [9] Theorem 6. It follows that under P^*_0 the Poisson point process of excursions from 0 of R has the same law as that under $P^{*,\uparrow}_{0+}$ of $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$. Furthermore the former is the image under ϱ of the P.P.P. of excursions of $\overline{X}^* - X^*$ under P^*_0. Therefore, if $\underline{\underline{n}}$ is the excursion measure of $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$, we have that the image under time reversal of n is $\underline{\underline{n}}$. We borrow the following lemma from Chaumont [9] Theorem 5.

Lemma (Chaumont [9]). *Let $m = \sup\{t > 0 : X^{*,\uparrow}_t = \inf_{s \leq t} X^{*,\uparrow}_s\}$, and $X^{*,\uparrow}_m$ the absolute minimum. Under $P^{*,\uparrow}_x$, $x > 0$, the process $X^{*,\uparrow}$ reaches $X^{*,\uparrow}_m$ once only and the processes $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_0, 0 \leq s \leq m)$ and $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, m < s)$ are independent. Under $P^{*,\uparrow}_x$, conditionally on $\underline{X}^{*,\uparrow}_m = y$, $0 < y \leq x$, the law of the former is $P^{*,\downarrow}_{x-y}$ and the latter has the same law as $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > 0)$ under $P^{*,\uparrow}_{0+}$.*

Moreover, under $\underline{\underline{n}}$ the excursion process is Markovian with semigroup

$$p^{*,\downarrow}_t(x, dy) = \frac{p^{*,0}(x, dy) y^{a\rho-1}}{x^{a\rho-1}},$$

that is, the h–transform of $(X^{0,*}, P^{*,0})$ via the excessive function $h'_{1-\rho}(x) = x^{a\rho-1}, x \geq 0$. We denote by $P^{*,\downarrow}$ the law of this h–transform.

The law $P^{*,\downarrow}$ is that of a self–similar Markov process that hits 0 continuously and then dies at 0. Thus, associated to $\underline{\underline{n}}$ and $P^{*,\downarrow}$ there is a self–similar Markov process Z that is a recurrent extension of $(X^{*,\downarrow}, T^{*,\downarrow}_0)$; this is, indeed, the process $Z_{a(1-\rho)}$ of Theorem 2 (iii). We denote its law by $\widetilde{\mathbb{Q}}^{\wedge}_{\cdot}$. We claim that under $P^{*,\uparrow}_x, x > 0$, conditionally on $X^{*,\uparrow}_m = y$, $0 < y \leq x$, the process $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ has the same law as Z started at $x - y$. By a monotone class argument, to see this it suffices to prove that for all bounded measurable functionals F, G and all bounded measurable functions g we have that

$$\begin{aligned} &P^{*,\uparrow}_x(g(X_m)F(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s \leq m)G(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > m)) \\ &= P^{*,\uparrow}_x(g(X_m)\, \widetilde{\mathbb{Q}}^{\wedge}_{x-X_m}(F(Z_s, s \leq T_0)G(Z_s, s > T_0))). \end{aligned}$$

Indeed, due to the preceding lemma, the left-hand side of the above equation is equal to

$$P^{*,\uparrow}_x(g(X_m)P^{*,\downarrow}_{x-X_m}(F(X_s, s \leq T_0))P^{*,\uparrow}_{0+}(G(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > 0))),$$

and, by the Markov property applied at time T_0, the right–hand side is equal to

$$P^{*,\uparrow}_x(g(X_m)\, \widetilde{\mathbb{Q}}^{\wedge}_{x-X_m}(F(Z_s, s \leq T_0))\, \widetilde{\mathbb{Q}}^{\wedge}_0(G(Z_s, s > 0))),$$

Using the fact that Z is a recurrent extension of $X^{*,\downarrow}$ we have that for any $z > 0$

$$\widetilde{\mathbb{Q}}^{\wedge}_z(F(Z_s, s \leq T_0)) = P^{*,\downarrow}_z(F(X_s, s \leq T_0)),$$

and, given that the processes Z and $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ can be recovered from their respective Poisson point processes of excursions from 0, and that these have the same law since they have the same excursion measure, we get that

$$\widetilde{\mathbb{Q}}_0^{\wedge}(G(Z_s, s > 0)) = P_{0+}^{*,\uparrow}(G(X_s^{*,\uparrow} - \underline{\underline{X}}_s^{*,\uparrow}, s > 0)),$$

and the claim follows.

Another consequence of the results of Silverstein [25] is that the function $h_\rho^{'}(x) = x^{a(1-\rho)-1}, x > 0$ is excessive for the semigroup of (X^0, P^0). Then the process $(X^\downarrow, P^\downarrow)$ which is the h–transform of (X^0, P^0) via the function $h_\rho^{'}$ is a self–similar Markov process that hits 0 continuously, see Chaumont [10]. Thus according to Lamperti [21], the Lévy process $(\xi^\downarrow, \mathbb{Q}^\downarrow)$ associated to $(X^\downarrow, P^\downarrow)$ is a Lévy process with infinite lifetime and that drifts to $-\infty$, namely it is the Lévy process $(\xi, \mathbf{Q})$ conditioned to drift to $-\infty$. To see this, we claim that the function $e^{(a(1-\rho)-1)x}, x \in \mathbb{R}$ is invariant for $(\xi, \mathbf{Q})$. Indeed, by properties of h-transformations we have for the stopping time $\varphi^{-1}(1)$ that

$$\begin{aligned} P_x^\downarrow(\varphi^{-1}(1) < T_0) &= P_x^0\left(X^{a(1-\rho)-1}_{\varphi^{-1}(1)}, \varphi^{-1}(1) < T_0\right)/h_\rho^{'}(x) \\ &= \mathbf{Q}(\exp\{(a(1-\rho)-1)\xi_1\}, 1 < \zeta) \\ &= \mathbf{Q}(\exp\{(a(1-\rho)-1)\xi_1\}). \end{aligned}$$

The leftmost term in the preceding equality is equal to $P_x^\downarrow(1 < \varphi(T_0)) = 1$ since $\varphi(T_0) = \infty$ $P_x^\downarrow$–a.s. for any $x > 0$, see [21] Lemma 3.3. Therefore, the law

$$\mathbf{Q}^\downarrow|_{D_t} = e^{(a(1-\rho)-1)\xi_t}\,\mathbf{Q}|_{D_t}, \qquad t \geq 0,$$

is that of a Lévy process with infinite lifetime. We also have that $\mathbf{Q}^\downarrow(e^{\xi_1}) = 1$, since $1 = a(1-\rho) - (a(1-\rho)-1)$, and as a consequence under $\mathbf{Q}^\downarrow$ the Lévy process $\xi^\downarrow$ drifts to $-\infty$. By arguments similar to those given in Section 2 we verify that the self–similar Markov process associated to $(\xi^\downarrow, P^\downarrow)$ is equivalent to $(X^\downarrow, P^\downarrow)$. Observe that, in general, $a(1-\rho)-1 < 0$ and thus $\gamma = a(1-\rho)-1$ is the only candidate to satisfy the hypotheses (HI–d,e) under $\mathbf{Q}$. We have already verified (HI–d) and using an argument similar to that used to verify that (HI-c) holds we get that (HI-e) holds. In this case the measure $\mathbf{n}^\downarrow$ constructed in Section 5 is equal to the one constructed by Chaumont [8] section 2.4 and plays an important rôle in obtaining pathwise transformations.

6.2 On the excursions that leave 0 by a jump and hit 0 continuously

Let $\mathbb{P}_x, x \geq 0$, be the law of a self–similar Markov process X such that under $\mathbb{P}_x$, X hits 0 continuously in a finite time:

$$\mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0) = 1 \qquad \text{for all } x > 0,$$

and that 0 is a cemetery point. Assume that the Lévy process associated to X via Lamperti's transformation satisfies the hypothesis (H2) in Chapter III. Then in Chapter III we proved that the recurrent extension of (X, T_0) that leaves and hits 0 continuously admits a weak dual whose excursion measure is the image under time reversal of $\mathbf{n}$. A similar result can be established for the recurrent extensions that leave 0 by a jump. In order to give a precise statement we next recall and introduce some notation.

We will use the notation of Chapter III. We denote by $\mathbf{P}$ the law of the Lévy process ξ associated to X. We assume henceforth that $\mathbf{E}(\xi_1^-) < \infty$ and that the law $\mathbf{P}$ satisfies the hypotheses (H2) in Chapter III. We denote θ the Cramér exponent of $\mathbf{P}$ and by $\mathbf{P}^\natural$ the h–transform of $\mathbf{P}$ via the invariant function $h(x) = e^{\theta x}, x \in \mathbb{R}$. Let $\widehat{\mathbf{P}^\natural}$ be the law of $\widehat{\xi^\natural} = -\xi^\natural$ under $\mathbf{P}^\natural$. The probability measures

$\mathbb{P}_x, \widehat{\mathbb{P}}_x, \mathbb{P}^{\natural}_x$ and $\widehat{\mathbb{P}^{\natural}_x}$ are the laws of the self–similar Markov processes associated to the Lévy processes with laws $\mathbf{P}, \widehat{\mathbf{P}}, \mathbf{P}^{\natural}$ and $\widehat{\mathbf{P}^{\natural}}$ respectively.

By the hypotheses (H2) it follows that the measure $\widehat{\mathbf{P}^{\natural}}$ has some finite exponential moments; in fact

$$e^{\widehat{\psi^{\natural}}(\beta)} := \widehat{\mathbf{P}^{\natural}}(e^{\beta\xi_1}) \leq 1, \qquad \beta \in [0, \theta],$$

where the inequality is an equality only for $\beta = 0, \theta$. This implies that for any $\beta \in]0, \theta[$ the function $h_\beta(x) = e^{\beta x}, x \in \mathbb{R}$, is excessive for the semi–group of the process $\widehat{\xi^{\natural}}$. Thus the h–transform ${}^{\beta}\mathbf{Q}$, of $\widehat{\mathbf{P}^{\natural}}$ via the excessive function h_β is a probability measure over the space of càdlàg trajectories with a finite lifetime. Under ${}^{\beta}\mathbf{Q}$ the canonical process is a Lévy process with finite lifetime since ${}^{\beta}\mathbf{Q}(t < \zeta) = e^{t\widehat{\psi^{\natural}}(\beta)},\ t > 0$ and, conditionally on $\{t < \zeta\}$, the increment $\xi_{t+s} - \xi_t$ is independent of $(\xi_r, r \leq t)$ and has the same law as ξ_s under ${}^{\beta}\mathbf{Q}$. Furthermore, we have constructed the measure ${}^{\beta}\mathbf{Q}$ in such way that it satisfies the hypotheses (HI). Indeed, under ${}^{\beta}\mathbf{Q}$ the canonical process is not an arithmetic process since by hypothesis it is not under $\mathbf{P}$. For $\theta_\beta = \theta - \beta$ we have that

$${}^{\beta}\mathbf{Q}(e^{\theta_\beta \xi_1}) = \widehat{\mathbf{P}^{\natural}}(e^{\theta\xi_1}) = 1,$$

and

$${}^{\beta}\mathbf{Q}(\xi_1^+ e^{\theta_\beta \xi_1}) = \mathbf{E}(\xi_1^-) < \infty.$$

Let ${}^{\beta}\mathbb{Q}_x$ be the law of the α–self–similar Markov process $Y_\beta = (Y_{\beta,t}, t \geq 0)$ associated to the Lévy process with law ${}^{\beta}\mathbf{Q}$ via Lamperti's transformation. By Theorem 1 the process (Y_β, T_0) admits a unique self–similar recurrent extension $\widetilde{Y}_\beta = (\widetilde{Y}_{\beta,t}, \geq 0)$ that leaves 0 continuously. We denote ${}^{\beta}\overline{\mathbf{n}}$ the associated excursion measure.

By the results in Section III.3 we know that there exists a unique self–similar recurrent extension $X_\beta = (X_{\beta,t}, t \geq 0)$ of (X, T_0) that leaves 0 by a jump according to the jumping-in measure

$$\nu_{\theta-\beta}(dx) = d_{\alpha,\theta-\beta} x^{-(1+\theta-\beta)} dx, x > 0,$$

with $d_{\alpha,\theta-\beta} = (\theta - \beta)/\mathbf{E}(I^{\alpha(\theta-\beta)})\Gamma(1 - \alpha(\theta - \beta))$.

We now have all the elements required to establish the main result of this section, which is a corollary to Theorems 1 & 2.

Proposition 3. *Let $\beta \in]0, \theta[$.*

(i) For any $x > 0$ and T stopping time for the filtration $(\mathcal{G}_t, t \geq 0)$ we have that

$${}^{\beta}\mathbb{Q}_x(F_T, T < T_0) = x^{-\beta} \widehat{\mathbb{P}^{\natural}}_x(F_T X_T^{\beta}, T < T_0), \qquad F_T \in \mathcal{G}_T.$$

(ii) The process X_β is in weak duality with the process $\widetilde{Y}_\beta$ w.r.t. $x^{1/\alpha-1-\theta+\beta}dx, x > 0$.

(iii) The image under time reversal of the excursion measure ${}^{\beta}\overline{\mathbf{n}}$ is given by

$$\rho({}^{\beta}\overline{\mathbf{n}}(\cdot)) = d_{\alpha,\theta-\beta} \int_0^\infty x^{-1-\theta+\beta} \mathbb{E}_x(\cdot) dx.$$

(iv) The excursion measure ${}^{\beta}\overline{\mathbf{n}}$ is such that for every $t > 0$

$${}^{\beta}\overline{\mathbf{n}}(F_t, t < T_0) = (c_\beta)\widehat{\mathbf{n}}(F_t X_t^{\beta}, t < T_0),$$

with $\widehat{\mathbf{n}}$ the unique normalized excursion measure compatible with the self–similar Markov process $(\widehat{X}^{\natural}, T_0)$ such that $\widehat{\mathbf{n}}(X_{0+} > 0) = 0$, and c_β a normalizing constant.

Proof. The proof of (i) is a straightforward consequence of the fact that the measure ${}^{\beta}\mathbf{Q}$ is an h-transform of the measure $\widehat{\mathbf{P}^{\natural}}$. The statements in (ii) and (iii) are consequences of the following claim: the measure $\mathbf{P}$ is equal to the measure $\widehat{{}^{\beta}\mathbf{Q}^{\natural}}$. To see this recall that the former is the dual of the measure ${}^{\beta}\mathbf{Q}^{\natural}$ which is in turn the h–transform of ${}^{\beta}\mathbf{Q}$ via the invariant function $h_{\theta-\beta} = e^{(\theta-\beta)x}, x \in \mathbb{R}$. Since under the measures $\mathbf{P}$ and $\widehat{{}^{\beta}\mathbf{Q}^{\natural}}$ the canonical process is a Lévy process with infinite lifetime, all that we have to do to prove the claimed fact is to verify that both have the same 1-dimensional marginals. This is proved in the following sequence of equalities: for every $t > 0, \lambda \in \mathbb{R}$,

$$\widehat{{}^{\beta}\mathbf{Q}^{\natural}}(e^{i\lambda\xi_t}) = {}^{\beta}\mathbf{Q}^{\natural}(e^{-i\lambda\xi_t}) = {}^{\beta}\mathbf{Q}(e^{i\lambda\xi_t}e^{(\theta-\beta)\xi_t}, t < \zeta) = \widehat{\mathbf{P}^{\natural}}(e^{-i\lambda\xi_t}e^{\theta\xi_t}) = \mathbf{P}(e^{i\lambda\xi_t}).$$

Therefore, the laws $\mathbb{P}_x$ and $\widehat{{}^{\beta}\mathbb{Q}^{\natural}}_x$ are equal for all $x > 0$, and the self–similar recurrent extension X_β is equal to the process $Z_{\theta-\beta}$ in Theorem 2 (iii). The statement in (ii) and (iii) follows from Theorem 2.

To prove (iv) recall from Theorem 1 that for every $t > 0$

$${}^{\beta}\overline{\mathbf{n}}(A_t, t < T_0) = (a_{\alpha,\theta_\beta})^{-1}\, {}^{\beta}\mathbb{Q}^{\natural}{}_{0+}(A_t Y_t^{-\theta+\beta}), \quad A_t \in \mathcal{G}_t,$$

with $a_{\alpha,\theta_\beta} = \alpha\, \mathbf{Q}^{\natural}(J^{-(1-(\theta-\beta))})\Gamma(1-\alpha\theta+\alpha\beta)/m^{\natural}$, $m^{\natural} = \mathbf{Q}^{\natural}(\xi_1)$. On the other hand, since the measure ${}^{\beta}\mathbf{Q}^{\natural}$ is equal to $\widehat{\mathbf{P}}$ we have that ${}^{\beta}\mathbb{Q}^{\natural}{}_x$ is equal to $\widehat{\mathbb{P}}_x$ for all $x > 0$. Which implies ${}^{\beta}\mathbb{Q}^{\natural}{}_{0+} = \widehat{\mathbb{P}}_{0+}$ over $\mathcal{G}$. The result is then obtained using the fact that the excursion measure $\widehat{\mathbf{n}}$ is such that for every $t > 0$

$$\widehat{\mathbf{n}}(A_t, t < T_0) = (\widehat{a}_{\alpha,\theta})^{-1}\widehat{\mathbb{E}}_{0+}(A_t, X_t^{-\theta}), \qquad A_t \in \mathcal{G}_t,$$

with $\widehat{a}_{\alpha,\theta} = \alpha\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m$, $m = -\mathbf{E}(\xi_1)$. The constant c_β is determined by $c_\beta = \widehat{a}_{\alpha,\theta}/a_{\alpha,\theta_\beta}$. □

As a final comment, observe that the measure $\mathbf{Q}^{\downarrow}$ of section 5 satisfies the assumptions (H2) of Chapter III and we can therefore apply the construction and results obtained in that section to study the self–similar Markov process $(X^{\downarrow}, \mathbb{Q}^{\downarrow})$ associated to $\mathbb{Q}^{\downarrow}$. In particular, in the a-stable process setting for $a \in]1, 2[$, if $\mathbb{Q}^{\downarrow}_{\cdot}$ is the law of the process (X^0, P^0) conditioned to hit 0 continuously we have that the hypotheses (H2) are satisfied for $\theta = 1$, Then, for $\beta = 1 - a\rho$, we have that the process X_β corresponds to the stable process conditioned to stay positive and reflected at its future infimum under the law P and the process Y_β corresponds to the stable process reflected as its infimum under P^*. The latter is equal to the stable process reflected at its supremum under the law P. We leave the details to the interested reader. The restriction $a \in]1, 2[$ is just used to ensure that $0 < (1/a)\theta = 1/a < 1$ and thus the existence of the excursion measure $\widehat{\mathbf{n}}$ in (iv) in Proposition 3. The same result holds without the condition $a \in]1, 2[$, but in (iv) we will have a pseudo excursion measure.

6.3 The case where the process Y has increasing paths

Assume that the Lévy process ξ' of section 2 has increasing paths, that is ξ' is a subordinator. It is well known that the law of a subordinator has negative exponential moments:

$$]-\infty, 0[\subseteq C := \{\lambda \in \mathbb{R} : \mathbf{Q}'(e^{\lambda\xi_1}) < \infty\}.$$

In this case, the Laplace exponent ψ of ξ' is given by

$$\psi(\lambda) = d\lambda + \int_0^\infty (e^{\lambda x} - 1)\Pi(dx).$$

We assume that there is a $\theta \in \widehat{C} \cap]0,\infty[\neq \emptyset$ such that $\mathbf{Q}'(\xi_1 e^{\theta \xi_1}) < \infty$ and let $\mathbf{Q}$ be the law of the subordinator ξ' killed at rate $\boldsymbol{k} = \psi(\theta)$. Observe that instead of taking ∞ as cemetery point for the subordinator as usually, we are taking a point Δ such that $e^{\Delta} = 0$. Therefore, the α–self–similar Markov process Y associated to ξ is a process with a.s. increasing paths that suddenly jumps to 0 at some finite time and then dies. By construction, the law $\mathbf{Q}$ satisfies the hypotheses (HI) for $\widehat{\theta}$ and therefore we can construct a self–similar recurrent extension $\widetilde{Y}$ of Y that leaves 0 continuously a.s. By time reversal the dual process $\widehat{Y}^{\natural}$ is the self–similar Markov process associated to the negative of a subordinator whose Laplace exponent is easily derived from ψ. The recurrent extension of the self–similar Markov process $\widehat{Y}^{\natural}$ is that constructed in Example III.1 and is in weak duality with $\widetilde{Y}$.

Observe that in this case the process $\xi^{\natural}$ is a subordinator but it is not equal to ξ'. In general, even if the Lévy process ξ' drifts to ∞, the process $\xi^{\natural}$ is not equal to ξ'.

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