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Balanced Contributions Axiom in the Solution of Cooperative Games

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In a game v in characteristic function form, suppose the Banzhaf value ψ is used to pay a coalition S already formed. Then coalition S no longer receives $v(S)$; instead it receives $R^{\psi}(S) = \Sigma_{i \in S} \psi_i(v_s)$, where v_S denotes the subgame of coalition S. Surprisingly, the Shapley value of this new game $\mathrm{Sh}(N, R^{\psi})$ is equal to the Banzhaf value of v. In this paper we establish a similar result for all values satisfying balanced contributions axiom. Additionally, we introduce player's weights to obtain the corresponding result in the nonsymmetric case. *Journal of Economic Literature* Classification Number: C71 © 1997 Academic Press

By a *game* in characteristic form, in what follows just a game, we mean a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of players and v is a function $v : 2^N \to \mathbb{R}$ with the property that $v(\varnothing) = 0$ ($\mathbb{R}$ denotes the set of real numbers and 2^N the set of subsets of N). We usually refer to subsets S of N as *coalitions* and to the number $v(S)$ as the *worth* of S. Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance $s = |S|$, $t = |T|$, and so on.

Given a game (N, v) and a coalition $S \subseteq N$, we write (S, v_S) for a subgame obtained by restricting v to the subsets of S; that is, the domain of the function v is restricted to 2^S.

We denote by G^N the set of all games with set of players N as their first coordinate. By a *solution* φ we mean a family of functions

$$\varphi = \{\varphi^N : G^N \to \mathbb{R}^N | N \text{ is a finite set of players, one function for each } N\}.$$

In order to simplify the notation we will say that $\varphi^N(N, v) = (\varphi_i^N(N, v))_{i \in N} \in \mathbb{R}^N$ is the solution associated to the game $(N, v) \in G^N$.

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Also, when given a fixed game (N, v) and a subset $S \subseteq N$, we will write $\varphi(S)$ to denote $\varphi^S(S, v_S)$.

Undoubtedly, the most important solution in this direction is the Shapley value, followed probably by the Banzhaf value. They are defined as follows.

The *Shapley value* is the function $\mathrm{Sh}: G^N \to \mathbb{R}^N$ having the explicit representation

$$\mathrm{Sh}_i(N, v) = \sum_{T \subseteq N \setminus \{i\}} \frac{t!(n-t-1)!}{n!} \big(v(T \cup \{i\}) - v(T)\big),$$

and the *Banzhaf value* B corresponding to $(N, v) \in G^N$ is the vector $B(N, v)$ in $\mathbb{R}^N$, defined as follows:

$$B_i(N, v) = \frac{1}{2^{n-1}} \sum_{T \subseteq N} \big(v(T \cup \{i\}) - v(T)\big),$$

See Shapley (1953) and Lehrer (1988) respectively for further details.

Myerson (1980) introduced the *balanced contributions axiom*. He uses this axiom in a different structure but in the same spirit. In this context, we will say that a solution φ satisfies the balanced contributions axiom if for any game (N, v),

$$\varphi_i(S) - \varphi_i(S \setminus \{j\}) = \varphi_j(S) - \varphi_j(S \setminus \{i\})$$

for all $\{i, j\} \subseteq S$ and any nonempty subset $S \subseteq N$.

The idea behind this axiom is the following: suppose that players agree to use a solution φ whenever coalition S forms. Then $\varphi_i(S) - \varphi_i(S \setminus \{j\})$ is the amount player i gains or loses when S is already formed and player j resigns. The Balanced Contributions Axiom says that the amounts that each player would gain or lose by the other's withdrawal from the coalition should be equal.

Now, if φ is used to solve the subgame of a coalition S, then

$$R^\varphi(S) = \sum_{i \in S} \varphi_i(S)$$

is the expected amount for S. We will consider these quantities to form a new game (N, R^φ).

THEOREM 1. *If solution φ satisfies the Balanced Contributions Axiom then $\varphi(S) = \mathrm{Sh}(S, R^\varphi_S)$, where R^φ_S is the subgame of R^φ corresponding to the coalition S.*

Theorem 1 says that if we use a value which satisfies the balanced contributions axiom to solve each (S, v_S), $S \subseteq N$, the total amounts shared form a game (N, R^{φ}) whose Shapley value is equal to the original value of (N, v).

Remarks. (a) Myerson (1980) proved that the Balanced Contributions Axiom and efficiency characterize the Shapley value. Therefore if φ verifies the Balanced Contributions Axiom and φ is not the Shapley value, then φ is not efficient.

(b) Both Shapley and Banzhaf values satisfy the Balanced Contributions Axiom. Moreover, for any probability measure ξ in $[0,1]$ we can define a semivalue (for any finite number of players n) by

$$\psi_i(N,v) = \sum_{s \subseteq N \setminus \{i\}} p_s^n [v(S \cup \{i\}) - v(S)],$$

$$\text{where } p_s^n = \int_0^1 x^s (1-x)^{n-s-1} d\xi(x),$$

which satisfies the Balanced Contributions Axiom (see the appendix for proofs).

As an example, if we use the Banzhaf value for the three-person game, $v(\{1\}) = 1$, $v(\{2\}) = 4$, $v(\{3\}) = 3$, $v(\{1,2\}) = 5$, $v(\{1,3\}) = 6$, $v(\{2,3\}) = 9$, and $v(\{1,2,3\}) = 10$, we get

	$B(S, v_S)$			
S	1	2	3	$R^B(S)$
{1}	1	—	—	1
{2}	—	4	—	4
{3}	—	—	3	3
{1,2}	1	4	—	5
{1,3}	2	—	4	6
{2,3}	—	5	4	9
{1,2,3}	1.5	4.5	4.5	10.5

and $\mathrm{Sh}(N, R^B) = (1.5, 4.5, 4.5)'$, which is the Banzhaf value of (N, v) (see last row of the table).

We now turn to the nonsymmetric case.

For every $S \subseteq N$, the unanimity game (N, u_S) is defined as follows: $u_S(T) = 1$ if $S \subseteq T$, and $u_S(T) = 0$ otherwise.

For a vector $w \in \mathbb{R}^N$, $w > 0$, the weighted Shapley value $\mathrm{Sh}^w : G^N \to \mathbb{R}^N$ is defined as the unique linear operator satisfying for each unanimity game

(N, u_S):

$$\mathrm{Sh}_i^w(N, u_S) = \begin{cases} w_i/w(S) & \text{if } i \in S \\ 0 & \text{otherwise,} \end{cases}$$

where $w(S) = \sum_{i \in S} w_i$. See Kalai and Samet (1987) for further details.

Now, in order to introduce the result equivalent to Theorem 1 in the nonsymmetric case, we introduce player's weights. The solution φ satisfies the *Weighted Balanced Contributions Axiom* with $w \in \mathbb{R}^N$ if and only if

$$\big(\varphi_i(S) - \varphi_i(S\backslash\{j\})\big)w_j = \big(\varphi_j(S) - \varphi_j(S\backslash\{i\})\big)w_i$$

for all $\{i, j\} \subseteq S$ and any nonempty subset $S \subseteq N$. This solution will be denoted by φ^w.

THEOREM 2. *Given $w \in \mathbb{R}^n$, $w > 0$, a solution φ^w verifies the weighted balanced contributions property if and only if*

$$w(S)\varphi_i^w(S) = \big(R^{\varphi^w}(S) - R^{\varphi^w}(S\backslash\{i\})\big)w_i + \sum_{k \in S\backslash\{i\}} w_i \varphi_i^w(S\backslash\{k\}).$$

If additionally $\sum_{k \in S} \varphi_k^w(S) = v(S)$, then $\varphi_i^w(S) = \mathrm{Sh}_i^{w^S}(S, v_S)$.

Remarks. (c) $\varphi_i(S)$ is a weighted average of s quantities, the amounts that player i gets if any other player leaves the coalition S and the difference $R^{\varphi^w}(S) - R^{\varphi^w}(S\backslash\{i\})$ which corresponds to player i leaving S.

(d) Note that the first equation of Theorem 2 can be used to define other weighted values besides Shapley's.

In the previous example, with $w' = (3, 2, 3)$ we get

	$\varphi^w(S)$		
S	1	2	3
{1}	1	—	—
{2}	—	4	—
{3}	—	—	3
{1, 2}	1	4	—
{1, 3}	2	—	4
{2, 3}	—	4.8	4.2
{1, 2, 3}	1.25	4.3	4.45

and the weighted Shapley value $\mathrm{Sh}^w(N, v) = (1.25, 4.3, 4.45)'$ is equal to the last row of the table again, which could be defined as the weighted Banzhaf value.

APPENDIX

Proof of Theorem 1. In the proof of this theorem we will use Theorem 2. The proof is by induction on the number of players. Remember that $R^{\varphi}(S) = \sum_{i \in S} \varphi_i(S)$, so $\varphi_i(\{i\}) = R^{\varphi}(\{i\}) = \mathrm{Sh}_i(\{i\}, R^{\varphi}_{\{i\}})$. Now, suppose $\varphi_i(S\backslash\{k\}) = \mathrm{Sh}_i(S\backslash\{k\}, R^{\varphi}_{S\backslash\{k\}})$ for $k \in S$. Thus, by Theorem 2, φ satisfies

$$s\varphi_i(S) = R^{\varphi}(S) - R^{\varphi}(S\backslash\{i\}) + \sum_{k \in S\backslash\{i\}} \varphi_i(S\backslash\{k\})$$

$$= R^{\varphi}(S) - R^{\varphi}(S\backslash\{i\}) + \sum_{k \in S\backslash\{i\}} \sum_{T \subseteq S\backslash\{i,k\}} \frac{t!(s-1-t-1)!}{(s-1)!}$$

$$\times (R^{\varphi}(T \cup \{i\}) - R^{\varphi}(T)).$$

Note that each term corresponding to a proper subset of $S\backslash\{i\}$ is repeated $(s-t-1)$ times. Then

$$= R^{\varphi}(S) - R^{\varphi}(S\backslash\{i\}) + \sum_{T \subsetneq S\backslash\{i\}} \frac{t!(s-t-1)!}{(s-1)!}(R^{\varphi}(T \cup \{i\}) - R^{\varphi}(T)),$$

therefore

$$\varphi_i(S) = \sum_{T \subseteq S\backslash\{i\}} \frac{t!(s-t-1)!}{s!}(R^{\varphi}(T \cup \{i\}) - R^{\varphi}(T)).$$

Proof of the First Part of Theorem 2. Let φ^w be a solution satisfying a weighted balanced contributions axiom with $w > 0$ and consider the expression

$$(\varphi_i^w(S) - \varphi_i^w(S\backslash\{k\}))w_k = (\varphi_k^w(S) - \varphi_k^w(S\backslash\{i\}))w_i$$

with fixed i and $k \in S\backslash\{i\}$. Now, if we add these equations over $k \in S\backslash\{i\}$,

$$w(S\backslash\{i\})\varphi_i^w(S) - \sum_{k \in S\backslash\{i\}} w_k \varphi_i^w(S\backslash\{k\})$$

$$= w_i\left(R^{\varphi^w}(S) - \varphi_i(S) - \sum_{k \in S\backslash\{i\}} \varphi_k^w(S\backslash\{i\})\right),$$

thus

$$w(S)\varphi_i^w(S) = w_i(R^{\varphi^w}(S) - R^{\varphi^w}(S\backslash\{i\})) + \sum_{k \in S\backslash\{i\}} w_k \varphi_i^w(S\backslash\{k\}).$$

Let φ^w be a solution satisfying the expression of the theorem. The proof is by induction on the number of players.

The proof for coalitions with two players is direct. Now, suppose $'\varphi^w$ satisfies

$$\big(\varphi_i(S\backslash\{k\}) - \varphi_i(S\backslash\{j,k\})\big)w_j = \big(\varphi_j(S\backslash\{k\}) - \varphi_j(S\backslash\{i,k\})\big)w_i$$

for distinct players $i, j, k \in S$. Then

$$\begin{aligned}
&w(S)\big(w_j\varphi_i(S\backslash\{j\}) - w_i\varphi_j(S\backslash\{i\})\big)\\
&\quad = w_j\big[w(S\backslash\{j\}) + w_j\big]\varphi_i(S\backslash\{j\}) - w_i\big[w(S\backslash\{i\}) + w_i\big]\varphi_j(S\backslash\{i\})\\
&\quad = w_jw(S\backslash\{j\})\varphi_i(S\backslash\{j\}) - w_iw(S\backslash\{i\})\varphi_j(S\backslash\{i\})\\
&\qquad + w_j^2\varphi_i(S\backslash\{j\}) - w_i^2\varphi_j(S\backslash\{i\})\\
&\quad = w_j\Bigg[w_i\big(R^{\varphi^w}(S\backslash\{j\}) - R^{\varphi^w}(S\backslash\{i,j\})\big)\\
&\qquad\qquad + \sum_{k\in S\backslash\{i,j\}} w_k\varphi_i(S\backslash\{j,k\})\Bigg] + w_j^2\varphi_i(S\backslash\{j\})\\
&\qquad - w_i\Bigg[w_j\big(R^{\varphi^w}(S\backslash\{i\}) - R^{\varphi^w}(S\backslash\{i,j\})\big)\\
&\qquad\qquad + \sum_{k\in S\backslash\{i,j\}} w_k\varphi_j(S\backslash\{i,k\})\Bigg] - w_i^2\varphi_j(S\backslash\{i\})\\
&\quad = -w_jw_iR^{\varphi^w}(S\backslash\{i\}) + w_j^2\varphi_i(S\backslash\{j\})\\
&\qquad + w_iw_jR^{\varphi^w}(S\backslash\{j\}) - w_i^2\varphi_j(S\backslash\{i\})\\
&\qquad + \sum_{k\in S\backslash\{i,j\}} w_k\big[w_j\varphi_i(S\backslash\{j,k\}) - w_i\varphi_j(S\backslash\{i,k\})\big]
\end{aligned}$$

by the induction hypothesis,

$$\begin{aligned}
&= -w_jw_iR^{\varphi^w}(S\backslash\{i\}) + w_j^2\varphi_i(S\backslash\{j\}) + w_iw_jR^{\varphi^w}(S\backslash\{j\}) - w_i^2\varphi_j(S\backslash\{i\})\\
&\quad + \sum_{k\in S\backslash\{i,j\}} w_k\big[w_j\varphi_i(S\backslash\{k\}) - w_i\varphi_j(S\backslash\{k\})\big]
\end{aligned}$$

and by adding and subtracting $w_iw_jR^{\varphi^w}(S)$,

$$\begin{aligned}
&= w_jw_i\big[R^{\varphi^w}(S) - R^{\varphi^w}(S\backslash\{i\})\big] + \sum_{k\in S\backslash\{i\}} w_kw_j\varphi_i(S\backslash\{k\})\\
&\qquad - w_iw_j\big[R^{\varphi^w}(S) - R^{\varphi^w}(S\backslash\{j\})\big] - \sum_{k\in S\backslash\{j\}} w_kw_i\varphi_j(S\backslash\{k\})\\
&\quad = w(S)\big(w_j\varphi_i(S) - w_i\varphi_j(S)\big).
\end{aligned}$$

Thus we get

$$w_j\varphi_i(S\backslash\{j\}) - w_i\varphi_j(S\backslash\{i\}) = w_j\varphi_i(S) - w_i\varphi_j(S).$$

Proof of the Second Part of Theorem 2. We have to prove that $\varphi^w(S,\cdot)$ is linear and $\varphi_i^w(S,u_T) = w_i/w(T)$ if $i \in T$ and zero otherwise, for $T \subseteq S$. Both proofs are straightforward by induction on the cardinality of the coalitions. We just present the induction step of the second one. Suppose $\varphi_i^w(S\backslash\{k\},u_T) = w_i/w(T)$ if $i \in T$ and 0 otherwise, then

$$w(S)\varphi_i^w(S,u_T) = w_i(u_T(S) - u_T(S\backslash\{i\})) + \sum_{k\in S\backslash\{i\}} w_k\varphi_i^w(S\backslash\{k\},u_T);$$

if $i \notin T$ all terms are zero, and

$$\begin{aligned} w(S)\varphi_i^w(S,u_T) &= w_i + \sum_{k\in S\backslash T} w_k\frac{w_i}{w(T)} \\ &= \frac{w_i(w(T) + w(S\backslash T))}{w(T)} = \frac{w_i w(S)}{w(T)}. \end{aligned}$$

Proof of Remark (*b*). It is well known that the Banzhaf and Shapley values are a particulaer case of those semivalues associated with probability measures. These semivalues are of the form

$$\psi_i(N,v) = \sum_{S\subseteq N\backslash\{i\}} p_s^n[v(S\cup\{i\}) - v(S)],$$

and it is straightforward to see that the weights verify the relation

$$p_t^s + p_{t+1}^s = p_t^{s-1} \qquad \text{for } s \le n \text{ and any } t < s.$$

Thus,

$$\begin{aligned} &\psi_i(S,v_S) - \psi_j(S,v_S) \\ &\quad= \sum_{\{T:T\subseteq S\backslash\{i\}\}} p_t^s v(T\cup\{i\}) - \sum_{\{T:T\subseteq S\backslash\{i\}\}} p_t^s v(T) \\ &\qquad - \sum_{\{T:T\subseteq S\backslash\{j\}\}} p_t^s v(T\cup\{j\}) + \sum_{\{T:T\subseteq S\backslash\{j\}\}} p_t^s v(T) \\ &\quad= \sum_{\{T:T\subseteq S\backslash\{i\},\, j\notin T\}} p_t^s v(T\cup\{i\}) - \sum_{\{T:T\subseteq S\backslash\{i\},\, j\notin T\}} p_t^s v(T) \\ &\qquad - \sum_{\{T:T\subseteq S\backslash\{j\},\, i\notin T\}} p_t^s v(T\cup\{j\}) \\ &\qquad + \sum_{\{T:T\subseteq S\backslash\{j\},\, i\notin T\}} p_t^s v(T) + \sum_{\{T:T\subseteq S\backslash\{i\},\, j\in T\}} p_t^s v(T\cup\{i\}) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\{T:T\subseteq S\setminus\{i\},\, j\in T\}} p_t^s v(T) \\
&- \sum_{\{T:T\subseteq S\setminus\{j\},\, i\in T\}} p_t^s v(T\cup\{j\}) + \sum_{\{T:T\subseteq S\setminus\{j\},\, i\in T\}} p_t^s v(T) \\
&= \sum_{\{T:T\subseteq (S\setminus\{j\})\setminus\{i\}\}} p_t^s v(T\cup\{i\}) - \sum_{\{T:T\subseteq (S\setminus\{i\})\setminus\{j\}\}} p_t^s v(T\cup\{j\}) \\
&- \sum_{\{T:T\subseteq S\setminus\{i\},\, j\in T\}} p_t^s v(T) + \sum_{\{T:T\subseteq S\setminus\{j\},\, i\in T\}} p_t^s v(T) \\
&= \sum_{\{T:T\subseteq (S\setminus\{j\})\setminus\{i\}\}} (p_t^s + p_{t+1}^s) v(T\cup\{i\}) \\
&- \sum_{\{T:T\subseteq (S\setminus\{i\})\setminus\{j\}\}} (p_t^s + p_{t+1}^s) v(T\cup\{j\}).
\end{aligned}
$$

Taking into account that $p_t^s + p_{t+1}^s = p_t^{s-1}$, and adding and subtracting $\Sigma_{\{T:T\subseteq S\setminus\{i,j\}\}} p_t^{s-1} v(T)$, we get

$$\psi_i(S, v_S) - \psi_j(S, v_S) = \psi_i\big(S\setminus\{j\}, v_{S\setminus\{j\}}\big) - \psi_j\big(S\setminus\{i\}, v_{S\setminus\{i\}}\big).$$

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TÓPICOS EN ECONOMÍA MATEMÁTICA Y ECONOMETRÍA

Fernando Barceinas Paredes
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Azcapotzalco

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About inheritance distribution

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Abstract

In this work, we propose two axiomatic procedures to distribute an inheritance based on monetary compensations. In a cooperative context, the differences among the agents' evaluations of the goods are used to increase everyone's participation. When the goods' valuations are heterogeneous each agent gets much more than $1/n$ of the value he believes the inheritance has.

The two procedures are characterized by three criteria, namely: money preservation, Pareto optimality, and a certain kind of proportionality. The difference of the procedures is determined by the kind of proportionality used.
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Keywords: Inheritance; Axiomatic procedure; Pareto optimality

1. Introduction

In this work, we present two axiomatic procedures in order to distribute an inheritance. The first procedure is uniquely determined by three properties: preservation of money (implicit in the definition of the solution), Pareto optimality and being in proportion with the agents' monetary valuations of items. The second one is uniquely determined by almost the same three properties, the last one is changed by being in proportion with the agents' monetary valuations of the inheritance. In both cases, after each agent evaluates each item, the heterogeneity of the agents' preferences is used to increase everybody's participation.

Both procedures have several properties in common: let n be the number of agents, then every agent gets at least $1/n$ of what he believes is the value of the inheritance and this fraction increases when the items evaluations are heterogeneous. The solutions are monotonic with respect to the items. Another pair of common properties are that a change in the scale of items' evaluation yields an equivalent change in the total amounts agents receive and also, the solutions are continuous functions of the item's evaluations.

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The first procedure could be applied in steps: first some items and later others, and money could be distributed "in" or "out" of the procedure with the same result. We can think of this procedure as if every item is assigned to the highest bidder and the collected money is redistributed in proportion to the bids. The second procedure does not have the previous properties, but with their solutions every agent gets the same percentage of the value he believes is the whole value of the inheritance and this percentage is the highest that can be guaranteed for all the agents. Besides, when a new heir is included, keeping the same inheritance, the agents already present all benefit or all suffer. In this case, the collected money is redistributed in such a way that a common percent of the inheritance is as big as possible, each one in his corresponding evaluation.

The axiomatization of both procedures are also generalized in the case where the agents have different rights over the inheritance in a simple and natural way. Furthermore, all procedures are easy to use in real life and the solutions change in a smooth way as a function of the matrix A.

Both procedures are similar to the procedure of Knaster–Steinhaus (see, for example, Brams and Taylor (1996), p. 68). This procedure assign the items by an auction, gives to the agents an initial fair share ($1/n$ of the value he believes the items has) and then assign equal share of the surplus. Raith (2000) formulates three similar two-step fair division algorithms for negotiations: in the first step ensures an Pareto outcome, and in the second step he establishes 'fairness' through a distribution of gains. Tadenuma and Thomson (1993) consider the problem of fairly allocating an indivisible good when monetary compensations to the agents who do not receive the good are possible. The idea of assigning objects to the highest bidders was present in Steinhaus (1946). Adjusted Winner procedure of Brams and Taylor (1996, p. 68) allocates a set of divisible's goods among two agents without using money.

In Section 2 we provide a formal characterization of both procedures based on several definitions that describe individual features of the procedures. The main results are given in Theorems 1 and 3, with two further variants characterized by Theorems 2 and 4. In Section 3, we list further properties of the procedures and ends with a brief discussion of the results. All mathematical proofs are provided in the Appendix A.

Before we start, we want to clarify the framework of this work. We proceed, according to what is used in cooperative games by using any disagreement about the item's values to improve everyone's participation. Our main goal is to establish normative solutions based on desirable properties. Another option is that each agent tries to take advantage of any knowledge he has over the preferences of the others. This situation can be modeled as a non-cooperative game, see for example, Thomson (1988) or Tadenuma and Thomson (1993), but we leave this approach for future work.

2. Axiomatic characterization

By an Inheritance's Distribution problem (hereafter a "problem") we mean a triplet (M, N, A), where $N = \{1, \ldots, n\}$ is a finite set of agents that inherits a finite set $M = \{1, \ldots, m\}$ of goods and A is a $m \times n$ positive matrix, where a_{ij} entry is the value that agent j gives to the item i. In order to generate this matrix, a judge can ask the agents to give a bid for each item; each agent independently from the others.

Now, we turn our attention to the solution concept. First, we propose monetary compensations for an arbitrary assignment of the goods, and then we use the Pareto optimality property to get an assignment of the goods. Suppose for a moment that the goods are already assigned, i.e. suppose a partition $\{M_1, \ldots, M_n\}$ of M is given, where M_j is the set of goods agent j has received and let us denote by $u_j = \sum_{i \in M_j} a_{ij}$, the value that agent j gives to it. Some of these sets could be empty (in such a case, the corresponding entry for u is defined to be zero). Let $x \in \mathbb{R}^N$ be a vector of the amounts of money agents receive as compensation corresponding to u. So, the agent j receives x_j in money (or he pays, in case x_j is negative) and the set M_j of goods that he evaluated in u_j units of money.

Definition 1. We will say that $u \in \mathbb{R}^N$ is the result of an assignment of the goods in M, if there exists a partition $\{M_1, \ldots, M_n\}$ of M such that $u_j = \sum_{i \in M_j} a_{ij}$ for every $j \in N$.

Definition 2. We will say that u came from the highest bidders if u is the result of an assignment of the goods in M with a partition $\{M_1, \ldots, M_n\}$ of M such that if $k \in M_j$ then $a_{kj} \geq a_{ki}$ for every $i \in N$.

Definition 3. By a solution of (M, N, A), we mean a pair (u, x) where u is the result of an assignment of the goods in M and $x \in \mathbb{R}^N$ is a pay-off vector whose coordinates add up to zero, i.e. $\sum_{j \in N} x_j = 0$.

Now, we establish the properties that characterize the first procedure. One of the properties we will use to determinate our procedures is Pareto optimality. A solution is Pareto optimal if there is no other solution that will make one or more agents better off leaving the others as well off as they were. The next definition gives shape to this idea.

Definition 4. A solution (u, x) is Pareto optimal if and only if there does not exist any other solution $(\tilde{u}, \tilde{x})$ such that

$$\tilde{u} + \tilde{x} \geq u + x$$

with strict inequality holding true for at least one coordinate.

Lemma 1. *In the solution (u, x), u came from the highest bidders if and only if (u, x) is Pareto optimal.*

Proof. Let (u, x) be a solution where u came from the highest bidders and suppose there exists a solution $(\tilde{u}, \tilde{x})$ such that $\tilde{u} + \tilde{x} \geq u + x$ with strict inequality for at least one coordinate. Then, adding the n inequalities we get $\sum_{j \in N} \tilde{u}_j > \sum_{j \in N} u_j$ because $\sum_{j \in N} x_j = \sum_{j \in N} \tilde{x}_j = 0$. But, as u came from the highest bidders, $\sum_{j \in N} \tilde{u}_j \leq \sum_{j \in N} u_j$,which is a contradiction. So (u, x) is Pareto optimal.

For the converse, let (u, x) be a Pareto optimal solution and assume u does not come from the highest bidders. Then, there exists a good assigned to an agent who is not the highest bidder. In order to get a solution where two agents improve, the corresponding agent could sell this good to any agent who gives it a higher value with a price between their valuations. This contradicts the Pareto optimality of (u, x), so u comes from the highest bidders. □

For any $P \subseteq M$, let us denote by A_P the matrix A restricted to the rows in P. Consider the set of problems $\{(P, N, A_P)|P \subseteq M\}$ and a corresponding family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A). For a given solution $(u(P), x(P))$ of the problem (P, N, A_P) where $P \subseteq M$, we will denote by $\varphi(P) = u(P) + x(P)$ the total value agents receive. In case we need to specify the set of agents, we will include it as a second coordinate, i.e. $\varphi(P, N) = u(P, N) + x(P, N)$.

The goods have no value by themselves. This value is generated by the agents' evaluations and so, we feel that it is according to them that it must be distributed. So, we suppose that the changes in the value agents receive, when one item is omitted, are in proportion with the value they give to this item.

Definition 5. We will say that a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in proportion with A if and only if

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))a_{kj} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki} \tag{2.1}$$

for every $i, j \in N$ and $k \in P$.

Lemma 2. *Let $\{(u(P), x(P))|P \subseteq M\}$ be a family of solutions for (M, N, A) in proportion with A and $m(P) = \sum_{j\in N} \varphi_j(P)$ then*

(a) $\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]a_{ki}/\sum_{j\in N} a_{kj}$, *for every $P \subseteq M$ and $k \in P$.*
(b) $m(P) = \sum_{k\in P} m(\{k\})$.
(c) *Moreover, if the family is Pareto optimal then $m(\{k\}) = \max_{i\in N} a_{ki}$ for every $k \in M$.*

Proof.

(a) Consider a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A), an arbitrary subset $P \subseteq M$ and $k \in P$. Adding the corresponding equalities in (2.1) over $j \in N$, we get

$$\begin{aligned}(\varphi_i(P) - \varphi_i(P\backslash\{k\})) \sum_{j\in N} a_{kj} &= \sum_{j\in N} (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki} \\ &= [m(P) - m(P\backslash\{k\})]a_{ki},\end{aligned}$$

thus

$$\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\frac{a_{ki}}{\sum_{j\in N} a_{kj}}. \tag{2.2}$$

(b) After adding (2.2) over $i \in N$, the proof is direct by induction on the cardinality of P using the facts that $m(\emptyset) = 0$ and $\varphi_i(\emptyset) = 0$ for all $i \in N$.

(c) By (b), we have

$$m(\{k\}) = \sum_{i\in N} \varphi_i(P) = \sum_{i\in N} u_i(P) + x_i(P) = \sum_{i\in N} u_i(P) = a_{kj}.$$

Since, $u(P)$ came from the highest bidders by Lemma 1, $m(\{k\}) = a_{kj} = \max_{i\in N} a_{ki}$. □

Consider the following solution: every item is assigned to its highest bidder and the monetary compensations are calculated according to

$$x_i = \sum_{k\in M}\left[c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}\right] - u_i,$$

where $c_k = \max_{j\in N} a_{kj}$.

In case of a tie in the highest bid, any mechanism could be used to break it.

One of the main contributions of this work is established in the next theorem: the previous solution is the only one that is Pareto optimal and in proportion with A.

Theorem 1. *The family of solutions* $\{(u(P), x(P))|P \subseteq M\}$ *for* (M, N, A) *is in proportion with* A *and each* $(u(P), x(P))$, $P \subseteq M$ *is Pareto optimal if and only if for every* $P \subseteq M$, $u(P)$ *came from the highest bidders and*

$$x_i(P) = \sum_{k\in P}\left[c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}\right] - u_i(P). \tag{2.3}$$

Proof. Suppose a family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) in proportion with A and where each $(u(P), x(P))$, with $P \subseteq M$ is Pareto optimal. Notice first that for $P = \{k\}$, using (2.2) and the fact that $c_k = \max_{j\in N} a_{kj}$,

$$\varphi_i(\{k\}) = m(\{k\})\frac{a_{ki}}{\sum_j a_{kj}} = c_k \frac{a_{ki}}{\sum_j a_{kj}}. \tag{2.4}$$

Now, for arbitrary $P \subseteq M$ and $i \in N$, by Lemma 2(b) and (c),

$$\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\frac{a_{ki}}{\sum_{j\in N} a_{kj}} = \varphi_i(P\backslash\{k\}) + c_k \frac{a_{ki}}{\sum_{j\in N} a_{kj}}$$

so, if we repeat the previous argument for the items in $P\backslash\{k\}$ then we get (2.3).

Now, consider the family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) where $u(P)$ came from the highest bidders and $x(P)$ is given by (2.3). First notice that,

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))a_{kj} = c_k \frac{a_{ki}a_{kj}}{\sum_{j\in N} a_{kj}} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))a_{ki}$$

so, the family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in proportion with A. By Lemma 1, each $(u(P), x(P))$, $P \subseteq M$ is Pareto optimal. This proves the theorem. □

Note that with the solution of Theorem 1 agent i gets $c_k a_{ki}/\sum_{j\in N} a_{kj}$ from item k, so, every agent gets the same percentage $c_k/\sum_{j\in N} a_{kj}$ of the value a_{ki} that he believes the item k has.

Example. Suppose the set $N = \{1, 2, 3\}$ inherits the set of goods $M = \{1, 2\}$ and the agents in N evaluate the goods according to the matrix

$$A = \begin{bmatrix} 25 & 20 & 50 \\ 75 & 180 & 200 \end{bmatrix}$$

Let us calculate the solution established in Theorem 1 for this example. Both goods are assigned to agent 3, since he is the highest bidder in both cases, so agent 3 gets US$ 250 in goods and nothing for the others. Hence, we have that $u^{\mathrm{T}} = (0, 0, 250)$ and the corresponding monetary compensations are given by

$$x = \frac{50}{95}\begin{pmatrix}25\\20\\50\end{pmatrix} + \frac{200}{455}\begin{pmatrix}75\\180\\200\end{pmatrix} - \begin{pmatrix}0\\0\\250\end{pmatrix} = \begin{pmatrix}46.12\\89.64\\-135.77\end{pmatrix}$$

The result is excellent: agent 1 gets US$ 46.12 when he just expected US$ 33.33, agent 2 gets US$ 89.64 when he expected US$ 66.66 and agent 3 gets a benefit of US$ 114.23 (250–135.77) when according to his estimations, he would have received only US$ 83.33. Furthermore, every agent gets the same proportion of what he believes each item has. The next table summarizes the result. The first number in each cell is the value agent receives from the corresponding item and the number in paranthesis is its percentage with respect to the value he gives to the item.

Item	Agent 1 (percentage)	Agent 2 (percentage)	Agent 3 (percentage)
1	13.15 (52.6)	10.52 (52.6)	26.32 (52.6)
2	32.97 (43.96)	79.12 (43.96)	87.91 (43.96)
Total	46.12 (46.12)	89.64 (44.82)	114.23 (45.69)

Now, we consider the case where the agents do not have the same right over the inheritance. Let α_i be the percentage of the inheritance corresponding to agent i, e.g. a husband could leave 60% of the inheritance to his wife and 20% to each of his two sons, so $\alpha_1 = 0.6$, $\alpha_2 = 0.2$ and $\alpha_3 = 0.2$ in this case.

Definition 6. We will say that a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in a weighted proportion with A if and only if

$$(\varphi_i(P) - \varphi_i(P\backslash\{k\}))\alpha_j a_{kj} = (\varphi_j(P) - \varphi_j(P\backslash\{k\}))\alpha_i a_{ki} \tag{2.5}$$

for every $i, j \in N$ and $k \in P$.

Lemma 3. *If a family $\{(u(P), x(P))|P \subseteq M\}$ of solutions for (M, N, A) is in a weighted proportion with A then*

(a) $\varphi_i(P) = \varphi_i(P\backslash\{k\}) + [m(P) - m(P\backslash\{k\})]\alpha_i a_{ki} / \sum_{j\in N} \alpha_j a_{kj}$, *for every $P \subseteq M$ and $k \in P$.*
(b) $m(P) = \sum_{k\in P} m(\{k\})$.
(c) *Moreover, if the family is Pareto optimal then $m(\{k\}) = \max_{i\in N} a_{ki}$ for every $k \in M$.*

The proof of the lemma is similar to that of Lemma 2.

Theorem 2. *The family of solutions $\{(u(P), x(P))|P \subseteq M\}$ for (M, N, A) is in a weighted proportion with A and each $(u(P), x(P))$, $P \subseteq M$ is Pareto optimal if and only if for every*

$P \subseteq M$, $u(P)$ came from the highest bidders and

$$x_i(P) = \sum_{k \in P} \left[c_k \frac{\alpha_i a_{ki}}{\sum_{j \in N} \alpha_j a_{kj}} \right] - u_i(P) \tag{2.6}$$

The proof of the theorem is similar to that of Theorem 1.

Notice that the solutions of Theorems 1 and 2 could be applied in steps, if we divide the set M, the sum of the solutions of the parts coincide with the solution of the whole set.

Example (Continuation). For the previous example, suppose that agents 1, 2 and 3 have 20, 70, and 10%, respectively of the rights over the inheritance. As before we have that $u^{\mathrm{T}} = (0, 0, 250)$ and the corresponding monetary compensations are given by

$$\begin{aligned} x = {} & \frac{50}{0.2 \times 25 + 0.7 \times 20 + 0.1 \times 50} \begin{pmatrix} 0.2 \times 25 \\ 0.7 \times 20 \\ 0.1 \times 50 \end{pmatrix} \\ & + \frac{200}{0.2 \times 75 + 0.7 \times 180 + 0.1 \times 200} \begin{pmatrix} 0.2 \times 75 \\ 0.7 \times 180 \\ 0.1 \times 200 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 250 \end{pmatrix} \\ = {} & \begin{pmatrix} 29.05 \\ 185.689 \\ -214.74 \end{pmatrix} \end{aligned}$$

As before, the amounts agents receive are bigger than they expect. The result of using the solution of Theorem 2 is summarized in the following table

	Agent 1	Agent 2	Agent 3
α	0.2	0.7	0.1
Amount received in goods	0	0	250
Amount received in cash	29.05	185.69	−214.74
Total inheritance's value	100	200	250
Expected assignment	20	140	25
Total assignment	29.05	185.69	35.26
Inheritance's percentage	29.05	92.85	14.10

Now, we modify the definition of being in proportion with A to get a solution where every agent gets the same percentage of the value he believes the whole inheritance has. Moreover, this percentage is the highest that can be guaranteed for all the agents. We denote by $v \in \mathbb{R}^N$ the vector whose j-entry is the value assigned by agent j to the total inheritance, i.e. $v_j = \sum_{i \in M} a_{ij}$.

Definition 7. We will say that the solution (u, x) is in proportion with v if and only if

$$(u_i + x_i)v_j = (u_j + x_j)v_i \tag{2.7}$$

for every $i, j \in N$.

Theorem 3. *The solution (u, x) is Pareto optimal and in proportion with v if and only if u came from the highest bidders and $x = ((\iota^{\mathrm{T}}u)/(\iota^{\mathrm{T}}v))v - u$ where ι denotes the column vector of all 1's.*

Proof. Suppose a solution (u, x) is Pareto optimal and in proportion with v. By Lemma 1, we just have to prove that $x = ((\iota^{\mathrm{T}}u)/(\iota^{\mathrm{T}}v))v - u$. If we add the equations in (2.7) over $i \in N$, we get

$$(\iota^{\mathrm{T}}u + 0)v_j = (u_j + x_j)\iota^{\mathrm{T}}v$$

and it is follows that, $(\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v)v_j = u_j + x_j$.

For the converse, suppose $x = (\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v)v - u$ where u came from the highest bidders. Since,

$$\iota^{\mathrm{T}}x = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v}\iota^{\mathrm{T}}v - \iota^{\mathrm{T}}u = 0,$$

we see that (u, x) is a solution. Now, from the coordinates i and j of x we get that $(u_i + x_i)/v_i = (u_j + x_j)/v_j$. Therefore (u, x) is in proportion with v. The solution (u, x) is Pareto optimal by the Lemma 1. □

Example (Continuation). Let us calculate the solution established in Theorem 3 for the initial example. As before, both goods are assigned to agent 3, since he is the highest bidder in both cases. Hence, we have that $u^{\mathrm{T}} = (0, 0, 250)$, $v^{\mathrm{T}} = (100, 200, 250)$ and the corresponding monetary compensations are given by

$$x = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v}v - u = \frac{250}{550}\begin{pmatrix}100\\200\\250\end{pmatrix} - \begin{pmatrix}0\\0\\250\end{pmatrix} = \begin{pmatrix}45.45\\90.90\\-136.36\end{pmatrix}.$$

so we have,

	Agent 1	Agent 2	Agent 3
Amount received in goods	0	0	250
Amount received in cash	45.45	90.90	−136.36
Total inheritance's value	100	200	250
Inheritance's percentage	45.45	45.45	45.45

Every agent gets the same proportion of what he believes is the value of the inheritance (45.45%) and there is no other solution where the agent with the lowest percent could be higher (see proof of Theorem 6).

Now, we consider the case when the agents do not have the same right over the inheritance. As before, let α_i be the percentage of the inheritance corresponding to agent i.

Definition 8. We will say that the solution (u, x) is in a weighted proportion with v if and only if $(u_i + x_i)\alpha_j v_j = (u_j + x_j)\alpha_i v_i$ for every $i, j \in N$.

We will denote by $\hat{\alpha}$ the diagonal matrix corresponding to the vector $\alpha \in \mathbb{R}^N$.

Theorem 4. *The solution (u, x) is Pareto optimal and in a weighted proportion with v if and only if u came from the highest bidders and $x = (\iota^{\mathrm{T}} u/\alpha^{\mathrm{T}} v)\hat{\alpha} v - u$.*

The proof of Theorem 4 is similar to that of Theorem 3.

3. Some properties and remarks

In this section, we establish some properties of the previous solutions. We start describing four properties that we adapt from Moulin (1992) to the present context: weak individual rationality, resource and population monotonicity and population solidarity. We will say that a procedure has the *weak individual rationality* property when each agent gets with it, at least $1/n$th of what he believes is the value of the whole inheritance. The procedure has a *resource monotonicity* property if the welfare of no agent goes down when the goods to be divided increase and we will say it has the *population monotonicity* property if when we incorporate an additional heir, keeping the same inheritance, none of the agents already present should benefit. Furthermore, we will say the procedure has the *population solidarity* property if when a new agent shows up to share the same goods, the previous agents all benefit or all suffer. A deeper discussion of these properties could be found in Moulin (1992).

Moreover, we will say a procedure is *homogeneous of degree* 1 if a change in the scale of the items's evaluation yield an equivalent change in the total amounts agents receive.

Theorem 5. *Let (u, x) be the solution of (M, N, A) given by Theorem 1 then*

(a) *Weak individual rationality, $x_i + u_i \geq (1/n)v_i$ for every $i \in N$.*
(b) *Resource monotonicity, if $P \subseteq Q \subseteq M$ then $\varphi_i(P) \leq \varphi_i(Q)$ for every $i \in N$.*
(c) *Homogeneous of degree 1, the pair $(\mu u, \mu x)$ is a solution of $(M, N, \mu A)$ where μ is a positive real number.*
(d) *Continuity on A, the solution (u, x) of (M, N, A) is a continuous function of A.*

Proof.

(a) By the additivity of $\varphi_i(M, N)$ in its first coordinate, it is enough consider just one item, say item k. Then

$$x_i + u_i = \frac{c_k}{\sum_{j \in N} a_{kj}} a_{ki} \geq \frac{c_k}{nc_k} a_{ki} = \frac{1}{n} a_{ki}.$$

(b) It follows from the additivity of $\varphi_i(M, N)$ in its first coordinate and by the positivity of A.
(c) It is straightforward.
(d) Continuity follows from the fact that $c_k a_{ki}$ and $\sum_{j \in N} a_{kj}$ are continuous as functions of the a_{ij}'s and by the positivity of the denominator. □

The solution of Theorem 1 does not have the population solidarity property: in the problem with the matrix

$$A = \begin{bmatrix} 10 & 5 & 4 \\ 5 & 10 & 50 \end{bmatrix}$$

agent 2 is better off and agent 1 gets worse when agent 3 leaves.

In the solution of Theorem 1, there is a common percentage

$$\frac{c_k}{\sum_{j\in N} a_{kj}}$$

for each item each agent gets from the value he gives to it and in the solution of Theorem 3 there is a common percentage $\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v$ each agent gets from the value he gives to the inheritance.

Theorem 6. *The solution* (u, x) *of* Theorem 3 *has the following properties*

(a) *Weak individual rationality,* $x_i + u_i \geq (1/n)v_i$ *for every* $i \in N$.
(b) *Resource monotonicity, if* $P \subseteq Q \subseteq M$ *then* $\varphi_i(P) \leq \varphi_i(Q)$ *for every* $i \in N$.
(c) *Population solidarity, if* $N \subseteq N'$ then $\varphi_i(M, N') \leq \varphi_i(M, N)$ *for every* $i \in N$ *or* $\varphi_i(M, N) \leq \varphi_i(M, N')$ *for every* $i \in N$.
(d) *Homogeneous of degree* 1, *the pair* $(\mu u, \mu x)$ *is a solution of* $(M, N, \mu A)$ *where* μ *is a positive real number*.
(e) *Continuity on* A, *the solution* (u, x) *of* (M, N, A) *is a continuous function of* A.
(f) *There exists a* $\lambda \in \mathbb{R}$ *such that* $u + x = \lambda v$.
(g) $\max_{(u,x)\in\mathcal{S}} \min_{i\in N}(u_i + x_i)/v_i = \lambda$ *where* $\mathcal{S}$ *is the set of solutions*.

Proof.

(a) Clearly $\sum_{l\in N} u_l = \sum_{i\in M} \max\{a_{ik} | k \in N\}$, so,

$$\iota^{\mathrm{T}}v = \sum_{j\in N}\sum_{i\in M} a_{ij} \leq \sum_{j\in N}\sum_{i\in M} \max\{a_{ik} | k \in N\} = \sum_{j\in N}\sum_{l\in N} u_l = n\,\iota^{\mathrm{T}}u$$

and its follows that

$$\lambda = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v} \geq \frac{1}{n}.$$

(b) Let us consider two disjoint sets of goods, let u_1, u_2, v_1 and v_2 be the values for the corresponding matrices and suppose λ_1 and λ_2 such that

$$\lambda_1\iota^{\mathrm{T}}v_1 = \iota^{\mathrm{T}}u_1$$

$$\lambda_2\iota^{\mathrm{T}}v_2 = \iota^{\mathrm{T}}u_2$$

without loss of generality, we can suppose $\lambda_2 \geq \lambda_1$ so

$$\frac{\iota^{\mathrm{T}}u_1}{\iota^{\mathrm{T}}u_1 + \iota^{\mathrm{T}}u_2} = \frac{\lambda_1\iota^{\mathrm{T}}v_1}{\lambda_1\iota^{\mathrm{T}}v_1 + \lambda_2\iota^{\mathrm{T}}v_2} \leq \frac{\iota^{\mathrm{T}}v_1}{\iota^{\mathrm{T}}v_1 + \iota^{\mathrm{T}}v_2}$$

So,

$$\frac{\iota^{\mathrm{T}}u_1}{\iota^{\mathrm{T}}v_1} \leq \frac{\iota^{\mathrm{T}}u_1 + \iota^{\mathrm{T}}u_2}{\iota^{\mathrm{T}}v_1 + \iota^{\mathrm{T}}v_2}$$

and therefore

$$\frac{\iota^{\mathrm{T}}u_1}{\iota^{\mathrm{T}}v_1}v_1 \leq \frac{\iota^{\mathrm{T}}u_1 + \iota^{\mathrm{T}}u_2}{\iota^{\mathrm{T}}v_1 + \iota^{\mathrm{T}}v_2}(v_1 + v_2).$$

(c) There is a common percentage associated with the solution when we consider N and another when we consider N', so all the agents get more with the solution with higher percentage.
(d) It is straightforward.
(e) Continuity follows from the fact that $\iota^{\mathrm{T}}u$ and $\iota^{\mathrm{T}}v$ are continuous as functions of the a_{ij}'s.
(f) If we take $\lambda = \iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v$ then every agent gets $\iota^{\mathrm{T}}u/\iota^{\mathrm{T}}v$ of what he believes is the value of the whole inheritance (v_j).
(g) Suppose a solution $(\tilde{u}, \tilde{x})$ such that

$$\min_{i\in N} \frac{\tilde{u}_i + \tilde{x}_i}{v_i} > \lambda = \frac{\iota^{\mathrm{T}}u}{\iota^{\mathrm{T}}v}$$

then $(\tilde{u}_i + \tilde{x}_i)/v_i > \lambda$ for every $i \in N$. So

$$\sum_{i\in N} \tilde{u}_i = \sum_{i\in N} (\tilde{u}_i + \tilde{x}_i) > \lambda \sum_{i\in N} v_i = \sum_{i\in N} u_i$$

therefore $\iota^{\mathrm{T}}\tilde{u} > \iota^{\mathrm{T}}u$. But this is impossible, because u came from the highest bidders. □

In particular, (g) proves that there is no solution where everybody gets a bigger percentage of the inheritance. Among all solutions, where each player receives the same percentage share of the inheritance in terms of his own valuation, Theorem 3 gives the one with the highest common percentage.

Some of the properties of Theorem 3 still hold for the more general solution of Theorem 4. For example, the percentage agent j gets is no lower than α_j and the solution is homogenous of degree 1.

Formally, we can say that the procedure has the population monotonicity property if $N \subseteq N'$ then $\varphi_i(M, N') \leq \varphi_i(M, N)$ for every $i \in N$, but neither the solution of Theorem 1 nor the solution of Theorem 3 has it: Consider $M = \{1\}$, $N = \{1, 2\}$, $N' = \{1, 2, 3\}$ and $A = [2, 3, 20]$ then we get $\varphi(M, N) = (1\frac{1}{5}, 1\frac{4}{5})$ and $\varphi(M, N') = (1\frac{1}{3}, 2, 13\frac{1}{3})$ using the solution of Theorem 1 and $\varphi(M, N) = (1\frac{1}{5}, 1\frac{4}{5})$ and $\varphi(M, N') = (1\frac{3}{5}, 2\frac{2}{5}, 16)$ if we use the solution of Theorem 3. The reason is that agent 3 is the only one that is willing to pay a big amount for the item.

Now, suppose the rules had been explained to the agents and consider the moment they have to declare their item's value. They can lie if they want to, but if one of them says that an item has a lower value than he believes it has, then it might happen that another agent

gets the item for a lower price than the one he is willing to pay. And if he says the item has a bigger value then he take a risk of getting the item at this price.

Both procedures could be applied even if some of the agents feel they do not have enough money to pay an overassignment of goods. If one agent does not have enough money, he can invite another person, such as an antique dealer, to present a bid for each item. This agent could present the highest value between what he can afford and the bid of the other person. Following this strategy, agents could hide his preferences from the others and discourage them to manipulate the outcome. Moreover, the agents could invite jointly more people to participate in the process with a corresponding zero percent over the rights of the inheritance.

Implicit in the model, we are assuming that agents value a collection of goods by adding their individual worts and also that they have a quasi-linear utilities, i.e. linear utilities in money.

Besides, both procedures could be used to solve the chore division problem (this problem was posed by Gardner (1978, p. 124)): suppose the agents need to divide a set of chores. In this case, the agents would like to receive the smallest portion of the chores as possible. So, it is enough to request $a_{ij} < 0$ and to interpret $-a_{ij}$ as how much agent j is willing to receive for doing the chore i. Also, the problems of divorce could be solved without further modifications.

A solution is envy-free if no agent thinks that someone else has a better share. Clearly, the solutions presented here do not have this property, but it could be considered an advantage. In a problem with just one item and three or more agents, assuming an envy-free solution we need to assign the same monetary compensation to every agent that does not receive the item. In the case $A = (1, 1000, 1001)$, it is debatable that any couple of agents get the same monetary compensation. However, the procedures have a weaker property, neither the highest bidder envies the other agents, nor any other agent envies the highest bidder.

4. Further reading

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A salary system for the assignment problem

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CIMAT

Abstract

We propose a modification to the concept of the potential of agame à la Hart−Mas Colell to determine a salary system for theassignment problem. We obtain explicit formulas for the potentialof the assignment problem and for it's corresponding salarysystem. Also, we establish some properties of this salary systemand we give an interpretation in terms of the Shapley value.

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1. Introduction

Let us consider a finite set of workers, a set of jobs with equal cardinality and values a_{ij} which measure the efficiency of the assigning worker j to the job i. In the assignment problem each worker is to be assigned to a different job in such a way that the sum of corresponding efficiencies is maximum. In other words,

$$\begin{array}{lll} \max & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ x_{ij} & \\ \text{s.t.} & & \\ & \sum_{i=1}^{n} x_{ij} = 1 & j = 1,\ldots,n \\ & \sum_{j=1}^{n} x_{ij} = 1 & i = 1,\ldots,n \\ & x_{ij} = 0 \text{ or } 1 & \end{array}$$

where $x_{ij} = 1$ if the worker j is assigned to job i and $x_{ij} = 0$ otherwise.

In this work, we propose a fair salary system for the previous problem. We have three main assumptions: First, we suppose that the maximum efficiency of the system (the optimal value of the objective function) is equal to the amount to be shared in salaries among the workers. Alternatively, the total amount to be shared in salaries could be given exogenously and most of the results remain valid. Second, we only consider problems where the number of workers is equal to the number of jobs. Lastly, we suppose that each worker has the same right to be assigned to any of the jobs, so every worker must agree with the final assignment.

Notice that as a general rule it is not fair to pay according to the efficiency that each worker generates in the optimal solution since, in that case, some workers are usually sacrificed for the sake of global efficiency. For example in the optimal solution of the problem with the following matrix,

$$\begin{array}{c} \\ \text{Jobs} \end{array} \begin{array}{c} \text{Workers} \\ \left[\begin{array}{cc} 8 & 0 \\ 15 & 10 \end{array}\right] \end{array} \tag{1.1}$$

the first worker is assigned to job 1 and the second to job 2, however it is not fair to pay 8 units to the first worker and 10 units to the second since the first worker is more productive than the second worker, job by job. So, in order to define a salary for the workers it will be necessary to consider their relative abilities to do the different jobs.

Shapley and Shubik[4] suppose an optimal assignment of sellers to buyers in two-sided markets and proposed a solution based on linear programming to share the maximum total monetary payoff among sellers and buyers. In this work, we suppose an optimal assignment of a set of workers to a set of jobs and propose a solution to share the efficiency of the system only among the set of workers.

We use the concept of potential of a cooperative game to establish this salary system. Hart and Mas-Colell ([2] and [3]) established a new form to generate the value of a game. They start with the definition of a set of games in which a real function is defined, called the potential of the game. This function must be defined in such a way that the following two conditions are satisfied: i) the gain or participation of each player is equal to the difference in potential between the game that is considered and the one that results when he abandons the game, and, ii) the sum of gains or participations add to what all the players get in that game.

Surprisingly, these two conditions determine the gain or participation of the players and also these participations coincide with it's corresponding Shapley value.

For a brief revision of the concepts of cooperative games that are mentioned here, such as the Shapley value, see Appendix A (Finite games and their values) in Aumman and Shapley [1].

Now, we present the previous ideas formally. Let (N, v) be a cooperative game and let us denote by $G = \{(S, v_S) : S \subseteq N \text{ and } v_S = v|_S\}$ the family of subgames of (N, v). The potential of a game is defined as a function

$$P : G \to \mathbb{R}$$

such that

$$\sum_{j \in S} \left[P(S, v_S) - P(S \backslash \{j\}, v_{S \backslash \{j\}}) \right] = v(S) \tag{1.2}$$

with the condition that $P(\emptyset, v_\emptyset) = 0$. Furthermore, it is assumed that the loss of potential

$$P(S, v_S) - P(S \backslash \{j\}, v_{S \backslash \{j\}}) \tag{1.3}$$

when the player j abandons the game, is equal to the amount that corresponds to this player in the game (S, v_S).

Notice that (1.2) could be rewritten as

$$s \cdot P(S, v_S) = v(S) + \sum_{j \in S} P(S \backslash \{j\}, v_{S \backslash \{j\}}) \tag{1.4}$$

where $s = |S|$. This expression determines P and allows it's calculation in a recursive form. Moreover, it turns out that the difference (1.3) is equal to the Shapley value of player j in the game (S, v_S).

Furthermore, Hart and Mas-Colell give an explicit formula for the potential of a game:

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} v(S). \tag{1.5}$$

In section 2 we show that these ideas could be applied, with appropriate modification, to determine a salary system for the assignment problem. In section 3 we present some properties of the salary system proposed in this work. First, we see this salary system as a particular case of a more general framework and then we consider the case where the workers do not have any bargaining power.

2. Potential for the assignment problem

We assume that each worker is assigned to one and only one job. Let us also suppose that we have defined a potential on the set of this problems, i.e., a real function on this set.

We need to define the loss of potential when one worker abandons a problem, so we need also to eliminate a job. A priori it could be any job, so we propose the average loss of potential instead. Let us formalize this idea.

Denote by $\mathcal{M}$ the set of jobs, by $\mathcal{N}$ the set of workers and by $\mathcal{A}$ the matrix which entry a_{ij} represents the efficiency of the worker j doing the job i. Notice that rows of $\mathcal{A}$ are indexed by the elements of $\mathcal{M}$ and its columns by the elements in $\mathcal{N}$. We

suppose that $\mathcal{M}$ and $\mathcal{N}$ have the same cardinality and we will denote by $\mathcal{A}|(M,N)$ the matrix $\mathcal{A}$ restricted to the rows in M and columns in N. The problem,

$$\begin{array}{lll} \max & \sum_{i\in M}\sum_{j\in N} a_{ij}\ x_{ij} & \\ \text{s.t.} & & \\ & \sum_{i\in M} x_{ij} = 1 & j \in N \\ & \sum_{j\in N} x_{ij} = 1 & i \in M \\ & x_{ij} = 0 \text{ or } 1 & \end{array}$$

defined by the parameters $M \subseteq \mathcal{M}$, $N \subseteq \mathcal{N}$ and $A = \mathcal{A}|(M,N)$ will be denoted by (M,N,A). So that this does not produce confusion, we will use only (M,N) instead of (M,N,A) and sometimes we will represent it just by it's matrix A.

The set of problems that we will consider is as follows,

$$\mathcal{P} = \{(M,N,A) : M \subseteq \mathcal{M}, N \subseteq \mathcal{N} \text{ such that } |M| = |N| \text{ and } A = \mathcal{A}|(M,N)\},$$

It is well-known that these kinds of problems always have at least one optimal solution, therefore the following function is well defined,

$$L_P : \mathcal{P} \to \mathbb{R}$$

where $L_P(M,N)$ is the optimal value of the objective function for the problem (M,N). Let us denote

$$\overline{Pot(N\backslash\{j\})} = \sum_{i\in M} \frac{Pot(M\backslash\{i\}, N\backslash\{j\})}{n}.$$

Definition 1. *We define the potential of the assignment problem as a function,*

$$Pot : \mathcal{P} \to \mathbb{R}$$

such that $Pot(\emptyset,\emptyset) = 0$ *and*

$$\sum_{j\in N} [Pot(M,N) - \overline{Pot(N\backslash\{j\})}] = L_P(M,N). \tag{2.1}$$

As in Hart and Mas-Colell [2] and [3], we will suppose that

$$Pot(M,N) - \overline{Pot(N\backslash\{j\})}$$

is the salary which corresponds to the worker j in the problem (M,N). This number is the average of the loss of potential when the worker j abandons the system. It is easy to see that (2.1) is equivalent to the following recursive expression,

$$Pot(M,N) = \frac{n\ L_P(M,N) + \sum_{j\in N}\sum_{i\in M} Pot(M\backslash\{i\}, N\backslash\{j\})}{n^2} \tag{2.2}$$

which determines the potential of the assignment problem. We will denote indistinctly for $\varphi(M,N)$ or for $\varphi(A)$ the vector of salaries that corresponds to the problem (M,N,A), i.e., the vector with entry j is given by

$$\varphi_j(M,N) = Pot(M,N) - \overline{Pot(N\backslash\{j\})} \tag{2.3}$$

and we will call it a salary system for the assignment problem (M,N).

Example. In this example, we calculate the salary system for the problem with the following matrix,

$$\begin{bmatrix} 9 & 10 & 2 \\ 7 & 8 & 6 \\ 10 & 1 & 3 \end{bmatrix} \tag{2.4}$$

For simplicity of the exposition we will represent the problem (M, N, A) in the following form:

$$A^{L_P(M,N)}_{Pot(M,N)}$$

In order to calculate $Pot(M, N)$ using (2.2) we need to generate all sub problems as the result of removing one worker and one job. We repeat this process in each case until there are no workers. Then we calculate L_P for each of the obtained problems and their potential beginning with the lower cardinality. Notice that for matrices with one element, say $A = [a]$, $L_P(M, N) = Pot(M, N) = a$. While if the matrix is 2×2, then $Pot(M, N)$ is equal to the sum of the entries of the matrix, plus $2L_P(M, N)$, all divided by 4. Thus, the problems that are derived from (2.4) when one worker is missing are:

When worker 1 is missing:

$$\begin{bmatrix} 10 & 2 \\ 8 & 6 \end{bmatrix}^{16}_{14.5} \quad \begin{bmatrix} 10 & 2 \\ 1 & 3 \end{bmatrix}^{13}_{10.5} \quad \begin{bmatrix} 8 & 6 \\ 1 & 3 \end{bmatrix}^{11}_{10} \tag{2.5}$$

When worker 2 is missing:

$$\begin{bmatrix} 9 & 2 \\ 7 & 6 \end{bmatrix}^{15}_{13.5} \quad \begin{bmatrix} 9 & 2 \\ 10 & 3 \end{bmatrix}^{12}_{12} \quad \begin{bmatrix} 7 & 6 \\ 10 & 3 \end{bmatrix}^{16}_{14.5} \tag{2.6}$$

When worker 3 is missing:

$$\begin{bmatrix} 9 & 10 \\ 7 & 8 \end{bmatrix}^{17}_{17} \quad \begin{bmatrix} 9 & 10 \\ 10 & 1 \end{bmatrix}^{20}_{17.5} \quad \begin{bmatrix} 7 & 8 \\ 10 & 1 \end{bmatrix}^{18}_{15.5} \tag{2.7}$$

The potential of the problem with matrix (2.4) is the result of adding 3 times the optimal value of the objective function $(3 \cdot 26)$ with the potential of each of the nine previous problems and divide the result by 9. Thus,

$$\begin{bmatrix} 9 & 10 & 2 \\ 7 & 8 & 6 \\ 10 & 1 & 3 \end{bmatrix}^{26}_{22.555}$$

Now, we can calculate the salaries for the problem with matrix (2.4) using (2.3).

$$\begin{aligned} \overline{Pot(S\backslash\{1\})} &= (14.5 + 10.5 + 10)/3 = 11.667 \\ \overline{Pot(S\backslash\{2\})} &= (13.5 + 12 + 14.5)/3 = 13.333 \\ \overline{Pot(S\backslash\{3\})} &= (17 + 17.5 + 15.5)/3 = 16.667 \end{aligned}$$

giving us

$$\varphi(A) = \begin{bmatrix} 22.555 - 11.667 \\ 22.555 - 13.333 \\ 22.555 - 16.667 \end{bmatrix} = \begin{bmatrix} 10.888 \\ 9.222 \\ 5.888 \end{bmatrix}.$$

As another example, the wages for the problem given in the introduction (see 1.1) are 12.25 for worker 1 and 5.75 for worker 2.

3. Properties of the potential of the assignment problem

The next theorem establishes an explicit expression for the potential of the assignment problem. We will denote by

$$\widehat{L_M(S)} = \sum_{\{R\subseteq M:|R|=|S|\}} L_P(R,S). \tag{3.1}$$

Theorem 1. *For each $(M,N)\in\mathcal{P}$ we have that*

$$Pot(M,N) = \sum_{S\subseteq N} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}. \tag{3.2}$$

Theorem 2. *The salary for worker j in the assignment problem (M,N) is given by*

$$\varphi_j(M,N) = \sum_{\{S\subseteq N:j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \left[\widehat{L_M(S)} - \widehat{L_M(N\backslash S)}\right].$$

The previous theorem provides us an alternative way to calculate the salary system for the assignment problem. Using (3.1) and (2.5) - (2.7) we have that

$$\begin{aligned}
&\widehat{L_M(\{1,2,3\})} = 26\\
&\widehat{L_M(\{1,2\})} = 17+20+18 = 55\\
&\widehat{L_M(\{1,3\})} = 15+12+16 = 43\\
&\widehat{L_M(\{2,3\})} = 16+13+11 = 40\\
&\widehat{L_M(\{1\})} = 9+7+10 = 26\\
&\widehat{L_M(\{2\})} = 10+8+1 = 19\\
&\widehat{L_M(\{3\})} = 2+6+3 = 11
\end{aligned}$$

and therefore,

$$\varphi_1(A) = \frac{1}{9}(26-40)+\frac{1}{18}(55-11)+\frac{1}{18}(43-19)+\frac{1}{3}26 = 10.888.$$

The next result establishes a relation between the salary system given by (2.3) and the Shapley value. Let us denote by

$$\Theta = \{\theta : N\to M | \theta \text{ bijective}\}$$

the set of the $n!$ possible ways to assign the jobs to the workers. Let us select $\theta\in\Theta$ randomly (each case with probability $\frac{1}{n!}$), in such a way that worker j gets the job $\theta(j)$ and suppose the workers have the possibility of exchanging their assigned jobs. The process of bargaining generates a cooperative game in a natural way: when the coalition of workers S is formed they can generate an efficiency $v^\theta(S) = L(\theta(S),S)$. Thus, when they distribute the efficiency according to the value ϕ, they get an average $\frac{1}{n!}\sum_{\theta\in\Theta}\phi(v^\theta)$. The next theorem establishes that the salary system coincides with the previous process when they use the Shapley value. We will denote by $Sh(N,v)$ the Shapley value of the game (N,v).

Theorem 3. *For $A \in \mathcal{P}$ we have that*

$$\varphi(A) = \frac{1}{n!}\sum_{\theta\in\Theta} Sh(N, v^\theta) = Sh(N, \frac{1}{n!}\sum_{\theta\in\Theta} v^\theta).$$

The following theorem establishes some properties of the salary system. The first property is that the solution does not change when re-naming the jobs. The second one establishes the symmetry with regard to the workers, in particular for two workers with corresponding equal columns, the same salary is obtained. Also we have a change in the scale of efficiency that gives us an equivalent change in the salary. Lastly, we establish a weak monotony property with respect to the efficiency of the workers and a logical solution for permutation matrices.

Let P be a permutation matrix, i.e., a $n \times n$ matrix where $n-1$ of the entries of each column and of each row are zero and the other one is equal to one. It is well known that for any matrix A, PA is a permutation of rows of A and AP is a permutation of the columns of A. Let $\mathbf{1}$ be the column vector in $\mathbb{R}^n$ of all 1's and A_j the column j of the matrix A.

Theorem 4. *For $A \in \mathcal{P}$ we have that*

- **a)** $\varphi(PA) = \varphi(A)$ *for every permutation matrix* P.
- **b)** $\varphi(AP) = P\varphi(A)$ *for every permutation matrix* P.
- **c)** $\varphi(\lambda A) = \lambda\varphi(A)$ *for every real number* λ.
- **d)** *If* $A_k \leq A_j$ *then* $\varphi_k(A) \leq \varphi_j(A)$.
- **e)** $\varphi(P) = \mathbf{1}$ *for every permutation matrix* P.

In general, we have that $\varphi(diag(d)) \neq d$, where d is a vector and $diag(d)$ it's corresponding diagonal matrix.

It may also happen for the bargaining power of the workers that a positive salary is assigned to a worker with zero efficiency in all jobs. For example in the problem

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}^2_{\frac{7}{4}}$$

we get the salary system $(\frac{7}{4}, \frac{7}{4} - \frac{3}{2})^T = (\frac{7}{4}, \frac{1}{4})^T$. This could be explained with Theorem 3, with one of the $\theta \in \Theta$ there is no bargaining but with the other, worker 2 has some bargaining power.

4. Appendix

Proof of Theorem 1. The proof is by induction on the common cardinality of M and N. For $|N| = |M| = 1$, both (2.2) and (3.2) are equal to $L_P(M, N)$. Let us suppose that (2.2) and (3.2) are equal for $(M\backslash\{i\}, N\backslash\{j\})$ for every $i \in M$ and $j \in N$. Then,

$$\frac{1}{n^2}\sum_{i\in M}\sum_{j\in N} Pot(M\backslash\{i\}, N\backslash\{j\}) =$$

$$= \frac{1}{n^2}\sum_{i\in M}\sum_{j\in N}\sum_{S\subseteq N\backslash\{j\}}\sum_{\{R\subseteq M\backslash\{i\}:|R|=s\}} \frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R, S)$$

Now, since the double sum $\sum_{i\in M}\sum_{\{R\subseteq M\backslash\{i\}:|R|=s\}}$ is equivalent to $\sum_{\{R\subseteq M:|R|=s\}}\sum_{i\in M\backslash R}$ and the double sum $\sum_{j\in N}\sum_{S\subseteq N\backslash\{j\}}$ is equivalent to $\sum_{\{S\subset N:S\neq N\}}\sum_{j\in N\backslash S}$ we have that

$$= \frac{1}{n^2} \sum_{\{S\subset N: S\neq N\}} \sum_{j\in N\setminus S} \sum_{\{R\subseteq M:|R|=s\}} \sum_{i\in M\setminus R} \frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

$$= \sum_{\{S\subset N: S\neq N\}} \sum_{\{R\subseteq M:|R|=s\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 L_P(R,S) = Pot(M,N) - \frac{1}{s}L_P(M,N)$$

□

Proof of Theorem 2. By the previous theorem we have that

$$\frac{1}{n}\sum_{i\in M} Pot(M\setminus\{i\}, N\setminus\{j\}) = \frac{1}{n}\sum_{i\in M}\sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subseteq M\setminus\{i\}:|R|=s\}} \frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

Since the double sum $\sum_{i\in M}\sum_{\{R\subseteq M\setminus\{i\}:|R|=s\}}$ is equivalent to $\sum_{\{R\subset M:|R|=s\}}\sum_{i\in M\setminus R}$ we have that

$$= \sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subset M:|R|=s\}} \frac{(n-s)}{n}\frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

$$= \sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subset M:|R|=s\}} \frac{n}{n-s}\frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 L_P(R,S)$$

$$= \sum_{S\subseteq N\setminus\{j\}} \frac{n}{n-s}\frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

therefore

$$\varphi_j(M,N) = Pot(M,N) - \frac{1}{n}\sum_{i\in M} Pot(M\setminus\{i\}, N\setminus\{j\})$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} + \sum_{S\subseteq N\setminus\{j\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$- \sum_{S\subseteq N\setminus\{j\}} \frac{n}{n-s}\frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} +$$

$$+ \sum_{S\subseteq N\setminus\{j\}} \left[1 - \frac{n}{n-s}\right]\frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} - \sum_{S\subseteq N\setminus\{j\}} \frac{1}{n-s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} - \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(N\setminus S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \left[\widehat{L_M(S)} - \widehat{L_M(N\setminus S)}\right]$$

□

Proof of Theorem 3. Let μ be the cooperative game given by

$$\mu(S) = \frac{1}{n!} \sum_{\theta \in \Theta} v^{\theta}(S)$$

then

$$\mu(S) = \frac{1}{n!} \sum_{\theta \in \Theta} L(\theta(S), S)$$

$$= \sum_{\{R \subseteq M : |R| = s\}} \frac{s!(n-s)!}{n!} L(R, S)$$

Now, by (3.2) we have that

$$Pot(M, N) = \sum_{S \subseteq N} \sum_{\{R \subseteq M : |R| = s\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!} \right]^2 L(R, S)$$

$$= \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} \sum_{\{R \subseteq M : |R| = s\}} \frac{s!\,(n-s)!}{n!} L(R, S)$$

$$= \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} \mu(S) = P(N, \mu)$$

which is the potential of the game μ.

In a similar way

$$\overline{Pot(N \backslash \{j\})} = \frac{1}{n} \sum_{i \in M} Pot(M \backslash \{i\}, N \backslash \{j\}) =$$

$$= \frac{1}{n} \sum_{i \in M} \sum_{S \subseteq N \backslash \{j\}} \sum_{\{R \subseteq M \backslash \{i\} : |R| = s\}} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!} \right]^2 L(R, S)$$

$$= \sum_{S \subseteq N \backslash \{j\}} \frac{1}{n} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!} \right]^2 \sum_{i \in M} \sum_{\{R \subseteq M \backslash \{i\} : |R| = s\}} L(R, S)$$

$$= \sum_{S \subseteq N \backslash \{j\}} \frac{1}{n} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!} \right]^2 \sum_{\{R \subseteq M : |R| = s\}} (n-s) L(R, S)$$

$$= \sum_{S \subseteq N \backslash \{j\}} \frac{(s-1)!\,(n-s-1)!}{(n-1)!} \frac{s!\,(n-s)!}{n!} \sum_{\{R \subseteq M : |R| = s\}} L(R, S)$$

$$= \sum_{S \subseteq N \backslash \{j\}} \frac{(s-1)!\,(n-s-1)!}{(n-1)!} \mu(S) = P(N \backslash \{j\}, \mu_{N \backslash \{j\}})$$

□

Proof of Theorem 4.

a) It follows by Theorem 1 and using the fact that

$$\widehat{L_M(S)} = \widehat{L_{\theta(M)}(S)}$$

for every $\theta \in \Theta$.

b) It follows by $\varphi(A) = Sh(N, \frac{1}{n!} \sum_{\theta \in \Theta} v^{\theta})$ and the symmetry of the Shapley value.

c) The proof is straightforward since

$$L_P(R, S, \lambda B) = \lambda L_P(R, S, B).$$

d) Notice that the marginal contributions are monotonic, so, d) follows by Theorem 3.

e) Using b) and a) we have that

$$P\varphi(I) = \varphi(IP) = \varphi(P) = \varphi(PI) = \varphi(I)$$

Thus, $\varphi_i(I) = \varphi_j(I)$ for every $i, j \in N$. By the efficiency property $\varphi_j(I) = 1$ for every $j \in N$. □

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ORIGINAL PAPER

Dissection of solutions in cooperative game theory using representation techniques

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Abstract We compute a decomposition for the space of cooperative TU-games under the action of the symmetric group S_n. In particular we identify all irreducible subspaces that are relevant to the study of symmetric linear solutions – namely those that are isomorphic to the irreducible summands of $\mathbb{R}^n$. We then use such decomposition to derive, in a very economical way, some old and some new results for linear symmetric solutions.

1 Introduction

In this article we study linear, symmetric solutions for the space of cooperative TU games with n players using basic representation theory of the group of permutations S_n.

Representation theory is a general tool for organizing linear algebra data in the presence of a group of symmetries. It makes sense to use it, first, as a "bookkeeping" tool, converting arguments that would typically require some ingenuity on the part of the researcher into routine exercises. More importantly, it presents the information in a more clear and concise way, thus shedding new light into the relations (sometimes hidden until then) between the elements

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that participate in a given problem. We believe we have shown how this is true in the context of linear symmetric solutions in cooperative game theory.

Briefly, what we do is the following. We derive direct sum decompositions of the space of games and the space of payoffs into "elementary" pieces. Moreover, any linear, symmetric solution when restricted to any such elementary subspace is either zero or multiplication by a single scalar, regardless of the dimension of the elementary subspace – this follows from the so called Schur's lemma; therefore, all linear, symmetric solutions may be written (simultaneously) as a sum of trivial maps.

Once we have such a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by other conditions or axioms, for example: efficiency, dummy player axiom, self and anti-self-duality, etc.

The reader will find here new proofs, following the above scheme, of well known results as well as new theorems and characterizations for certain classes of linear symmetric solutions. Besides presenting these results, one of the main objectives of the present work is to advertise representation theory as a natural tool for research in cooperative game theory. We believe it is natural and powerful and nevertheless its use has been neglected (with the notable exception of Kleinberg and Weiss (1985), we have not been able to find a single reference where the theory is used).

2 Framework

Consider $\mathbb{R}^n$, the space of payoff vectors for n players. Every permutation σ of $N = \{1, 2, \ldots, n\}$ may be thought of as a linear map $L_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by permuting the coordinates of any vector $x \in \mathbb{R}^n$ according to σ. The assignment $\sigma \mapsto L_\sigma$ is called a representation, since we represent each permutation by a certain linear map. One can also say that there is a linear "action" on $\mathbb{R}^n$ by the group of permutations.

From linear algebra we know that the matrix of the linear map L_σ may have a simple block decomposition with respect to some basis of $\mathbb{R}^n$ (e.g., L_σ might be diagonalizable). But we are not really interested in only finding a basis that will set a single map L_σ in a simpler form. What we want is to write them all in a simpler way at the same time.

For example, consider the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, then $\mathbf{1}$ is at the same time an eigenvector to every linear map L_σ, for all σ, since permuting its coordinates does nothing to it. Let $\Delta_n = \{(t, t, \ldots, t) \mid t \in \mathbb{R}\}$ be the diagonal generated by $\mathbf{1}$; we say that Δ_n is "trivial", in the sense that every map L_σ is the identity map when restricted to Δ_n .

Look also at the orthogonal complement to $\mathbf{1}$, which we denote by $\Delta_n^\perp$. Clearly $\Delta_n^\perp = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n y_i = 0\}$. Now, if we pick $y \in \Delta_n^\perp$ and apply to it any linear map L_σ we obtain another vector also inside of $\Delta_n^\perp$; that is

$$y \in \Delta_n^\perp \Rightarrow L_\sigma(y) \in \Delta_n^\perp, \qquad \text{for every } \sigma,$$

since permuting the coordinates of a vector does not change the fact that their sum is zero. We say that $\Delta_n^\perp$ is an invariant subspace.

Thus, we can "decompose" $\mathbb{R}^n$ as a direct sum, $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$, in such a way that each summand is "invariant". So if we were to choose a basis of $\mathbb{R}^n$ consisting of $\mathbf{1}$ and a basis of $\Delta_n^\perp$, then all L_σ would have a matrix, with respect to that basis, of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A_\sigma \end{pmatrix}.$$

We show in the Appendix – although it is perhaps intuitively clear – that there is no further decomposition; that is, the subspace $\Delta_n^\perp$ does not contain any invariant subspace smaller than itself (and different to $\{0\}$). We say that $\Delta_n^\perp$ is "irreducible".

The space of cooperative games $G = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ is also a vector space and the group S_n has a natural action on G as follows: Given a permutation σ let $T_\sigma : G \to G$ be the linear map defined by

$$[T_\sigma(v)](S) = v(\sigma^{-1}S)$$

for every game $v \in G$ and every coalition $S \subset N$, where $\sigma^{-1}S$ is the coalition that contains a player i if and only if σ_i is a player in S.

The main result of Sect. 3 is a statement about a decomposition of G into invariant subspaces for this action.

We first observe the following obvious decomposition

$$G = \bigoplus_{j=1}^{n} G_j,$$

where G_j consists of those games that vanish on every coalition not containing exactly j players. Clearly, every G_j is invariant under all permutations of the players.

Let us identify certain types of games within each G_j, $j < n$ (G_n is a 1-dimensional trivial subspace generated by the game that assigns 1 to the grand coalition, and zero to every other one). For each $j < n$, and each $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ define the game $x^j \in G_j$ as follows

$$x^j(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = j; \\ 0 & \text{if } |S| \neq j. \end{cases}$$

Let $h_j : \mathbb{R}^n \to G_j$ denote the map $h_j(x) = x^j$; h_j is a linear 1–1 map that commutes with the actions of S_n, that is, $h_j \circ L_\sigma = T_\sigma \circ h_j$.

Set $C_j = h_j(\Delta_n)$ and $U_j = h_j(\Delta_n^\perp)$. We note that h_j is an isomorphism between U_j and $\Delta_n^\perp$ (similarly, between C_j and Δ_n) since it is a linear map which is 1–1 and onto and it commutes with the respective actions of S_n. Isomorphic spaces

are indistinguishable from the point of view of linear algebra together with an action of the group of permutations. Thus we may consider C_j as the same as Δ_n and U_j same as $\Delta_n^\perp$. In particular, both spaces are irreducible.

Typically (as soon as $n > 3$ and $1 < j < n-1$), within G_j there are games not necessarily lying in $C_j \oplus U_j$. In G we have defined a natural inner product:

$$\langle v_1, v_2 \rangle = \sum_{S \subset N} v_1(S) v_2(S);$$

moreover, all T_σ are orthogonal transformations with respect to this inner product: $\langle T_\sigma(v_1), T_\sigma(v_2) \rangle = \langle v_1, v_2 \rangle$, for all $v_1, v_2 \in G$ (when this happens it is said that the inner product is invariant under the action).

Call W_j the orthogonal complement to $C_j \oplus U_j$ within G_j. We have

$$G_j = C_j \oplus U_j \oplus W_j,$$

and we have arrived to the desired decomposition

$$G = C_1 \oplus \cdots \oplus C_n \oplus U_1 \oplus \cdots \oplus U_{n-1} \oplus W,$$

where $W = \bigoplus_{j=1}^{n-1} W_j$, and every C_j is isomorphic to Δ_n and every U_j is isomorphic to $\Delta_n^\perp$. Note that $C = C_1 \oplus \cdots \oplus C_n$ is precisely the space of symmetric games (that is, games whose values depend only on the cardinality of the given coalition).

Each W_j could be further decomposed so that in the end we have expressed W as a sum of irreducible subspaces. Now, the main theorem (Proposition 1) asserts that none of the irreducible summands of W is isomorphic to either Δ_n or to $\Delta_n^\perp$.

We use the above decomposition to study solutions, more precisely we look at solutions $\phi : G \to \mathbb{R}^n$ which are linear and symmetric (i.e., $\phi \circ T_\sigma = L_\sigma \circ \phi$, $\forall \sigma$). Rather than keep on carrying the L_σ's and T_σ's, one abuses notation and denotes the linear transformations by the same letter σ; thus, symmetry is simply expressed as

$$\phi(\sigma \cdot v) = \sigma \cdot \phi(v).$$

Schur's lemma (see the Appendix for a precise statement) implies that every linear, symmetric ϕ is zero on W, is a multiple of $h_j^{-1} : C_j \to \Delta_n$ when restricted to C_j and is a multiple of $h_k^{-1} : U_k \to \Delta_n^\perp$ when restricted to U_k. This is the punch line of the decomposition theorem.

As a result, the block decomposition (relative to the decomposition of G) of every symmetric, linear solution

$$\phi : G = C_1 \oplus \cdots \oplus C_n \oplus U_1 \oplus \cdots \oplus U_{n-1} \oplus W \to R^n = \Delta_n \oplus \Delta_n^\perp$$

is as simple as possible: with most entries zeroes, and where the non-zero blocks are all diagonal and multiples of the identity matrix:

$$\phi = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_1 I_{n-1} & \cdots & \mu_{n-1} I_{n-1} & 0 & \cdots & 0 \end{pmatrix}.$$

In particular, the space of all such solutions is $(2n-1)$-dimensional and W is the common kernel of all of them.

The rest of the article consists of using this information to study solutions satisfying further hypotheses.

In Sect. 4 we give some applications of this method. First we add to linearity and symmetry, the efficiency axiom and characterize very easily all such solutions:

A linear symmetric solution is efficient if and only if its restriction to the symmetric games C coincides with the egalitarian solution.

Next, we study and characterize the linear symmetric solutions that further satisfy the null (or dummy player) axiom.

In Sect. 4.3, we turn to Shapley's value and its relationship to additive games. It turns out that there is an invariant inner product on G (should this be the "natural" inner product on G?) that makes Shapley's value, Sh, the adjoint to the additive games map, i.e., the map $\mathbb{R}^n \to G$ such that

$$x \mapsto [S \mapsto x(S) = \sum_{j \in S} x_j].$$

Equivalently, there is an invariant inner product on G such that Shapley's value is the same as orthogonal projection – with respect to this inner product – onto the subspace of additive games.

Another application is to the study of the notion of self-duality. Recall that the duality operator on games $* : G \to G$, is defined by $(*v)(S) = v(N) - v(N \setminus S)$. A self-dual solution is one for which $\phi(*v) = \phi(v)$, for every game v. We study and characterize self-duality, and its relationship to the other most common axioms (efficiency and nullity).

The last section of the article is devoted to a few results regarding the kernel of linear symmetric solutions: we compute the common kernel of all linear, symmetric and efficient solutions; we give an expression for the kernel of any given linear symmetric solution, in particular we compute it for the Shapley value.

3 Group representation preliminaries

Precise definitions and some proofs for this section may be found in the Appendix at the end of the article. Nevertheless, for the sake of easier reading we repeat here a few definitions and give an idea of proofs, sometimes in a less rigorous but more accessible manner.

Let $N = \{1, \ldots, n\}$ be the set of players. Let $G = G^{(n)} = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ be the real vector space of games in n players.

The group of permutations of N, S_n, acts naturally on G via linear transformations (i.e., G is a representation of S_n). That is, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space G; namely, let

$$(\theta \cdot v)(S) := (\rho(\theta)(v))(S) = v(\theta^{-1} \cdot S)$$

for every $\theta \in S_n$, $v \in G$ and $S \subset N$, where $\theta \cdot S$ is the set obtained from S by permuting its elements according to θ (i.e., $\theta \cdot S = \{\theta_i := \theta(i) \mid i \in S\}$).

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\theta\sigma$ is the product (or composition) of the maps corresponding to θ and σ, in that order. We will sometimes say that S_n acts (linearly) on G.

Similarly, $\mathbb{R}^n$ is also a representation space for S_n:

$$\theta \cdot (x_1, x_2, \ldots, x_n) = (x_{\theta_1}, x_{\theta_2}, \ldots, x_{\theta_n}).$$

Definition 1 *A linear symmetric solution ϕ is a linear map $\phi : G \to \mathbb{R}^n$ that is symmetric in the following sense: for every $\theta \in S_n$ and $v \in G$ we have that*

$$\phi(\theta \cdot v) = \theta \cdot \phi(v).$$

In other words, a symmetric ϕ "commutes" with the actions on the domain and range of ϕ.

We denote by $\mathcal{LS}(G)$ the vector space of all linear symmetric solutions on G.

In the language of representation theory, what we are calling a linear, symmetric map is usually referred to as an S_n-equivariant map.

3.1 Decomposition of G under S_n

Definition 2

- *A subspace V of G or $\mathbb{R}^n$ is invariant (for the action of S_n) if for every vector $v \in V$ and every permutation $\theta \in S_n$ we have that*

$$\theta \cdot v \in V.$$

- *A subspace V of G or $\mathbb{R}^n$ is irreducible if V itself has no invariant subspaces other than $\{0\}$ and V itself.*

We begin with the decomposition of $\mathbb{R}^n$ into irreducible representations, which is easier, and then proceed to do the same thing for G; that is, we wish to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations in S_n

and in such a way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, set $\mathbf{1} = (\overbrace{1, 1, \ldots, 1}^{n}) \in \mathbb{R}^n$, $\Delta_n = \{(t, t, \ldots, t) \in \mathbb{R}^n\} = \mathbb{R}\mathbf{1}$ and $\Delta_n^{\perp} = \{x \in \mathbb{R}^n \mid x \cdot \mathbf{1} = 0\}$ the orthogonal complement to the diagonal Δ_n. The space $\Delta_n^{\perp}$ is usually called the "*standard representation*" of S_n. Notice that Δ_n is a "trivial" subspace in the sense that every permutation acts as the identity transformation.

Every permutation fixes every element of the diagonal line Δ_n, so, in particular, this line is an invariant subspace of $\mathbb{R}^n$. Being 1-dimensional, it is automatically irreducible. Its orthogonal complement, $\Delta_n^{\perp}$, consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence $\Delta_n^{\perp}$ is also an invariant subspace. The next result tells us that this subspace is also irreducible.

Lemma 1 *The decomposition of $\mathbb{R}^n$, under S_n, into irreducible subspaces is*

$$\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}.$$

Thus, Lemma 1 tells us that $\mathbb{R}^n$ as a vector space with group of symmetry S_n as defined above, can be written as an orthogonal sum of two subspaces (Δ_n and $\Delta_n^{\perp}$) which are invariant under permutations and which can no longer be further decomposed.

The proof of this lemma is an induction argument that can be found in the Appendix.

For each $j : 1, \ldots, n$, let $G_j = \{v \in G \mid v(S) = 0 \text{ if } |S| \neq j\}$. G_j is a vector subspace of G and, moreover, $G = \bigoplus_{j=1}^{n} G_j$, each G_j is invariant under S_n and the direct sum is orthogonal with respect to the invariant inner product on G given by $\langle v, w\rangle = \sum_{S \subset N} v(S)w(S)$. [1]

Here, invariance of the inner product means that every permutation $\theta \in S_n$ is not only a linear map of G, but an orthogonal map with respect to this inner product. Formally,

$$\langle \theta \cdot v, \theta \cdot w\rangle = \langle v, w\rangle$$

for every $v, w \in G$.

The following games play an important role in describing the decomposition of the space of games G.

For each $j : 1, \ldots, n$ define $c_j \in G_j$ as follows

$$c_j(S) = \begin{cases} 1 & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

[1] This seems like the natural inner product to consider. Nevertheless, later on we will see that there is another choice for which the Shapley value can be characterized as the adjoint of the map $\mathbb{R}^n \to G$ that takes $x \in \mathbb{R}^n$ to the game $\hat{x}(S) = \sum_{i \in S} x_i$.

Notice that $G_n = \mathbb{R}c_n$.

Also, for each $j : 1, \ldots, n$, and for each $x \in \mathbb{R}^n$, let $x^j \in G_j$ be given by

$$x^j(S) = \begin{cases} x(S) & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

where $x(S) = \sum_{i \in S} x_i$.

Definition 3 *Suppose V_1 and V_2 are two representations for the group S_n i.e., we have two vector spaces V_1 and V_2 where the group S_n is acting by linear maps. We say that V_1 and V_2 are isomorphic if there is a linear map between them, which is* 1–1 *and onto and that commutes with the respective S_n-actions. Formally, there is an invertible linear map $h : V_1 \to V_2$, such that*

$$h(\theta \cdot v_1) = \theta \cdot (h(v_1)) \qquad \textit{for all } \theta \in S_n.$$

We then write $V_1 \simeq V_2$.

For our purposes, V_1 will be an irreducible subspace of G and V_2 an irreducible subspace of $\mathbb{R}^n$

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions are equivalent under some linear invertible map between them. A concrete example may be found in the Appendix.

Proposition 1 *For $j < n$,*

$$G_j = C_j \oplus U_j \oplus W_j,$$

where $C_j = \mathbb{R}c_j \simeq \mathbb{R}$, $U_j = \{x^j \mid x \in \Delta_n^\perp\} \simeq \Delta_n^\perp$ and W_j does not contain any summands isomorphic to either $\mathbb{R}$ nor $\Delta_n^\perp$. The decomposition is orthogonal.

Let us give an idea of how the proof goes; the complete proof of this proposition can be found at the end of the Appendix as the proof of Proposition 9.

We define the map $T_j : \mathbb{R}^n \to G_j$ by $T_j(x) = x^j$. This map is linear (and S_n-equivariant) and 1–1. From Lemma 1 we have the splitting $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$. Thus, inside of G_j, we have the images of these two subspaces: $C_j = T_j(\Delta_n)$ and $U_j = T_j(\Delta_n^\perp)$.

Denote by W_j the orthogonal complement to $C_j \oplus U_j$ within G_j. The last, and hardest, part is to show that W_j does not contain any summands isomorphic to either C_j or U_j.

Proposition 1 does not quite give us a decomposition of G_j into irreducible summands. The subspaces C_j and U_j are irreducible (each isomorphic to Δ_n and $\Delta_n^\perp$, respectively) and together give us a copy of $\mathbb{R}^n$ inside of G. Whereas W_j may or may not be irreducible (depending on j and N), but as we shall see the exact nature of this subspace plays no role in the study of linear, symmetric solutions since (as it will be promptly proved) it lies in the kernel of any such solution.

Proposition 1 gives us a decomposition of the space of games that is a key ingredient in our subsequent analysis.

Set $C = \bigoplus_{j=1}^{n} C_j$. This is the space of **symmetric** games, i.e., games whose value on a given set depends only on its cardinality. According to Proposition 1, C is the largest subspace of G where S_n acts trivially. Let $U = \bigoplus_{j=1}^{n-1} U_j$ and $W = \bigoplus_{j=1}^{n-1} W_j$. Then

$$G = C \oplus U \oplus W.$$

The following result gives a good example of how Proposition 1 is to be used (in conjunction with Schur's Lemma) to gain information about linear symmetric solutions.

Corollary 1

- *Every linear symmetric solution vanishes in W.*
- $\dim \mathcal{LS}(G) = 2n - 1$.

Proof Let $\phi : C \oplus U \oplus W \to \Delta_n \oplus \Delta_n^{\perp}$ be a linear symmetric solution. Assume $X \subset W$ is an irreducible summand in the decomposition of W (even while we do not know the decomposition of W as a sum of irreducible subspaces, it is known that such a decomposition exists). Let $\pi_\alpha, \alpha \in \{1, 2\}$, denote orthogonal projection of $\mathbb{R}^n$ onto Δ_n and $\Delta_n^{\perp}$, respectively. Now, $\phi : G \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$, may be written as $\phi = (\pi_1 \circ \phi, \pi_2 \circ \phi)$. Denote by $\iota : X \to G$ the inclusion, then, the restriction of ϕ to X may be expressed as

$$\phi_{|X} = \phi \circ \iota = (\pi_1 \circ \phi \circ \iota, \pi_2 \circ \phi \circ \iota).$$

Now, $\pi_\alpha \circ \phi \circ \iota$ is a linear symmetric map from X to either Δ_n or $\Delta_n^{\perp}$; according to Proposition 1, X is not isomorphic to either of these two spaces, thus Schur's Lemma (see Appendix for the statement) says that $\pi_\alpha \circ \phi \circ \iota$ must be zero. Since this is true for every irreducible summand X of W, ϕ is zero on all of W.

Schur's Lemma also implies that ϕ maps each C_j into Δ_n and each U_j into $\Delta_n^{\perp}$, and that its restriction to each C_j or U_j is unique up to multiplication by a scalar (i.e., any two linear symmetric solutions when restricted to C_j – or U_j – differ only by multiplication by a constant).

So, define the following linear symmetric solutions. For $j : 1, \ldots, n$ and $k : 1, \ldots, n-1$, we set

$$\phi_j(c_l) = \delta_{jl},\ \phi_{j_{|U \oplus W}} \equiv 0,$$

$$\psi_k(x^l) = \delta_{kl} x,\ \psi_{k_{|C \oplus W}} \equiv 0$$

and where δ_{jl} is Kronecker's delta.

Clearly $\{\phi_j\} \cup \{\psi_k\}$ is a linearly independent set. Moreover, as discussed above, Schur's Lemma tells us that for any $\phi \in \mathcal{LS}(G)$, every $j : 1, \ldots, n$ and

$k : 1, \ldots, n-1$, the restriction of ϕ to C_j is a multiple of ϕ_j and ϕ restricted to U_k is a multiple of ψ_k. Thus ϕ is a linear combination of the ϕ_j's and ψ_k's. Thus $\{\phi_j\} \cup \{\psi_k\}$ is a basis for $\mathcal{LS}(G)$ □

Remark 1 Proposition 1 and Corollary 2 imply that in order to study symmetric solutions, one needs to look only at those games inside $C \oplus U$; in the next section we refine this further to a subspace of dimension $2n-1$ inside of G.

3.1.1 The space W

Although we have remarked that to the study of linear symmetric solutions the space W plays only the role of the common kernel of every such solution, it may be interesting, nevertheless, to characterize the games that lie in this subspace. That is the content of Proposition 2 of this subsection.

Lemma 2

$$C \oplus U = \left\{ \sum_{j=1}^{n} x_j^j \mid x_j \in \mathbb{R}^n, \text{ for } j = 1, 2, \ldots, n \right\}.$$

Proof Let $V = \left\{ \sum_{j=1}^{n} x_j^j \mid x_j \in \mathbb{R}^n \right\}$. We first show $V \subset C \oplus U$.

Recall that to every vector $x \in \mathbb{R}^n$, and to every $j : 1, \ldots, n$, we have associated a game x^j via

$$x^j(S) = \begin{cases} x(S) & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

Thus, for the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$ we have that

$$\mathbf{1}^j = \begin{cases} j & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j; \end{cases}$$

which in particular shows that $\mathbf{1}^j = jc_j$.

Now, choose n arbitrary vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$. Each of them may be decomposed as a sum with respect to the direct sum decomposition $\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$. Thus we write $x_j = a_j \mathbf{1} + z_j$ where $z_j \in \Delta_n^{\perp}$.

Then,

$$\sum_{j=1}^{n} x_j^j = \sum_{j=1}^{n} a_j \mathbf{1}^j + \sum_{j=1}^{n} z_j^j = \sum_{j=1}^{n} j a_j c_j + \sum_{j=1}^{n-1} z_j^j \in C \oplus U,$$

since $z^n = 0$ for all $z \in \Delta_n^{\perp}$. This shows that $V \subset C \oplus U$.

Now, $c_j = \frac{1}{j} \mathbf{1}^j \in V$, thus $C_j \subset V$. Also, every element in U_j is, by definition, of the form x^j with $x \in \Delta_n^{\perp}$, thus $U_j \subset V$. Hence $C \oplus U \subset V$ □

Proposition 2

$$W = \left\{ w \in G \mid \forall i,j : 1,\ldots,n, \sum_{\substack{|S|=j \\ S \ni i}} w(S) = 0 \right\}.$$

Proof First, $w \in W \Leftrightarrow w \in (C \oplus U)^{\perp} \Leftrightarrow \langle w, \sum_1^n x_j^j \rangle = 0$, for all $x_j \in \mathbb{R}^n$. Thus, if $\{e_i\}$ stands for the standard basis in $\mathbb{R}^n$, then $w \in W \Leftrightarrow \langle w, e_i^j \rangle = 0$, for every $i,j : 1,\ldots,n$. Now,

$$\langle w, e_i^j \rangle = \sum_{|S|=j} w(S) e_i(S),$$

but

$$e_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}.$$

□

In Kleinberg and Weiss (1985) the number of irreducible summands appearing in W is computed, whereas in Amer et al. (2003) a basis for W is given.

Example 1 Let us compute a basis for W for the case of four players, i.e., $n = 4$.

If $w \in W$, then for the coalition $\{i\}$ we have

$$w(\{i\}) = \sum_{\substack{|S|=1 \\ S \ni i}} w(S) = 0.$$

For subsets of cardinality two we get a system of four equations (one for each i) in the six unknowns $w(\{1,2\}), w(\{1,3\}), w(\{1,4\}), w(\{2,3\}), w(\{2,4\}), w(\{3,4\})$. The solution space for this system is 2-dimensional with basis:

$$(u(\{1,2\}), u(\{1,3\}), u(\{1,4\}), u(\{2,3\}), u(\{2,4\}), u(\{3,4\})) = \begin{cases} (1,0,-1,-1,0,1) \\ (0,1,-1,-1,1,0) \end{cases}.$$

For subsets of cardinality three we again get four linearly independent equations in four unknowns. Thus the value of w is zero on all subsets with three elements. Finally, on the total set it also vanishes.

Thus, W consists of games that vanish on subsets of cardinality different to 2, and, on sets of cardinality two satisfy the relations:

$w(\{1,2\}) = w(\{3,4\}), w(\{1,3\}) = w(\{2,4\}),$

$w(\{1,4\}) = w(\{2,3\}) = -w(\{1,2\}) - w(\{1,3\}).$

Remark 2 It is not difficult to show that always $W_1 = W_{n-1} = 0$.

3.2 Symmetric solutions and the action of H_n

Every solution $\phi \in \mathcal{LS}(G)$ is determined by its nth-coordinate, ϕ_n. Moreover, ϕ_n is an H_n-invariant functional, where $H_n = \{\theta \in S_n \mid \theta_n = n\}$ is the subgroup fixing n. More precisely, let

$$\begin{aligned}\mathrm{Hom}_{H_n}(G,\mathbb{R}) = \{f : G \to \mathbb{R} \mid\ & f \text{ is linear and } f(\theta \cdot v) = f(v), \forall v \in G, \\ & \text{and } \forall \theta \in H_n\}\end{aligned}$$

denote the space of H_n-invariant linear functions on G. Then

Lemma 3 *The linear map $\mathcal{LS}(G) \to \mathrm{Hom}_{H_n}(G,\mathbb{R})$ given by $\phi \mapsto \phi_n$ is an isomorphism of vector spaces.*

Proof First, let us show that if $\phi \in \mathcal{LS}(G)$ then ϕ_n is an H_n-invariant functional. Let $v \in G$ be any game, and $\sigma \in H_n$ be a permutation fixing player n. Thus, symmetry of ϕ implies

$$\begin{aligned}(\phi_1(\sigma \cdot v), \ldots, \phi_{n-1}(\sigma \cdot v), \phi_n(\sigma \cdot v)) &= \phi(\sigma \cdot v) = \sigma \cdot (\phi_1(v), \ldots, \phi_{n-1}(v), \phi_n(v)) \\ &= (\phi_{\sigma_1}(v), \ldots, \phi_{\sigma_{n-1}}(v), \phi_n(v))\end{aligned}$$

since $\sigma_n = n$. Thus $\phi_n(\sigma \cdot v) = \phi_n(v)$ for every $\sigma \in H_n$.

In order to check that $\phi \mapsto \phi_n$, is an isomorphism it suffices to give the inverse: If $f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$, then let

$$\Psi(f) = (\phi_1, \phi_2, \ldots, \phi_{n-1}, f),$$

where $\phi_j(v) = f((jn) \cdot v)$, $v \in G$, and (jn) is the transposition interchanging j with n and fixing every other player. The map Ψ is clearly linear, and the inverse to $\phi \mapsto \phi_n$. The remaining question is whether the map $\Psi(f)$ is indeed symmetric. We finish the proof by showing this.

We use the fact that any permutation $\theta \in S_n$ is a composition of transpositions. Hence, to prove symmetry it is enough to check it for an arbitrary transposition (ik). There are two cases, when both i and k are different to n, and when (ik) is of the form (in).

Let us consider the first case, and for simplicity of exposition assume $(ik) = (12)$ and $n \geq 3$. For $j \geq 3$, we have

$$\phi_j((12) \cdot v) = f((jn)(12) \cdot v) = f((12)(jn) \cdot v) = f((jn) \cdot v);$$

where the second equality holds because (jn) and (12) commute and the last equality is the H_n-invariance of f.

Also,

$$\phi_1((12)\cdot v) = f((1n)(12)\cdot v) = f((12)(2n)\cdot v) = f((2n)\cdot v)$$

since $(1n)(12) = (12)(2n)$ and invariance of f. Similarly, $\phi_2((12)\cdot v) = f((1n)\cdot v)$. Hence,

$$\begin{aligned}\phi((12)\cdot v) &= (f((2n)\cdot v), f((1n)\cdot v), f((3n)\cdot v), \ldots, f(((n-1)n)\cdot v), f(v))\\ &= (12)\cdot\phi(v).\end{aligned}$$

Finally, we take the case of a transposition of the form (in). In this case, for $j \neq i$,

$$\phi_j((in)\cdot v) = f((jn)(in)\cdot v) = f((ij)(jn)\cdot v) = f((jn)\cdot v),$$

whereas

$$\phi_i((in)\cdot v) = f((in)(in)\cdot v) = f(v) = \phi_n(v).$$

Thus the ith and the nth coordinates of ϕ have been permuted. □

Just as we decomposed $\mathbb{R}^n$ and G under the action of S_n, we want to find a decomposition under the action of the smaller group $H_n \subset S_n$. Clearly, every space that is invariant under the larger group is still invariant under H_n, though it may happen that a space that was irreducible for S_n decomposes into more than one piece under H_n. Let us see first what happens to $\mathbb{R}^n$.

Recall that $\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$. Now, Δ_n is 1-dimensional so it is also irreducible for H_n. What about $\Delta_n^{\perp}$? Consider the vector $\omega \in \Delta_n^{\perp}$ given by

$$\omega = (1, 1, \ldots, 1, 1-n).$$

Clearly ω and its multiples are fixed by every element of H_n. Likewise, the orthogonal complement (within $\Delta_n^{\perp}$) to the line through ω is also invariant. Call that space A. The simplest way to see that A is invariant is by noticing that

$$A = \{(x_1, x_2, \ldots, x_{n-1}, 0) \mid x_1 + \cdots + x_{n-1} = 0\},$$

thus permuting the first $n-1$ entries of a vector leaves the vector inside of A.

The subgroup H_n can be identified with S_{n-1}, the group of permutations of the first $n-1$ players, and A is then seen to be isomorphic to the standard representation of S_{n-1}, i.e., $\Delta_{n-1}^{\perp}$. Therefore, it is irreducible.

Summarizing, the irreducible decomposition of $\mathbb{R}^n$ under the action of the group H_n is

$$\mathbb{R}^n = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\omega \oplus A.$$

We turn now to the decomposition of G under H_n.

Theorem 1 *Let $n \geq 3$.*

1. *The space of games G decomposes under H_n as*

$$G = C \oplus T \oplus V;$$

where
   (a) $C = \bigoplus_{j=1}^{n} C_j$ *as before, (in particular it is an n-dimensional trivial representation);*
   (b) $T = \bigoplus_{j=1}^{n-1} \mathbb{R}\omega^j$ *is a trivial (for H_n) representation of dimension $n-1$;*
   (c) *V does not contain any trivial summands;*
   (d) *The decomposition is orthogonal.*
2. *Any $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ vanishes in V, hence any symmetric solution $\phi \in \mathcal{LS}(G)$ is determined by the values of ϕ_n in the $(2n-1)$-dimensional trivial subspace $C \oplus T$.*

Proof Each C_j is 1-dimensional, so remains an irreducible piece under the action of H_n.

Each U_j is isomorphic to $\Delta_n^{\perp}$, which we have just seen splits as a sum of two subspaces, a trivial one and a space isomorphic to A. Set $A^j = \{z^j \mid z \in A\}$, then

$$U_j = \mathbb{R}\omega^j \oplus A^j$$

is the decomposition of U_j into irreducibles under the H_n-action.

Notice that, even when $n = 3$, the action of H_n on A (and thus the one on A^j) is not the trivial action (e.g., when $n = 3$, A is 1-dimensional, but the action can be seen to be multiplication by the sign of the permutation).

So we get a decomposition of G of the form

$$G = C \oplus T \oplus V,$$

C and T as in the statement of the theorem and $V = \bigoplus_{j=1}^{n-1} A^j \oplus W$. The A^j's are irreducible and non-trivial, so to prove (c) we need to check only that there are no trivial summands (for the H_n-action) within W.

Suppose W contains a 1-dimensional trivial subspace Y, say. Take any non-zero element $y \in Y$ and define the linear map $f : G \to \mathbb{R}$ by setting $f(y) = 1$ and $f \equiv 0$ on $Y^{\perp}$. Such an f is H_n-invariant (since H_n does nothing on Y and f is set to be zero on its orthogonal complement), and thus, as we have seen, it determines a linear symmetric solution ϕ. But we have proved that every linear symmetric solution vanishes on W. This is a contradiction. Therefore, W cannot contain trivial H_n-representations.

Orthogonality of the decomposition follows from that of the S_n -decomposition, plus the fact that $\mathbb{R}\omega^j \perp A^j$.

To prove 2, notice that, by Schur's Lemma, any H_n-invariant $f : G \to \mathbb{R}$ vanishes when restricted to any irreducible subspace other than a trivial one. Thus $f_{|V} \equiv 0$. On the other hand, any linear $f : C \oplus T \to \mathbb{R}$, when extended

as zero to V, is an H_n-invariant functional and gives rise to a linear symmetric solution. □

Remark 3 In other words, any symmetric solution can be uniquely determined as follows: given arbitrary real numbers $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$ set $\phi_n(c_j) = a_j$, $\phi_n(\omega^j) = b_j, \phi_{n|_V} \equiv 0$. Then ϕ_n is H_n-invariant and determines via Ψ a symmetric solution ϕ. Notice that this shows again that $\mathcal{LS}(G)$ is $(2n-1)$-dimensional.

Theorem 1 identifies the minimal possible subspace of games $(C \oplus T)$ that determines every possible linear symmetric solution. Thus, in principle, to understand any given linear symmetric solution one has only to know its values on $c_1, \ldots, c_n, \omega^1, \ldots, \omega^{n-1}$.

4 Applications

4.1 Efficient symmetric solutions

Definition 4 *A solution is efficient if*

$$\sum_{j=1}^{n} \phi_j(v) = \phi(v) \cdot \mathbf{1} = v(N), \qquad \text{for every } v \in G.$$

In the previous section we saw that any linear symmetric solution ϕ is completely determined by its nth coordinate ϕ_n. Also, at the end of that section, we saw that ϕ_n is itself determined by its values on $C \oplus T$. Thus one expects that efficiency can be translated into some simple condition on these values. Thus,

Proposition 3 *Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$. The linear symmetric solution determined by f is efficient if and only if*

1. $f(c_j) = 0$, *for all $j < n$; and*
2. $f(c_n) = \frac{1}{n}$.

In particular, the set of efficient symmetric solutions is an affine space of dimension $n-1$.

Proof Let ϕ be the symmetric solution corresponding to f. Thus, $\phi_k(v) = f((kn) \cdot v)$.

First of all, $C_n^{\perp}$ is exactly the subspace of games v for which $v(N) = 0$. Of these, those in $U \oplus W$ trivially satisfy $\phi(v) \cdot 1_n = 0$, since (by Schur) their image lies in $\Delta_n^{\perp}$.

Therefore, efficiency need only be checked on C.

Since c_j is fixed by all permutations in S_n,

$$\sum_{k=1}^{n} \phi_k(c_j) = nf(c_j)$$

and so, ϕ is efficient if and only if $nf(c_j) = c_j(N) = \delta_{jn}$.

Finally, for the last assertion, let $f_0 \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ be given by $f_0(c_j) = 0$, $j < n$, $f_0(c_n) = \frac{1}{n}$, and $f_{0|_T} \equiv 0$ (i.e., f_0 is the invariant functional that gives the "egalitarian solution" $\phi_0(v) = \frac{v(N)}{n}$). Then the set of efficient symmetric solutions corresponds to the following affine set of H_n-invariant functionals

$$\{f + f_0 \mid f \in \mathrm{Hom}_{H_n}(G, \mathbb{R}) \text{ and } f_{|_C} \equiv 0\}. \qquad \square$$

Those games v whose values, $v(S)$, depends only on the cardinality of the coalition, S, are called symmetric games. In our notation these are exactly the games $v \in C$. The next Corollary characterizes the solutions to these games in terms of linearity, symmetry and efficiency.

Corollary 2 *Among all linear, symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on symmetric games.*

Proof If we restrict a solution ϕ to the symmetric games space C, then efficiency is equivalent to $\phi_n(c_j) = \frac{1}{n}\delta_{jn}$, i.e., $\phi(v) = \frac{v(N)}{n}$. $\square$

In other words, all linear, symmetric, efficient solutions (e.g., Shapley's value) coincide with the egalitarian solution when restricted to the space of symmetric games.

4.1.1 Formula for all efficient symmetric solutions

We have seen that every efficient symmetric solution is uniquely determined via a functional $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ such that

1. $f(c_j) = 0, j < n$;
2. $f(c_n) = \frac{1}{n}$.

Now, we want to translate this information into more standard game theoretic terminology. We compute a formula for $f(v)$, for any game v, and, finally, for $\phi(v)$.

Observe that, since $\omega = (1, 1, \ldots, 1, 1 - n)$,

$$\omega^j(S) = \begin{cases} \sum_{i \in S} \omega_i & |S| = j \\ 0 & |S| \neq j \end{cases} = \begin{cases} 0 & |S| \neq j \\ j & |S| = j \quad \text{and} \quad n \notin S \\ j - n & |S| = j \quad \text{and} \quad n \in S \end{cases}.$$

Given a game $v \in G$ we first compute its orthogonal projection onto $C_n \oplus T$ (since it is the only part that contributes in the computation of any efficient symmetric solution). The projection of v onto $C_n = \mathbb{R}c_n$ is just $v(N)c_n$. Next we compute its projection, v^T, onto T: this is $v^T = \sum_j \frac{\langle v, \omega^j \rangle}{\langle \omega^j, \omega^j \rangle} \omega^j$, where

$$\langle v, \omega^j \rangle = \sum_{S \subset N} v(S)\omega^j(S) = \sum_{\substack{|S|=j \\ n \notin S}} jv(S) + \sum_{\substack{|S|=j \\ n \in S}} (j - n)v(S),$$

and

$$\langle \omega^j, \omega^j \rangle = \sum_{\substack{|S|=j \\ n \notin S}} j^2 + \sum_{\substack{|S|=j \\ n \in S}} (j-n)^2 = \binom{n-1}{j} j^2 + \binom{n-1}{j-1} (n-j)^2 = \frac{n!}{(j-1)!(n-j-1)!}.$$

Note that $\langle \omega^j, \omega^j \rangle = \langle \omega^{n-j}, \omega^{n-j} \rangle$.

Remark 4 At this point, we do not really care about the value of $\langle \omega^j, \omega^j \rangle$, since it is absorbed by the parameters $b_j = f(\omega^j)$. Nevertheless it will be used later on.

Since

$$f(v) = f(v(N)c_n + v^T) = \frac{v(N)}{n} + f(v^T) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} \frac{\langle v, \omega^j \rangle}{\langle \omega^j, \omega^j \rangle} f(\omega^j)$$

for all $v \in G$, we get:

Proposition 4 *All efficient linear symmetric solutions are given by* $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ *of the following form:*

$$f(v) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} t_j \left[\sum_{\substack{|S|=j \\ n \notin S}} j v(S) + \sum_{\substack{|S|=j \\ n \in S}} (j-n) v(S) \right],$$

where $t_1, \ldots, t_{n-1}$ *are arbitrary real numbers.*

In the previous formula, the coefficients t_j are related to $f(\omega^j)$ via

$$t_j = \frac{f(\omega^j)}{\langle \omega^j, \omega^j \rangle}.$$

Theorem 2 *The efficient linear symmetric solutions are precisely those* $\phi \in \mathcal{LS}(G)$ *of the form*

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} (n-s) \left[\beta_s v(S) - \beta_{n-s} v(N \setminus S) \right], \tag{1}$$

where $s = |S|$ *and* $\beta_s \in \mathbb{R}$ *are arbitrary.*

Proof From Proposition 4 it follows that

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} t_j \left[\sum_{\substack{|S|=j \\ i \notin S}} j v(S) + \sum_{\substack{|S|=j \\ i \in S}} (j-n) v(S) \right].$$

Now,

$$\sum_{j=1}^{n-1}\sum_{\substack{|S|=j\\ i\notin S}} t_j jv(S) = \sum_{j=1}^{n-1}\sum_{\substack{|T|=n-j\\ i\in T}} t_j jv(N\setminus T)$$
$$= \sum_{l=1}^{n-1}\sum_{\substack{|T|=l\\ i\in T}} t_{n-l}(n-l)v(N\setminus T)$$

from which we get the expression of the theorem by setting $\beta_s = -t_s$. □

We should mention that an equivalent formula to (1) has been obtained by Driessen and Radzik (2002).

4.2 Nullity

Definition 5

- *For a game $v \in G$, we say that player i is null for v if*

$$v(S) = v(S \cup \{i\})$$

  for all $S \subset N$.
- *A solution $\phi \in \mathcal{LS}(G)$ is said to be null (or to satisfy the nullity axiom) if $\phi_i(v) = 0$ for every game v for which i is a null player.*

Here we study the nullity axiom in the same spirit of the previous section. Namely, first we understand the nullity axiom as a condition on the the last coordinate, ϕ_n, of a linear symmetric solution. Thus, in terms of H_n-invariant functionals, the condition of being null may be restated as follows:

$f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$ gives rise to a null linear symmetric solution if and only if $f(v) = 0$ for every game v for which n is a null player.

Let $M_n = \{v \in G \mid n \text{ is null for } v\} \subset G$. Note that M_n is an invariant subspace for the action of H_n. It is in fact isomorphic to the representation space $G^{(n-1)}$ of all games in $n-1$ players:

If we set $L : G^{(n-1)} = \{g : 2^{\{1,2,\ldots,n-1\}} \to \mathbb{R} \mid g(\emptyset) = 0\} \to M_n$ by

$$L(g)(S) = \begin{cases} g(S) & \text{if } n \notin S \\ g(S\setminus\{n\}) & \text{if } n \in S \end{cases}$$

then L is an H_n-equivariant isomorphism.

Now, any $f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$ is determined by its values on the subspace $C \oplus T$, thus we want to identify the null games for n inside $C \oplus T$. By definition, this is the space $\mathcal{N} = (C \oplus T) \cap M_n$.

Since $C \oplus T$ is the largest trivial subspace – for H_n – in G, we conclude that $\mathcal{N}$ is the largest trivial subspace inside M_n. Therefore, it corresponds under the isomorphism L to the subspace of symmetric games in $n-1$ players, that is

$$\mathcal{N} = L(C^{(n-1)}).$$

So immediately we get that $\dim \mathcal{N} = n-1$ and, since $\dim(C \oplus T) = 2n-1$, that the space of null symmetric linear solutions is n-dimensional.

Next we compute an explicit basis for $\mathcal{N}$.

Lemma 4 *Let $\mathcal{N} = (C \oplus T) \cap M_n$. Then $\mathcal{N}$ is the $(n-1)$ -dimensional trivial subspace with basis $\{v_j\}$, $j : 1, \ldots, n-1$, where*

$$v_j = \frac{n-j}{n} c_j + \frac{j+1}{n} c_{j+1} + \frac{1}{n}\left(\omega_n^j - \omega_n^{j+1}\right).$$

Proof Let $\{z_j\}_1^{n-1}$ be the basis of $C^{(n-1)}$ obtained by restricting each c_j, $j : 1, \ldots, n-1$, to subsets of $\{1, \ldots, n-1\}$. Then, by the above observation, $v_j = L(z_j)$ is a basis of $\mathcal{N}$; more explicitly,

$$v_j(S) = \begin{cases} c_j(S) & \text{if } n \notin S \\ c_j(S \setminus \{n\}) & \text{if } n \in S \end{cases} = \begin{cases} 1 & \text{if } |S| = j \quad \text{and} \quad n \notin S \\ 1 & \text{if } |S| = j+1 \quad \text{and} \quad n \in S \\ 0 & \text{otherwise} \end{cases}.$$

Now, write $v_j = \sum a_i c_i + \sum b_k \omega^k$. To compute the coefficients a_i, b_k we proceed as follows:

$$\left.\begin{aligned} 1 = v_j(\{1, \ldots, j\}) &= a_j + j b_j \\ 0 = v_j(\{1, \ldots, j-1, n\}) &= a_j + (j-n) b_j \end{aligned}\right\} \Rightarrow \; b_j = \tfrac{1}{j} \text{ and } a_j = 1 - \tfrac{j}{n}$$

similarly, evaluating v_j on $\{1, \ldots, j+1\}$ and $\{1, \ldots, j, n\}$ one gets $a_{j+1} = \frac{j+1}{n}$ and $b_{j+1} = \frac{-1}{n}$ (if $j < n-1$, for $j = n-1$ one has $a_{j+1} = 1$, $b_{j+1} = 0$).

Finally, in the same way one computes that $a_k = b_k = 0$ for the remaining coefficients. □

Example 2 Let us assume that $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ is null and efficient. Efficiency implies $f(c_j) = 0$, for $j \in N \backslash \{n\}$, and $f(c_n) = \frac{1}{n}$. Therefore, if $r_j = f(\omega_n^j)$,

$$0 = f(v_{n-1}) = \frac{1}{n}\left(1 + r_{n-1}\right) \Rightarrow r_{n-1} = -1,$$

and so, if $j < n-1$

$$0 = f(v_j) = \frac{1}{n}(r_j - r_{j+1}) \Rightarrow r_j = r_{j+1};$$

thus $r_j = -1$, for all $j \in N \backslash \{n\}$.

Hence, we conclude that there is exactly one linear symmetric solution which is efficient and null (the Shapley value, Shapley (1953)). It is given by $f(c_n) = \frac{1}{n}$, $f(c_j) = 0$ and $f(\omega_n^j) = -1$ for $j \in N\backslash\{n\}$. □

We finish this subsection by stating the general formula for null linear symmetric solutions, i.e., solutions that satisfy all the axioms that traditionally characterize the Shapley value except for the efficiency axiom. Dubey et al. (1981) have given a similar formula. See also Weber (1988) for further discussion about solutions satisfying the nullity axiom.

Proposition 5 *The space of null linear symmetric solutions is n-dimensional. The general formula for such a solution is given by:*

$$\phi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S)\right].$$

for arbitrary $r_0, \ldots, r_{n-1} \in \mathbb{R}$*, and where* $s = |S|$.

Proof Define, for $j : 1, \ldots, n-1$,

$$\mu_j(S) = \begin{cases} 1 & \text{if } |S| = j \text{ and } n \notin S \\ -1 & \text{if } |S| = j+1 \text{ and } n \in S \\ 0 & \text{otherwise} \end{cases},$$

and

$$\chi_n(S) = \begin{cases} 1 & \text{if } S = \{n\} \\ 0 & \text{if } S \neq \{n\} \end{cases}.$$

The μ_j's and χ_n are H_n-invariant, and so they belong to $C \oplus T$. Let us show that they form a basis for the orthogonal complement to $\mathcal{N}$ inside $C \oplus T$.

Since for every game v,

$$\langle v, v_k \rangle = \sum_{S \subset N} v(S) v_k(S) = \sum_{\substack{|S|=k \\ n \notin S}} v(S) + \sum_{\substack{|S|=k+1 \\ n \in S}} v(S) = \sum_{\substack{|S|=k \\ n \notin S}} \left[v(S) + v(S \cup \{n\})\right]$$

it follows that $\langle \mu_j, v_k \rangle = 0$ and $\langle \chi_n, v_k \rangle = 0$. It is also not hard to check that $\{\mu_j\} \cup \{\chi_n\}$ is an orthogonal set.

Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ be an arbitrary null solution, and $v \in G$. Then

$$f(v) = f\left(\frac{\langle v, \chi_n \rangle}{\langle \chi_n, \chi_n \rangle} \chi_n + \sum_{j=1}^{n-1} \frac{\langle v, \mu_j \rangle}{\langle \mu_j, \mu_j \rangle} \mu_j \right).$$

Since

- $\langle v, \chi_n \rangle = v(\{n\})$,
- $\langle \chi_n, \chi_n \rangle = 1$,
- $\langle v, \mu_j \rangle = \sum_{\substack{|S|=j \\ n \notin S}} [v(S) - v(S \cup \{n\})]$ and
- $\langle \mu_j, \mu_j \rangle = 2 \binom{n-1}{j}$,

we obtain

$$f(v) = f(\chi_n) v(\{n\}) + \sum_{j=1}^{n-1} \left(\sum_{\substack{|S|=j \\ n \notin S}} [v(S) - v(S \cup \{n\})] \frac{f(\mu_j)}{2\binom{n-1}{j}} \right).$$

Set $r_0 = f(\chi_n)$, $r_j = \frac{-f(\mu_j)}{2\binom{n-1}{j}}$ for $j : 1, \ldots, n-1$. Then,

$$f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S) \right],$$

where the sum includes $S = \emptyset$ and where s stands for the cardinality of S.

Now recall that $\phi_i(v)$ is obtained from $f = \phi_n$ by interchanging i with n. □

For future computations, we state the formulas of the $\mu_k, k : 1, \ldots, n-1$, and χ_n, in terms of the orthogonal basis $\{c_i, \omega^j\}$, $i : 1, \ldots, n, j : 1, \ldots, n-1$:

$$\mu_k = \frac{n-k}{n} c_k - \frac{k+1}{n} c_{k+1} + \frac{1}{n} \left(\omega^k + \omega^{k+1} \right),$$

$$\chi_n = \frac{1}{n} \left(c_1 - \omega^1 \right).$$

4.3 Shapley's solution

Shapley's solution can be characterized (as is well known) as the unique linear symmetric solution which is both efficient and null. As seen in the previous section, it is characterized by saying that its nth coordinate functional Sh_n is the unique element of $\mathrm{Hom}_{H_n}(G, \mathbb{R})$ such that

- $Sh_n(c_n) = \frac{1}{n}$,
- $Sh_n(c_j) = 0, j < n$, and
- $Sh_n(\omega^j) = -1, j < n$.

From formula (1) for all efficient, symmetric and linear solutions, we obtain the following formula for Shapley's value [see Myerson (1991) for a similar expression]:

$$Sh_i(v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(N \setminus S)],$$

where $s = |S|$.

4.3.1 Shapley's value as the adjoint of the additive games map

Throughout the article we have been using an inner product on G which seems very "natural" (we call it the standard inner product on G); namely,

$$\langle v_1, v_2 \rangle = \sum_{S \subset N} v_1(S) v_2(S).$$

In fact, any other S_n-invariant inner product on G would have done as well. By Schur's Lemma, on each irreducible component these inner products are unique up to a constant. We see next, that a suitable choice of conformal factors can be chosen which make the new inner product have a unique relation to Shapley's value.

Look at the following S_n-equivariant map:$\hat{}: \mathbb{R}^n \to G$, which maps a vector x to the game $S \mapsto x(S) = \sum_{i \in S} x_i$, which we denote by $\hat{x}$. The image $\widehat{\mathbb{R}^n} \subset G$, of this map, is precisely the subspace of additive games (i.e., those games v such that $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$).

Set a new inner product $\langle\langle , \rangle\rangle$ on G as follows:

- On C define it by declaring $\{c_1, c_2, \ldots, c_{n-1}, \widehat{\mathbf{1}}\}$ orthogonal, and of length $\sqrt{n}$.
- On each U_j set $\langle\langle , \rangle\rangle$ with

$$\langle\langle x^j, y^j \rangle\rangle = \frac{1}{n-1} x \cdot y$$

  for every $x, y \in \Delta_n^{\perp}$.
- Set $U_i \perp U_j$, for all $i \neq j$.
- Set $\langle\langle , \rangle\rangle = \langle , \rangle$ on W.

Then $\langle\langle , \rangle\rangle$ is S_n-invariant; moreover

Theorem 3 *Shapley's value is characterized as being the adjoint, with respect to $\langle\langle , \rangle\rangle$, of the map $x \mapsto \hat{x}$. In other words, for every game $v \in G$ and every vector $x \in \mathbb{R}^n$ we have that*

$$Sh(v) \cdot x = \langle\langle v, \hat{x} \rangle\rangle.$$

Proof Let $\varphi : G \to \mathbb{R}^n$ be the adjoint to $\hat{}$ with respect to $\langle\langle , \rangle\rangle$, i.e., define the linear solution φ by

$$\varphi(v) \cdot x = \langle\langle v, \hat{x}\rangle\rangle \qquad \text{for every } v \in G, x \in \mathbb{R}^n.$$

First of all, notice that φ is symmetric. Thus $\varphi(\widehat{\mathbf{1}}) = \lambda\mathbf{1}$, for some $\lambda \in \mathbb{R}$, and since

$$\lambda n = \lambda\mathbf{1} \cdot \mathbf{1} = \varphi(\widehat{\mathbf{1}}) \cdot \mathbf{1} = \langle\langle \widehat{\mathbf{1}}, \widehat{\mathbf{1}}\rangle\rangle = n$$

then $\varphi(\widehat{\mathbf{1}}) = \mathbf{1}$.

Also, for every $j < n$, $\varphi(c_j) \cdot \mathbf{1} = \langle\langle c_j, \widehat{\mathbf{1}}\rangle\rangle = 0$, therefore $\varphi(c_j) = 0$.

To finish, we need only check that $\varphi_n(\omega^j) = -1$ for every $j < n$. Thus, it is enough to check that $\varphi(\omega_n^j) = \frac{1}{n-1}\omega$; but, for every $y \in \Delta_n^{\perp}$,

$$\varphi(\omega^j) \cdot y = \langle\langle \omega^j, \hat{y}\rangle\rangle = \langle\langle \omega^j, y^j\rangle\rangle = \frac{1}{n-1}\omega \cdot y. \quad \square$$

Remark 5 The proof shows that given any linear symmetric, efficient solution ψ, for which $\psi_n(\omega^j) < 0$, for every j, we can find an invariant positive definite inner product on G so that ψ is the adjoint of the map $\hat{}$.

The condition $\psi_n(\omega^j) < 0$, for every j, for efficient ψ, is a natural one, in that it reflects on the fact that the only negative values of the game ω^j happen for coalitions that contain player n.

Corollary 3 *For any $v \in G$,*

$$Sh_i(v) = \langle\langle v, \hat{e}_i\rangle\rangle$$

where $e_i \in \mathbb{R}^n$ is the usual ith vector of the standard basis of $\mathbb{R}^n$.

Remark 6 Equivalently, with respect to $\langle\langle , \rangle\rangle$, Shapley's value is characterized by saying that

- $Sh(\hat{x}) = x$, for every additive game $\hat{x}$, and
- $Sh(v) = 0$, for every game v perpendicular to the additive games.

So Shapley's value, in this sense, is nothing more than the orthogonal projection (with respect to $\langle\langle , \rangle\rangle$) to the additive games space. This should be compared with results by Kultti and Salonen (2005) relating certain efficient linear solutions with different types of inner products on G.

4.4 Duality

The interpretation of $v(S)$ changes accordingly to what people want to model. For example, $v(S)$ could be the joint benefit that the coalition S could generate if they decide to play together; in this case we would say that v is a benefit game. In a second interpretation, we could assume that the players in N want to hire a service, then $v(S)$ could be thought of as the joint cost (for the players in S) if they act together. In the latter case we say that v is a cost game. In both cases,

$v(S)$ is the "worth" assigned to the coalition S when it is formed, i.e., when the players in S decide to play together.

The duality operator, as defined next, allows us to move from one of these interpretations to the other. Thus it is a natural concept to study.

The duality operator $* : G \to G$ is defined by

$$(*v)(S) = v(N) - v(N \setminus S).$$

Observe that $*^2 = I$ and that $*$ is S_n-equivariant (i.e., symmetric).

Definition 6 *A solution $\phi \in \mathcal{LS}(G)$ is*

- *self-dual, if $\phi(*v) = \phi(v)$ for all $v \in G$;*
- *anti-self-dual, if $\phi(*v) = -\phi(v)$ for all $v \in G$.*

In order to understand better the (anti-)self-dual solutions we need to understand the action of $*$ on the H_n-invariant subspace $C \oplus T$. Since $*$ is S_n-equivariant, C is an invariant subspace: i.e., $*c \in C$ for every $c \in C$; similarly, T is also an invariant space for $*$.

Lemma 5 *The action of $*$ on $C \oplus T$ is as follows:*

1. *For $j < n$, $*c_j = -c_{n-j}$.*
2. *$*c_n = c_1 + \cdots + c_n$.*
3. *For $j < n$, $*\omega^j = \omega^{n-j}$.*

Proof 1. Let S be such that $|S| = k$, then

$$*c_j(S) = c_j(N) - c_j(N \setminus S) = -\delta_{j,n-k}.$$

2.

$$*c_n(S) = c_n(N) - c_n(N \setminus S) = 1 - \delta_{n,n-k},$$

thus, $*c_n(S) = 1$ for every $S \neq \emptyset$.

3.

$$*\omega^j(S) = -\omega^j(N \setminus S) = \begin{cases} 0 & \text{if } |N \setminus S| \neq j \\ -\omega(N \setminus S) & \text{if } |N \setminus S| = j \end{cases} = \begin{cases} 0 & \text{if } |S| \neq n-j \\ \omega(S) & \text{if } |S| = n-j \end{cases}$$
$$= \omega^{n-j}(S).$$

□

Remark 7 Note that $*\hat{\mathbf{1}} = \hat{\mathbf{1}}$ (in fact for every $x \in \mathbb{R}^n$ $*\hat{x} = \hat{x}$). Thus, one can check that $*$ is orthogonal with respect to the inner product $\langle\langle , \rangle\rangle$ (defined in the previous section) that sets Shapley's solution as the adjoint of $x \mapsto \hat{x}$. From this one can give a quick proof of the self-duality of Shapley's value: For every $v \in G, x \in \mathbb{R}^n$

$$Sh(*v) \cdot x = \langle\langle *v, \hat{x} \rangle\rangle = \langle\langle v, *\hat{x} \rangle\rangle = \langle\langle v, \hat{x} \rangle\rangle = Sh(v) \cdot x.$$

Duality of Shapley's value was already noticed in Kalai and Samet (1987).

Clearly, every symmetric solution decomposes uniquely as a sum of a self and an anti-self dual solution ($\phi(v) = \frac{\phi(v)+\phi(*v)}{2} + \frac{\phi(v)-\phi(*v)}{2}$), thus the spaces of these solutions have complementary dimensions. In fact,

Corollary 4 *Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$, with $f(c_j) = a_j$ and $f(\omega^j) = b_j$.*

1. *The symmetric solution corresponding to f is self-dual if and only if $a_j = -a_{n-j}$, $b_j = b_{n-j}$, $j < n$. In particular, the space of symmetric self-dual solutions has dimension n.*
2. *The symmetric solution corresponding to f is anti-self-dual if and only if $a_j = a_{n-j}$, $b_j = -b_{n-j}$, $j < n$, and $a_1 + \cdots + a_{n-1} + 2a_n = 0$. In particular, the space of symmetric anti-self-dual solutions has dimension $n - 1$.*

Proof Let us show the first statement, the other one is proven similarly.

Let ϕ be the linear symmetric solution such that $\phi_n = f$. Then, for every game v

$$\phi(*v) = \phi(v) \Leftrightarrow f(*v) = f(v) \Leftrightarrow \forall j \begin{cases} f(*c_j) = f(c_j) \\ f(*\omega^j) = f(\omega^j) \end{cases} \Leftrightarrow \begin{cases} -a_{n-j} = a_j,\ j < n \\ a_1 + \cdots + a_n = a_n \\ b_{n-j} = b_j \end{cases};$$

now, notice that the middle equation is a consequence of the first one.

Finally, for the dimension count, assume first that n is even, then $a_{n/2} = 0$, and we have the following "free" variables: $a_1, \ldots, a_{n/2-1}, a_n, b_1, \ldots, b_{n/2}$, i.e., n variables in total. Whereas if n is odd, then we can specify values for: $a_1, \ldots, a_{[n/2]}, a_n, b_1, \ldots, b_{[n/2]}$, so we have $2[n/2] + 1 = n$ variables. □

4.4.1 Efficient and self-dual solutions

An efficient and self-dual solution assigns equal importance to the amount that a particular coalition claims as well as to that amount which the players outside of the coalition fail to claim.

Corollary 5 *The space of linear symmetric solutions which are both efficient and self-dual has dimension $\left[\frac{n}{2}\right]$. This space is the space of solutions of the form*

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} (n - s)\beta_s \left[v(S) - v(N \setminus S)\right]$$

with $\beta_s = \beta_{n-s}$.

Proof With same notation as above, efficiency implies $a_j = 0, j < n$, and $a_n = \frac{1}{n}$; while self-duality imposes the extra $b_j = b_{n-j}$. Clearly, then, the dimension is $\left[\frac{n}{2}\right]$. The general formula follows from formula (1) for efficient solutions, since $\beta_{n-s} = \beta_s$. □

Since $a_1 + \cdots + a_{n-1} + 2a_n = 0$ for any anti-self-dual solution, there can be none which is also efficient.

4.4.2 Nullity and duality

Recall the bases $\{v_j\}$ and $\{\mu_j\} \cup \{\chi_n\}$ of the null space $\mathcal{N}$ and its orthogonal complement $\mathcal{N}^{\perp}$ (respectively) inside $C \oplus T$. The next lemma gives us the action of $*$ with respect to these bases.

Lemma 6

1. *For* $j = 1, \ldots, n-2$ *we have*

$$*v_j = -v_{n-j-1} \text{ and } *\mu_j = \mu_{n-j-1}.$$

2. $*v_{n-1} = *c_n - \chi_n$ *and* $*\mu_{n-1} = - * c_n - \chi_n$.
3. $*\chi_n = \frac{-1}{n}(c_{n-1} + \omega^{n-1})$.

Proof The proof follows from Lemma 4, the formulas at the end of 2.2 and Lemma 5. □

Recall (Proposition 5) that $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ gives rise to a null solution if and only if it is of the form:

$$f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S)\right]$$

where $r_0 = f(\chi_n)$, and

$$r_j = \frac{-f(\mu_j)}{2\binom{n-1}{j}}$$

for $j : 1, \ldots, n-1$.

Proposition 6 *The null solution* $f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S)\right]$ *is self-dual (respectively, anti-self-dual) if and only if* $r_j = r_{n-j-1}$ *(respectively,* $r_j = -r_{n-j-1}$*), for all* $j < n$.

Proof Let us give a proof for the self-duality case.

The null f gives a self-dual solution if and only if it satisfies self-duality on any basis for $C \oplus T$. Thus, f gives rise to a self-dual solution if and only if

$$f(*v_j) = f(v_j), f(*\mu_j) = f(\mu_j) \quad \text{and} \quad f(*\chi_n) = f(\chi_n).$$

A null f automatically satisfies $f(*v_j) = 0 = f(v_j)$ for $j \leq n-1$, thus the first $n-2$ of these equations impose no restriction.

The equation $f(*v_{n-1}) = f(v_{n-1}) = 0$ is equivalent to

$$0 = f(v_{n-1}) = f(*c_n) - r_0.$$

The set of equations $f(*\mu_j) = f(\mu_j)$ give

$$\begin{aligned} r_j &= r_{n-j-1}, \qquad \text{for } j < n-1, \\ -2r_{n-1} &= -f(*c_n) - r_0 \end{aligned}$$

which, when combined with the previous one, gives

$$\begin{aligned} r_j &= r_{n-j-1}, \qquad \text{for } j < n-1, \\ r_{n-1} &= r_0; \end{aligned}$$

i.e., $r_j = r_{n-j-1}$ for all $j \leq n-1$.

Finally, the last equation, $f(*\chi_n) = f(\chi_n)$, is already implied by the others. Notice that, since $2c_n = \nu_{n-1} - \mu_{n-1}$ then $f(*c_n) = f(c_n)$ follows from the first two sets of equations. Then,

$$f(\chi_n) = f(- * c_n - *\mu_{n-1}) = f(-c_n - \mu_{n-1}) = f(*\chi_n).$$

Anti-self-duality is treated in the same way. □

5 Kernels

Corollary 1 is the statement that the common kernel of all symmetric linear solutions is W, i.e.,

$$\bigcap_{\phi \in \mathcal{LS}(G)} \ker \phi = W.$$

From the basic results for efficient solutions discussed in Sect. 3.1 (namely Proposition 3), the next result follows easily.

Proposition 7 *The common kernel of all linear, symmetric and efficient solutions is*

$$C_1 \oplus \cdots \oplus C_{n-1} \oplus W.$$

Remark 8 In Amer et al. (2003) the common kernel of all linear, symmetric and null solutions is computed. It is shown that it coincides with W!

In what follows we concentrate on the kernel of a single solution.

Let $\phi : G \to \mathbb{R}^n$ be any linear symmetric solution. As we saw before it is uniquely determined by the numbers $a_i = \phi_n(c_i)$, $b_j = \phi_n(\omega^j)$.

Proposition 8 *The kernel of ϕ consists of all games $c + u + w \in C \oplus U \oplus W$, such that*

- $\langle c, c_\phi \rangle = 0$, *where* $c_\phi = \sum_{i=1}^{n} \frac{a_i}{\|c_i\|^2} c_i$.
- $u = \sum_{j=1}^{n-1} z_j^j$, *with* $z_j \in \Delta_n^\perp$ *such that* $\sum_j b_j z_j = 0$.
- $w \in W$ *is arbitrary.*

Compare the following corollary to a similar result in Kleinberg and Weiss (1985).

Corollary 6 *The kernel of Shapley's value consists of all games $c + u + w \in C \oplus U \oplus W$, such that*

- $c \in C_1 \oplus \cdots \oplus C_{n-1}$.
- $u = \sum_{j=1}^{n-1} z_j^j$, *with* $z_j \in \Delta_n^\perp$ *such that* $\sum_j z_j = 0$.
- $w \in W$ *is arbitrary.*

Proof of Corollary: For Shapley's value $a_1 = a_2 = \cdots = a_{n-1} = 0$, $a_n = \frac{1}{n}$ and $b_j = -1$. □

Remark 9 Recall that at the end of Sect. 2.3 we have shown that the kernel of Shapley's value is the orthogonal complement -with respect to the inner product $\langle\langle , \rangle\rangle$ defined there- to the subspace of additive games. This corollary is an equivalent formulation without mention to $\langle\langle , \rangle\rangle$.

Proof of Proposition: Since $\ker \phi$ is an invariant subspace of G then

$$\ker \phi = (\ker \phi \cap C) \oplus (\ker \phi \cap U) \oplus (\ker \phi \cap W);$$

to prove this, let A be an irreducible summand of $\ker \phi$, then A is an irreducible summand of G, and, thus, is contained in either C, U or W.

Now, for $c \in C$

$$\langle c, c_\phi \rangle = \sum_i \frac{a_i}{\|c_i\|^2} \langle c, c_i \rangle = \sum_i \frac{\langle c, c_i \rangle}{\|c_i\|^2} \phi_n(c_i) = \phi_n(c),$$

and thus, $c \in \ker \phi$ if and only if $\langle c, c_\phi \rangle = 0$.

Every $u \in U$ can be written as $u = \sum_{j=1}^{n-1} z_j^j$, with $z_j \in \Delta_n^\perp$. Then,

$$\phi(u) = \sum_j \phi(z_j^j) = \frac{1}{1-n} \sum_j b_j z_j.$$

Finally, $W \subset \ker \phi$. □

Appendix

A reference for basic representation theory is Fulton and Harris (1991). Nevertheless, we recall all the basic facts that we need.

Definition 7 *Let H be an arbitrary group. A representation for H is a homomorphism $\rho : H \to GL(V)$, where V is a vector space and $GL(V)$ denotes the group of invertible linear maps of V.*

In other words, a representation of H is a map assigning to each element $h \in H$ a linear map $\rho(h) : V \to V$ that respects multiplication:

$$\rho(h_1h_2) = \rho(h_1)\rho(h_2).$$

One usually abuses notation and talks about the representation V without explicitly mentioning the homomorphism ρ. Thus, when applying the linear transformation corresponding to $h \in H$ on a vector $v \in V$, we write $h \cdot v$ rather than $(\rho(h))(v)$.

Definition 8 *Let V and W be two representations for the group H.*

- *A linear map $T : V \to W$ is H-equivariant if $T(h \cdot v) = h \cdot (T(v))$, for every $v \in V$ and every $h \in H$.*
- *V and W are isomorphic H-representations, $V \simeq W$, if there exists an H-equivariant isomorphism between them.*

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra with a group of symmetries, the same. They are vector spaces of the same dimension where the actions are seen to correspond under a linear isomorphism.

Example 3 Let $A \subset G$ denote the subspace of additive games, i.e., $v \in A$ if and only if $v(S \cup T) = v(S) + v(T)$ for every pair of coalitions S, T with $S \cap T = \emptyset$. S_n acts on A; in other words, if $\theta \in S_n$ and $v \in A$ then $\theta \cdot v \in A$ as is readily seen.

We claim that A as a representation is isomorphic to $\mathbb{R}^n$ with the usual action $\theta \cdot (x_1, \ldots, x_n) = (x_{\theta_1}, x_{\theta_2}, \ldots, x_{\theta_n})$.

The S_n-equivariant isomorphism $\phi : \mathbb{R}^n \to A$ is given by $\phi(x) = \hat{x}$ where

$$\hat{x}(S) = \sum_{j \in S} x_j.$$

Clearly ϕ is a linear map and $\hat{x}$ is an additive game for every x. Let us show equivariance. We need to prove that

$$\phi(\theta \cdot x) = \theta \cdot \hat{x}.$$

Now,

$$(\theta \cdot \hat{x})(S) = \hat{x}(\theta^{-1}S) = \sum_{\{j \mid \theta_j \in S\}} x_j;$$

on the other hand, if we set $y = \theta \cdot x$, then

$$(\phi(\theta \cdot x))(S) = \widehat{\theta \cdot x}(S) = \hat{y}(S) = \sum_{i \in S} y_i,$$

but $y_i = x_{\theta^{-1}i}$ and, hence, both sums are identical (just set $i = \theta_j$).

Clearly ϕ is 1–1, and hence it is onto by counting dimensions of both spaces.

So, from the point of view of linear algebra with S_n-action, the space of additive games and $\mathbb{R}^n$ are the same. □

Definition 9 *A representation V is irreducible if it does not contain a nontrivial invariant subspace. That is, if $U \subset V$ is also a representation for H (meaning that $h \cdot u \in U$ for every $h \in H$ and every $u \in U$), then U is either $\{0\}$ or all of V.*

The following theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

Theorem 4 *(Schur's Lemma) Let V, W be irreducible representations of a group H. If $\phi : V \to W$ is H-equivariant, then $\phi \equiv 0$ or ϕ is an isomorphism.*

Moreover, if V and W are complex vector spaces, then ϕ is unique up to multiplication by a scalar $\lambda \in \mathbb{C}$.

Proof Kerϕ and Imϕ are invariant subspaces of V and W, respectively, thus it is zero or the total space. From this follows the first part.

If the vector spaces are complex, $\phi, \psi : V \to W$ H-equivariant, then $T = \phi^{-1} \circ \psi : V \to V$ must have an eigenvalue $\lambda \in \mathbb{C}$. Since the eigenspace corresponding to λ is invariant, it must be all of V, i.e., $T = \lambda I$ or $\psi = \lambda\phi$. □

Corollary 7 *Let V be a real irreducible representation, such that its complexification $V^{\mathbb{C}} = V \otimes \mathbb{C} = V \oplus iV$ is also irreducible (as a complex representation). Let W be a real irreducible representation. If $\phi : V \to W$ is equivariant, then ϕ is unique up to multiplication by a real scalar.*

Proof Schur's Lemma implies that ϕ is zero or an isomorphism. Suppose ϕ is an isomorphism, then $W^{\mathbb{C}}$ is also isomorphic to $V^{\mathbb{C}}$ and $\phi^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ (the complex-linear extension of ϕ) is an isomorphism. If $\psi : V \to W$ is any equivariant map, then by the previous theorem $\psi^{\mathbb{C}} = \lambda\phi^{\mathbb{C}}$, for some $\lambda \in \mathbb{C}$. Since $\phi^{\mathbb{C}}$ and $\psi^{\mathbb{C}}$ preserve real parts (i.e., send V to W) λ must be real. □

Lemma 7 *The decomposition of $\mathbb{R}^n$, under S_n, into irreducible subspaces is*

$$\mathbb{R}^n = \mathbb{R}\mathbf{1} \oplus \Delta_n^{\perp}.$$

Proof We need only to check that $\Delta_n^{\perp}$ is irreducible. This is done by induction on n. If $n = 2$, then $\Delta_n^{\perp}$ is 1-dimensional and thus, irreducible.

$\Delta_n^{\perp}$ is generated by $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. The isotropy subgroup of e_n, $H_n = \{\theta \in S_n \mid \theta_n = n\}$, is isomorphic to S_{n-1}, and $\Delta_n^{\perp}$ is the standard representation for H_n.

Therefore, induction hypothesis says that

$$\Delta_n^{\perp} \simeq \mathbb{R} \oplus 1_{n-1}^{\perp}$$

is the decomposition (up to isomorphism) into irreducible H_n-representations. Actually, the precise summands of this decomposition are $\mathbb{R}\omega$ and its orthogonal complement inside $\Delta_n^{\perp}$, where $\omega = (e_1 - e_n) + (e_2 - e_n) + \cdots + (e_{n-1} - e_n) = (1, 1, \ldots, 1, 1 - n)$.

Now, if $\Delta_n^\perp$ were a reducible S_n-representation, then it would split into subspaces also invariant under H_n. Thus, the only possible splitting would be into $\mathbb{R}\omega$ and its complement. But these subspaces are not S_n-invariant. □

Remark 10 Observe that the above proof also shows that the real irreducible representations $\mathbb{R}$ and $\Delta_n^\perp$ have irreducible complexifications.

Proposition 9 *For $j < n$,*

$$G_j = C_j \oplus U_j \oplus W_j,$$

where $C_j = \mathbb{R}c_j \simeq \mathbb{R}$, $U_j = \{x^j \mid x \in \Delta_n^\perp\} \simeq \Delta_n^\perp$ and W_j does not contain any summands isomorphic to either $\mathbb{R}$ nor $\Delta_n^\perp$. The decomposition is orthogonal.

Proof Let X_j be the subset of 2^N consisting of all subsets of cardinality j. So, we may identify G_j with $\mathbb{R}[X_j]$ the space of real functions on X_j. Let $K = S_j \times S_{n-j} \prec S_n$ be the subgroup fixing $\{1, 2, \ldots, j\}$. Thus, $X_j = S_n/K$ and

$$G_j \simeq \mathbb{R}[S_n/K].$$

Hence, if τ is the trivial 1-dimensional representation for K, then G_j is the induced representation for S_n. Now, Frobenius reciprocity implies that

$$\langle V, G_j \rangle_{S_n} = \langle V_{|K}, \tau \rangle_K$$

for every S_n-representation V. Here, $\langle V, W \rangle_G$ denotes the dimension of the space of G-equivariant maps from V to W; therefore, if V is irreducible, $\langle V, W \rangle_G$ is the number of copies of V in W.

For example, for $V = \mathbb{R}$, the trivial representation, we get

$$\langle \mathbb{R}, G_j \rangle_{S_n} = \langle \tau, \tau \rangle_K = 1.$$

And if $V = \Delta_n^\perp$, then again

$$\langle \Delta_n^\perp, G_j \rangle_{S_n} = \langle \Delta_n^\perp |_K, \tau \rangle_K = 1$$

since the standard representation for S_n, restricted to $S_j \times S_{n-j}$ contains exactly two copies of τ.

The invariant inner product $\langle , \rangle$ gives an equivariant isomorphism $G_j \to G_j^*$ via $v \mapsto [u \mapsto \langle v, u \rangle]$; in particular must preserve the decomposition. This implies orthogonality of the decomposition. □

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ORIGINAL PAPER

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Welfare and stability in senior matching markets

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Abstract We consider matching markets at a senior level, where workers are assigned to firms at an unstable matching—the status-quo—which might not be Pareto efficient. It might also be that none of the matchings Pareto superior to the status-quo are Core stable. We propose two weakenings of Core stability: status-quo stability and weakened stability, and the respective mechanisms which lead any status-quo to matchings meeting the stability requirements above mentioned. The first one is inspired by the Top trading cycle and Deferred Acceptance procedures, the other one belongs to the family of Branch and Bound algorithms. The last procedure finds a core stable matching in many-to-one markets whenever it exists, dispensing with the assumption of substitutability.

Keywords Matching · Core consistency · Status-quo stability · Weakened stability · Branch and bound algorithm

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1 Introduction

1.1 Motivation

The report by Roth (2002) leads to a non-ambiguous conclusion: matching institutions should provide core stable outcomes.[1] While in theoretical settings the normative appeal of the core yields from its characterization, the argument, here, is factual. Specifically, clearinghouses that produce core stable outcomes survive, others do not. In our view, the relevance of core stability for clearinghouses is partly tautological: a core stable outcome is robust to attempts of self-reassignment by coalitions of agents. Otherwise, groups of agents would have good reasons to oppose the outcome proposed by the central institution for their freedom to engage in economic activities. Thus, clearinghouses which design core stable outcomes make them easier to enforce.

Nevertheless, inefficiencies might prevent decentralized labor markets from reaching core stability. Among others, the agenda of offers and acceptances may bias the assignment of agents; a worker might accept an offer by a firm and, once committed, receive the offer of a preferred firm she cannot accept anymore. One might also think about changes in the preferences of agents. The adoption of centralized mechanisms in matching markets at a junior level allows to tackle these inefficiencies.

These are not the only difficulties experienced by decentralized markets at a senior level. Namely, markets where agents are matched to one another, and these matchings are disrupted by changes in the population of agents. The analysis is pioneered by Blum et al. (1997). Cantala (2004) observes two features that explain why instability might be persistent in those markets, as well as Pareto inefficiency: (1) in the case where workers do not have tenure, the market reaches stability again *only if* the disruption is the opening of positions and retirement of workers *and* firms make offers; (2) dynamics of offers might not find existing stable matchings.

We consider matching markets at a senior level, where workers and firms are matched to one another at a status-quo matching. The status-quo might be neither stable nor efficient. We elaborate two approaches—solution concepts and the respective mechanisms—that lead the market to a Pareto improvement.

1.2 Features of the problem

Our analysis is restricted in the sense that we guarantee to *all* agents in the market a match at least as preferred as the status-quo. The requirement is natural for senior workers who might have a protective status. We think of senior professors holding a tenure. This right to stay permanently in a job also ensures that any job switch will be for a preferred position. Assuming, however, that both sides of the market can enforce the status-quo unilaterally would be too strong an assumption. Our approach, instead, is motivated by the centralized nature of the problem: guaranteeing the status-quo is an incentive device for agents to take part in the mechanisms, comparable to the participation constraint in the contract theory literature.

[1] See Roth and Sotomayor (1990) for a complete introduction to matching markets.

Suppose that the set of matchings Pareto superior to the status-quo is non empty, is one of those matchings core stable? The answer is negative, as shown in Example 1.

Example 1 Consider the following market with three firms, f_1, f_2, and f_3, and three workers, w_1, w_2, and w_3, where preferences are given by the following profile

$$\begin{array}{ccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} \\ \hline w_3 & w_1 & w_1 \\ w_1 & w_2 & w_2 \\ & & w_3 \end{array} \quad \text{and} \quad \begin{array}{ccc} \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline f_3 & f_3 & f_1 \\ f_2 & f_2 & f_3 \\ f_1 & & \end{array}.$$

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

The only matching Pareto superior to μ^0 is

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_1 & w_2 \end{pmatrix},$$

which is blocked by (f_3, w_1).

Hence, we are restricted to looking for core consistent procedures, namely those which select a core stable matching whenever it exists.

1.3 On manipulability

Alcalde and Barberà (1994) extend a result by Roth (1982) and show that there is no matching rule that is strategy-proof, efficient and individually rational. The negative result applies to our approach since they deal with the particular case where the status-quo is the empty matching, namely it is such that all agents are unmatched. We believe, however, that clearinghouses should not worry so much about the negative result. Dubins and Freedman (1981) and Roth (1982, 1984) consider markets where preferences are strict and show that mechanisms which select the optimal stable matching for one side of the market is strategy-proof for this side of the market. Demange et al. (1986) establish a general result, when preferences might be not-strict and, thus, the optimal stable matching defined above may not exist. Strategic questions for the other side of the market are analyzed in Roth (1982, 1984) and Gale and Sotomayor (1985). More recently and specifically about the DA algorithm, Ehlers (2004) considers that workers assign a probability to be matched to desirable firms. In this setting, manipulating seems to be a very sophisticated behavior.

Our issue is also related to the literature on a one-sided assignment when agents own property rights, which is comparable to holding a tenure. While in these markets there is conflict between equal treatment of equals, Pareto optimality and strategy-proofness (Zhou 1990), there is literature, following Shapley and Scarf (1974) and

their Top trading cycle procedure, where the authors show that it is possible to combine core stability and group-strategy proofness (Roth 1982; Ma 1994; Svenson 1999; Bird 1984; Moulin 1995; Abdulkadiroğly and Sönmez 1998; Abdulkadiroğly and Sönmez 1999; Papaï 2000[2] and others). From Alcalde and Barberà (1994), in contrast, we know that there is no core consistent procedure that is strategy-proof. Furthermore, we exhibit in Example 2 that new ways to manipulate a mechanism arise from core consistency.

1.4 Two core consistent solutions

We propose two weakenings of the core. Both intend to capture the idea developed earlier: the "less" agents oppose a matching, as formalized by blocking coalitions, the easier it is to enforce. First, *status-quo stability*. We guarantee to all agents an outcome at least as preferred as the status-quo. Thus, a blocking coalition that is not compatible with a re-assignment of all agents to matches at least as preferred as their status-quo is not a valid objection. Hence, a matching where all blocking coalitions are not valid, faces no legitimate opposition. In this sense it is stable as a status-quo, or status-quo stable.

Notice that there is no conflict between status-quo stability and Pareto efficiency, moreover the solution concept itself is not core consistent. We define a two-step procedure, the Status-Quo Stable (SQS) procedure: first Pareto efficiency is reached by a graph representation of the problem inspired by the Top trading cycle; second, we adapt the Deferred Acceptance (DA) algorithm, from Gale and Shapley (1962), to our setting to reach core consistency. We show that our status-quo stable procedure finds a status-quo stable matching. In particular, whenever a core stable matching exists, the procedure picks the core stable matching unanimously preferred by workers among all status-quo matchings Pareto superior to the status-quo and status-quo stable. However, it does not single out an outcome. Moreover, the procedure only applies to one-to-one markets.

Second, *weakened stability*. Consider again academic markets. Suppose that a centralizer has to choose between two matchings, both Pareto superior to the status-quo and neither Pareto dominates the other. The first matching is such that a university with a micro position hires a micro specialist and blocks the matching with a micro professor. The second matching is such that a university with a macro position hires a micro specialist and blocks the matching with a macro professor. We argue that the first blocking coalition is a weaker opposition than the second one. It is so because it is desirable, from an educational point of view, that a position be held by the adequate specialist. Formally it means that blocking coalitions are comparable and that this comparison follows from a social objective: the more a blocking pair impacts the social objective, the stronger objection it is. Matching models do not capture this feature of the market. To do so, we endow the economy with a societal welfare function.[3] Among all matchings Pareto superior to the status-quo, we choose the one with the

[2] Papaï (2000) does not assume property rights.

[3] In our leading example, the function only takes into account the utility of universities.

weakest opposition. Specifically, for all such matchings, we sum all utility improvements for firms from all blocking coalitions, and pick the matching which entails the smallest such summation [See (1)].

How would DA algorithms perform in our setting? First, the procedures, adapted in Blum et al. (1997) and Cantala (2004) to senior markets, do not take into account the welfare restrictions above mentioned, except individual rationality. Second, it might cycle. One type of cycling is harmless: even if we consider a case where a status-quo matching exists, one can easily design an example where a DA algorithm would cycle. To solve the difficulty one might adopt the solution proposed by Roth and Vande Vate John (1990), namely introduce loop detectors in the algorithm that detect them and launch a new sequence of offers until finding the one that leads to a stable matching. The solution provides no clue if such a matching does not exist. Finally, these procedures require firms to have substitutable preferences.

We make use of a much more versatile family of procedures: Branch and Bound Algorithms. Four of their properties motivate the choice: (a) they do not require any restriction on the preferences of firms, (b) by construction they do not cycle, (c) they can compute all the possible solutions of the problem—which means, in the case of junior markets, that they might compute all the stable matchings, (d) whenever the problem to solve has no solution, they specify it.

We establish that the outcome matching of our Weakened Stability (WS) Algorithm is the solution to our problem and it is status-quo stable. Moreover, when the input matching is the empty one, it is core stable whenever a core stable matching exists, even if the preferences of firms are not substitutable.[4]

Section 2 introduces notations, Sect. 3 deals with manipulability, status-quo stability is presented in Sect. 4 and Weakened Stability in Sect. 5. The Appendix contains the proof of theorems, the details of the WS Algorithm and conditions that guarantee the existence of a core stable matching Pareto superior to any status-quo.

2 Preliminaries

2.1 The market

A many-to-one matching market is a quadruple $(\mathcal{F}, \mathcal{W}, q, \succ)$ where $\mathcal{F}$ and $\mathcal{W}$ are two disjoint finite sets of agents. $\mathcal{F} = \{f_1, \ldots, f_m\}$ is the set of firms and $\mathcal{W} = \{w_1, \ldots, w_n\}$ is the set of workers; generic firms and workers will be denoted by f and w, respectively. Subsets of $\mathcal{F}$ and $\mathcal{W}$ are denoted by F and W. The vector of quotas associated with each firm is $q = \left(q_f\right)_{f \in \mathcal{F}}$, where q_f is the maximum number of workers that can be assigned to firm f. Preference relations are not symmetrically defined between firms and workers since a firm can be assigned to many workers whereas a worker can be assigned to at most one firm. Each firm f has a strict, transitive and complete preference relation $\succ_f$ over the family of subsets of workers $2^{\mathcal{W}}$. We interpret the empty set as firm f not being assigned to any worker. When a firm

[4] Echenique and Oviedo (2004), among others, also dispense on the substitutability assumption using fixed point techniques.

ranks the empty set better than a subset, it means that it prefers remaining unmatched to being assigned to this subset. Each worker w has a strict, transitive and complete preference relation $\succ_w$ over the set $\mathcal{F} \cup \{\emptyset\}$. We interpret the empty set in $\succ_w$ as w being unemployed. Preference profiles are $(m+n)$-tuples of preference relations and they are represented by $\succ = \left(\succ_{f_1}, \ldots, \succ_{f_m}, \succ_{w_1}, \ldots, \succ_{w_n}\right)$.

For any firm f we define the *acceptable set of f under q and $\succ$* to be the subsets of workers with cardinality smaller or equal to q_f, strictly preferred to the empty set; namely $A_f(q, \succ) \equiv \left\{S \subseteq \mathcal{W} \mid S \succ_f \emptyset \text{ and } |S| \leq q_f\right\}$. Subsets in $A_f(q, \succ)$ are called acceptable. Since only acceptable subsets will matter, we represent the preferences of the firm as a list of acceptable subsets. Likewise, for any worker w we define the *acceptable set of w under $\succ$* to be the set of firms strictly preferred to $\emptyset$. We denote it by $A_w(\succ)$. Firms in $A_w(\succ)$ are called acceptable. We will represent the preferences of firms and workers by ordered lists of acceptable partners. A pair (w, f) is acceptable under q and $\succ$ if both agents are mutually acceptable.

Definition 1 A *matching* μ is a mapping from the set $\mathcal{F} \cup \mathcal{W}$ into the set of all subsets of $\mathcal{F} \cup \mathcal{W}$ such that for all $f \in \mathcal{F}$ and $w \in \mathcal{W}$:

(1) $\mu(f) \in 2^{\mathcal{W}}$ and $|\mu(f)| \leq q_f$,
(2) either $|\mu(w)| = 1$ and $\mu(w) \in \mathcal{F}$, or $\mu(w) = \emptyset$,
(3) $\mu(w) = f$ if and only if $w \in \mu(f)$.

Let $\mathcal{M}$ denote the space of all possible matchings.

2.2 Stability concepts

Let $\succ$ be a preference profile. Given a set $W \subseteq \mathcal{W}$, let the *Choice* of firm f, denoted as $Ch\left(W, q_f, \succ_f\right)$, be f's *most preferred* subset of W with cardinality at most q_f according to its preference ordering $\succ_f$.

A matching μ is *blocked by a worker w* if she prefers remaining alone than being matched to $\mu(w)$; i.e., $\emptyset \succ_w \mu(w)$. Similarly, μ is *blocked by a firm f* if $\mu(f) \neq Ch\left(\mu(f), q_f, \succ_f\right)$. We say that a matching is *individually rational* if it is not blocked by any individual agent. A matching is *blocked by a worker-firm pair (w, f)* if worker w prefers being matched to f than to $\mu(w)$ and f would like to hire w; i.e., $f \succ_w \mu(w)$ and $w \in Ch\left(\mu(f) \cup \{w\}, q_f, \succ_f\right)$.

Definition 2 A matching μ is *pair-wise stable* if it is not blocked by any individual agent or any worker-firm pair.

Let W be a subset of $\mathcal{W}$. A matching μ is *blocked by a workers-firm coalition (W, f)* if all workers w in W prefer being matched to f than to $\mu(w)$ and f would like to hire W; formally if for all $w \in W$, $f \succ_w \mu(w)$ and $W \subseteq Ch\left(\mu(f) \cup W, q_f, \succ_f\right)$. We say that (W, f) forms a blocking coalition of μ. Let $\mathcal{W}_{f,\mu}$ be the set of workers who prefer f to their match under μ and, thus, they are potential members of blocking coalitions of μ. Formally, $\mathcal{W}_{f,\mu} = \{w \in \mathcal{W} \mid f \succ_w \mu(w)\}$.

Definition 3 A matching μ is *group-stable* if it is not blocked by any individual agent or by any workers-firm coalition.

A group-stable matching is also pair-wise stable; moreover core stability defined by weak dominance and group stability coincide in such markets.[5]

3 Strategy proofness

We aim to design a core consistent procedure which assigns to all agents in the market a match at least as preferred as their status-quo, and Pareto undominated. Unfortunately, none of them is strategy-proof.

Definition 4 A mechanism is strategy-proof if it is a dominant strategy, for all agents, to report their true preferences.

We now state the negative result.

Theorem 1 *In senior matching markets, there is no core consistent and strategy-proof mechanism that chooses a Pareto undominated matching and which guarantees to all agents a match at least as preferred as the status-quo.*

With respect to Alcalde and Barberà (1994), Example 2 exhibits new manipulations due to core consistency.

Example 2 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}, q_{f_1} = q_{f_2} = q_{f_3} = 1, \mathcal{W} = \{w_1, w_2, w_3\}$ and true preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_3	w_3	w_1	f_3	f_1	f_2
w_2	w_2	w_3	f_1	f_2	f_1
w_1					f_3

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

There are two matchings which are Pareto superior to the status-quo:

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2 & w_3 & w_1 \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_2 & w_1 \end{pmatrix}.$$

Notice that μ^1 is stable while μ^2 is blocked by (f_2, w_3), thus a core consistent procedure should pick μ^1. Nevertheless, if f_1 reports $\succ'_{f_1}$ where w_3 is preferred to w_1 and w_2 is not acceptable, the only matching Pareto superior to the status-quo is μ^2, which has to be selected, even if it is not core stable. Thus, in this market, firm 1 would gain by misrepresenting its preferences through $\succ'_{f_1}$.

[5] See Roth (1984).

4 Status-quo stability

The status-quo is guaranteed to all agents. Thus, to be considered as a valid objection to a matching, blocking coalitions have to be compatible with a reassignment that makes all agents at least as well off as in the status-quo. In this sense, in Example 2, if μ^2 becomes the new status-quo, the blocking pair (f_2,w_3) to μ^2 is not valid since, if f_2 and w_3 are matched, f_1 cannot be reassigned to a firm preferred to her status-quo, w_3.

Definition 5 Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$, a matching μ is *status-quo stable* if for all blocking coalitions $(f, W) \subseteq \mathcal{F} \times 2^{\mathcal{W}}$ to μ, there is no matching where f and W are assigned to each other, possibly with other workers, which it is Pareto superior to μ.

As for status-quo stability, there is no conflict between blocking coalitions and Pareto optimality. Thus, given a status-quo μ_o, looking for matchings that are status-quo stable and Pareto superior to μ_o is equivalent to looking for the set of matchings Pareto superior to μ_o which are not Pareto dominated by another matching. Denote the set $\text{SQS}(\mu_o)$; by transitivity of preferences it is not empty whenever there is at least one matching Pareto superior to the status-quo. Example 1 also shows that picking a matching randomly in $\text{SQS}(\mu_o)$ is not a core consistent procedure since both μ^1 and μ^2 are status-quo stable.

Our aim is not only to reach a matching in $\text{SQS}(\mu_o)$ but to select a core stable matching, whenever it exists. The status-quo stability procedure performs the task for one-to-one markets.

4.1 The status-quo stability (SQS) procedure

The SQS procedure begins by a graph representation of our problem.

1. Each node represents a match as defined by the status-quo μ_o; if $\mu_o(f) = w$, (f, w) is assigned a node, if $\mu_o(f) = \emptyset$, f is assigned a node and if $\mu_o(w) = \emptyset$, w is assigned a node.
2. From each node with a worker w, draw all arrows towards[6] nodes with firms f such that both w and f prefer each other to their respective status-quo.
3. Identify all *cycles* and *paths* defined as follows.
   A *cycle* is an ordered set S of pairs (f, w) which appear only once in S, where, in the graph constructed as mentioned in 1 and 2
   (a) from each node (f, w) in S an arrow points to another node in S,
   (b) (f, w) is pointed by an arrow from another node in S, moreover
   (c) (f', w') follows (f, w) in S only if (f, w) points to (f', w') in the graph, finally the first pair in S is said to follow the last one.
   A *path* is an ordered set S with one and only one single worker w , one and only one firm f and possibly pairs (f', w'), they all appear only once in S and, in the graph constructed as mentioned in 1 and 2

[6] Thus, it is a directed graph.

(a) the node with the single worker w points to another node in S and is the first element in the set,
(b) for each node (f', w') in S there is one arrow that points to another node in S and (f', w') is pointed to by an arrow from another node in S,
(c) the node with the single firm f is pointed to by another node in S and is the last element in the set,
(d) $[(f', w')$ or $f']$ follows $[(f, w)$ or $w]$ in S only if $[(f, w)$ or $w]$ points to $[(f', w')$ or $f']$ in the graph.

Let $\mathcal{P}$ be the set of all paths and cycles and denote as p an element in $\mathcal{P}$. We are now ready to construct all possible Pareto improvements that may lead the market to status-quo stability, and select one of them.

4. A *composition* c is a subset of $\mathcal{P}$ such that:
   (a) for all $p, p' \in c$, $p \cap p' = \emptyset$ and
   (b) for all $p'' \in \mathcal{P}$ which do not belong to c, there is at least one $p \in c$ and $p'' \cap p \neq \emptyset$.

   Let $\mathcal{C}$ be the set of all compositions. We say that a worker w prefers composition c to composition c' if she prefers the firms which follow her in c to the one in c'.
5. Given a status-quo μ_o and a composition c in $\mathcal{C}$, the induced matching $\mu(\mu_o, c)$ is such that
   (a) if a firm f' belongs to a path or a cycle p in the composition c, it is assigned the worker w of the previous element (single worker or couple) in p;
   (b) otherwise it is assigned the same match as in μ_o.

   Let $I(\mu_o, \mathcal{C})$ be the set of induced matchings by all compositions in $\mathcal{C}$.
   5.1 If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$ then $SQ - S(\mu_o) := \mu_o$, otherwise let $i := 1$,
   5.2 If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$, go to 5.5.
   Otherwise, pick a worker and let her choose within $I(\mu_o, \mathcal{C})$ her favorite matching in $I(\mu_o, \mathcal{C})$; if she is indifferent between different matchings, pick a second worker to break the tie and so on and so forth until a single matching μ^i is selected. If $i := 1$, let $\overline{\mu} \equiv \mu^i$.
   5.3 Let all firms f make simultaneous offers to all groups of workers which are preferred to $\mu^i(f)$ for all f.
   5.4 If no offer is accepted by all workers in a group, $SQ - S(\mu_o) := \mu^i$, otherwise $I(\mu_o, \mathcal{C}) := I(\mu_o, \mathcal{C}) \setminus \mu^i$, $i := i + 1$; go to 5.2.
   5.5 $SQ - S(\mu_o) := \overline{\mu}$.

We now perform the SQS procedure in a simple market.

Example 3 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$, $q_{f_1} = q_{f_2} = q_{f_3} = q_{f_4} = 1$, $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ and preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{f_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_4	w_3	w_1	w_2	f_4	f_1	f_4	f_2
w_1	w_1	w_4	w_3	f_2	f_4	f_2	f_3
w_2	w_2	w_3	w_4	f_1	f_2	f_3	f_4

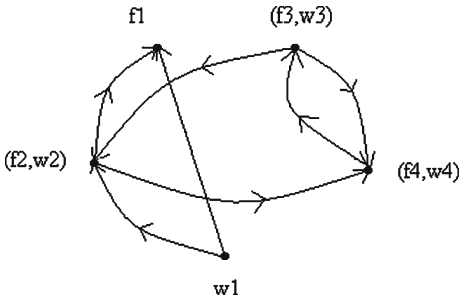


Fig. 1 Stages 1–2: the graph representation

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & \varnothing & f_2 & f_3 & f_4 \\ \varnothing & w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

The SQS works as follows:
1–2. We represent the status-quo by the graph in Fig. 1.
3. There are two cycles:

$$p_1 = [(f_2, w_2), (f_4, w_4), (f_3, w_3)] \quad \text{and}$$
$$p_2 = [(f_3, w_3), (f_4, w_4)],$$

and two paths:

$$p_3 = [(w_1), (f_2, w_2), (f_1)] \quad \text{and}$$
$$p_4 = [(w_1), (f_1)].$$

4. The set of compositions is $\mathcal{C} = \{c_1, c_2, c_3\}$ where

$$c_1 = ([(w_1), (f_1)], [(f_2, w_2), (f_4, w_4), (f_3, w_3)]),$$
$$c_2 = ([(w_1), (f_1)], [(f_3, w_3), (f_4, w_4)]), \quad \text{and}$$
$$c_3 = ([(w_1), ((f_2, w_2)), (f_1)], [(f_3, w_3), (f_4, w_4)]).$$

5. Then $I(\mu_o, \mathcal{C}) = \{\mu^1, \mu^2, \mu^3\}$ where

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_3 & w_4 & w_2 \end{pmatrix},$$
$$\mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_4 & w_3 \end{pmatrix} \text{ and}$$
$$\mu^3 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2 & w_1 & w_4 & w_3 \end{pmatrix}.$$

Now, if w_3 is selected to pick a matching in $I(\mu_o, \mathcal{C})$, she only discards μ^1, and if w_2 is selected, μ^3 is singled out. Offers are emitted by w_1 and w_4, respectively, to f_4 and f_2. Since none is accepted, μ^3 is stable, $SQ - S(\mu_o) = \mu^3$.

Theorem 2 states that our SQS procedure finds a status-quo stable matching and it is a core consistent procedure.

Theorem 2 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$, $q_f = 1$ *for all* $f \in \mathcal{F}$ *and a status quo* μ_o *then*

1 $SQ - S(\mu_o)$ *is status-quo stable, and Pareto superior to* μ_o *or is* μ_o,
2 *whenever the set of core stable matchings Pareto superior to* μ_o *is non-empty,* $SQ - S(\mu_o)$ *is the core stable matching unanimously preferred by workers (and worst for firms).*

4.2 Comments

If none of the status-quo stable matchings is core stable, workers might not agree on a ranking of matching in SQS(μ_o), thus the order in which they are picked in the procedure might affect the output matching.

The procedure is not monotonic in the status-quo in the sense that a preferred status-quo does not ensure a preferred outcome of the SQS procedure. In particular the advantage of being guaranteed the status-quo might well be balanced by the fact that switching to a better position is conditioned by the simultaneous improvement of the match. Indeed, if a worker/firm is the best alternative for her/its match, she/it cannot switch to another position.

This simultaneous improvement requires a central intervention since, unlike in the Top trading cycle procedures, agents might belong to two different paths or cycles, hence compatible reassignments are not likely to occur without coordination. Moreover, Stage 5 is necessary for the SQS procedure to be core consistent.

Finally, the status-quo stability procedure is not adaptable to many-to-one markets when firms have preferences which are not responsive. The Weakened Stability (WS) algorithm, in contrast, does not require any assumption about the preferences of firms over subsets of workers.

5 Branch and bound algorithms and weakened stability

5.1 The optimization problem

We assume that preferences of firms are represented by utility functions, generically denoted u_f for firm f. We choose the reversed order representation: the lower the utility, the better; and the best subset of workers is assigned utility 0.[7] In this approach blocking coalitions are comparable: the more a blocking coalition impacts the utility of the firm, the stronger it is.

Formally, let (S, f) be a blocking coalition of μ, then $u_f(\mu(f)) - u_f(S)$ measures the welfare improvement of f if the blocking is completed. Our stability index is the sum of utility improvements for firms from all blocking coalitions.

[7] This is convenient for the algorithm, although not necessary.

Definition 6 Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$; for a matching μ, let

$$i \equiv \sum_{\text{All blocking coalition } (S,f) \text{ of } \mu.} u_f(\mu(f)) - u_f(S),$$

then μ is said to be *weakened stable of order* i.

Notice that a matching weakened stable of order 0 is core stable. Denote by WS_i the set of matchings that are weakened stable of order i. We now define the utilitarian societal welfare function $W(\mu) = \sum_{f\in\mathcal{F}} u_f(\mu(f))$ that we aim to minimize, choosing a matching within the set of weakened stable matchings of the lowest order.

Formally, given a status-quo μ_o, our problem is

$$\begin{aligned} \min_{\mu \text{ is Pareto superior to } \mu_0} \quad & W(\mu) \\ \text{s.t.} \quad & \mu \in WS_i \quad \text{and} \\ & WS_j = \emptyset \quad \text{if } j < i. \end{aligned} \tag{1}$$

Hence, a matching μ is selected instead of another matching μ' in the following cases: (a) whenever the order of stability of μ is lower than the one of μ', (b) whenever the order of stability of μ or μ' are the same but $W(\mu) < W(\mu')$; otherwise μ and μ' are indifferent. Notice that the status-quo is the solution to the program when it is not Pareto dominated. The following algorithm finds this (these) optimal matching(s).

5.2 The weakened stability algorithm

For all firms $f \in \mathcal{F}$, let $B_f(\mu_o) = \{W \subseteq 2^{\mathcal{W}} | W \succeq_f \mu_o(f)\}$ be the set of subsets of workers f prefers to its status-quo, and for all workers $w \in \mathcal{W}$, let $B_w(\mu_o) = \{f \in \mathcal{W} | f \succeq_w \mu_o(w)\}$ be the set of firms w prefers to her status-quo. Let $A = \times_{f\in\mathcal{F}}(B_f(\mu_o) \cup \{\emptyset\})$, where for all elements in A, the profile of workers where the f^{th} entry is interpreted as being assigned to firm f. Notice that A contains all matchings Pareto superior to μ_o, that is why we will restrict our attention to assignments in A. We also observe that some of the matchings in A may not be Pareto superior to μ_o since preferences of workers are not taken into account in A. Finally, some assignments in A may not be matchings since, for instance, a worker might be assigned to many firms.

The WS algorithm belongs to the family of Branch and Bound (BB) algorithms. This technique is one of the most commonly used in optimization problems[8] when all or some of the decision variables are discrete (integer or mixed programing) and no characterization of optima exists; namely unlike first and second order conditions in differential calculus environments. As a consequence, the set of decision variables, A in our case, has to be scrutinized.

In our problem, there are as many decision variables as firms in the market, hence, the number of solutions can be very large: we call any matching a solution, and a

[8] Branch and Bound algorithms are used to solve, for instence, the classical assignment problem in operation research.

matching that solves (1) is an *optimal* solution. The efficiency of BB algorithms relies on the fact that, instead of analyzing a particular solution at a time, they discard sets of solutions. We denote $R \equiv (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$, $R \subseteq A$, the set of solutions where the subset of workers W_f is assigned firm f for $f = 1, \ldots, n$, and there is no specific subset assigned to firms $f = n+1, \ldots, F$.

For all $R = (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$ we define $Z_L(R)$, the upper bound of the objective function of problem (1)[9] reached by solutions in R. Formally,

$$Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min \left\{ u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f \right\}.$$

Thus, $Z_L(R)$ is the minimal value reached by the objective function when all firms $n+1, \ldots, F$ are assigned their favorite subset of workers among those not assigned at R. We call $\overline{R}$ the assignment in R for which the value of the objective function is $Z_L(R)$. It might be that $\overline{R}$ is neither a matching nor stable, in any case if this lower bound does not improve upon the tentative optimal solution when the last one is stable of order 0, no matching in R will be optimal, therefore solutions in R are discarded.[10]

We keep a record of the following information: in $WSP(\mu_0)$ the best current solution in the process, in i_t its order of weakened stability and in Z_U the value of its objective function.

The stack, S, is the set of solutions that the algorithm still has to scrutinize. At each iteration, the algorithm picks a set of solutions $R \equiv (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$ in S, deletes it from the stack ($S := S \setminus R$), and performs the following tests:

(a) When the tentative optimal solution[11] is core stable, is the value of the objective function of the tentative solution smaller than the upper bound of R?
(b) Can one assign to each of the unassigned workers in R a firm preferred to the status-quo?
(c) Can one assign to each of the unassigned firms in R a group of workers preferred to the status-quo?

If the answer to at least one question is positive, the optimal solution cannot belong to R, another set of solutions in the stack is considered. Otherwise, one cannot discard solutions in $R = (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$, we break off R in subfamilies of the form $R' = (W_1, \ldots, W_n, W_{n+1}, \emptyset, \ldots, \emptyset)$. There are as many subfamilies as subsets of workers unassigned in R preferred to the status-quo by firm f_{n+1}. Thus, for each W_{n+1} in B_{n+1} and $\mathcal{W} \setminus \cup_{f=1}^{n} W_f$ Pareto superior to the status-quo, a subfamily of

[9] $Z_L(R) \geq \min_{\mu \in R} W(\mu)$

[10] The use of the tentative optimal objective values motivates the term Bound in the expression "Branch and Bound Algorithm".

[11] The tentative optimal solution is the solution which is optimal within the set of solutions already scrutinized.

solutions $R' = (W_1, \ldots, W_{n+1}, \emptyset, \ldots, \emptyset)$ has to be inspected. These solutions are included in the stack; i.e., $S := S \cup \{R'\}$ for all such R'.

If a solution is such that all firms are assigned a subset of workers, its order of stability is computed as well as the value of the societal welfare function. If its order of stability of μ is lower than the one of the tentative solution, or the same but the value of the societal welfare function is lower, it becomes the new tentative solution. Otherwise, it is discarded.

The algorithm stops when the stack is empty. Then the last tentative solution becomes the outcome of the algorithm. The detail of the algorithm is in the Appendix.[12] Example 4 shows how our procedure works.

Example 4 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_1	f_3	f_1	f_1
w_3	w_2	w_2	f_2	f_3	f_3
w_1	w_3	w_3	f_1	f_2	f_2

Moreover,

$$u_{f_1}(w_2) = 0 = u_{f_2}(w_1) = u_{f3}(w_1)$$
$$u_{f_1}(w_3) = 1 = u_{f_2}(w_2) = u_{f3}(w_2)$$
$$u_{f_1}(w_1) = 2 = u_{f_2}(w_3) = u_{f3}(w_3)$$

The status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Thus, at $t = 1$:

$$\begin{aligned} WSP(\mu_0) = \mu_0, i &= u_{f_1}(w_1) + u_{f_2}(w_2) + u_{f3}(w_3) = 2 + 1 + 2 \quad \text{and} \\ Z_U &= [u_{f_1}(w_3) - u_{f_1}(w_1)] + [u_{f_1}(w_2) - u_{f_1}(w_1)] + [u_{f_2}(w_1) - u_{f_2}(w_2)] \\ &\quad + [u_{f_3}(w_2) - u_{f_3}(w_3)] + [u_{f_3}(w_1) - u_{f_3}(w_3)]. \\ &= [2-1] + [2-0] + [1-0] + [2-1] + [2-0] \\ &= 1 + 2 + 1 + 1 + 2 = 7 \\ S &= (\emptyset, \emptyset, \emptyset). \end{aligned}$$

Iteration 1: pick $R = \{(\emptyset, \emptyset, \emptyset)\}$.

(a) When the tentative optimal solution is core stable, is the value of the objective function of the tentative solution smaller than the upper bound of R?
The tentative solution is not stable.

[12] Notation in the Example differs slightly from the notation used in Appendix.

(b) Can one assign to each of the unassigned workers in R a firm preferred to the status-quo?
Yes.
(c) Can one assign to each of the unassigned firms in R a group of workers preferred to the status-quo?
Yes.

Then $S = \{(w_2, \varnothing, \varnothing), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 2: pick $R = (w_2, \varnothing, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_2, w_1, \varnothing), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 3: pick $R = (w_2, w_1, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_2, w_1, w_3), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 4: pick $R = (w_2, w_1, w_3)$.

$$i((w_2, w_1, w_3)) = [u_{f_3}(w_1) - u_{f_3}(w_3)] = 2 < 7$$
$$Z_L((w_2, w_1, w_3)) = 2$$

Thus

$$WSP(\mu_0) := (w_2, w_1, w_3)$$
$$i := 2$$
$$Z_U := 2.$$

and $S = \{(w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 5: pick $R = (w_3, \varnothing, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_2, \varnothing), (w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 6: pick $R = (w_3, w_2, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_2, w_1), (w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 7: pick $R = (w_3, w_2, w_1)$,

$$i((w_3, w_2, w_1)) = [u_{f_1}(w_2) - u_{f_1}(w_3)] = 1 = i$$
$$Z_L((w_3, w_2, w_1)) = 2$$

Thus

$$WSP(\mu_0) := (w_2, w_1, w_3)$$
$$i := 1$$
$$Z_U := 2.$$

and $S = \{(w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 8: pick $R = (w_3, w_1, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_1, w_2), (w_1, \varnothing, \varnothing)\}$.
Iteration 9: pick $R = (w_3, w_1, w_2)$

$$i((w_3, w_1, w_2)) = [u_{f_3}(w_2) - u_{f_3}(w_1)] = 1 = i$$
$$Z_L((w_3, w_1, w_2)) = 2 = Z_U$$

Thus

$$WSP(\mu_0) := \{(w_2, w_1, w_3), (w_3, w_1, w_2)\}$$
$$i := 1$$
$$Z_U := 2.$$

and $S = \{(w_1, \varnothing, \varnothing)\}$.
Iteration 10: pick $R = (w_1, \varnothing, \varnothing)$

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_1, w_2, \varnothing)\}$.
Iteration 11: pick $R = (w_1, w_2, \varnothing)$

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_1, w_2, w_3)\}$.
Iteration 12: pick $R = (w_1, w_2, w_3)$

$$i((w_1, w_2, w3)) = 7 > i$$
$$Z_L((w_3, w_1, w_2)) = 5 > Z_U$$

It is discarded and $S = \{\varnothing\}$ and $WSP(\mu_0) := \{(w_2, w_1, w_3), (w_3, w_1, w_2)\}$. This concludes Example 4.

We are now ready to state our main result.

Theorem 3 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$ *and a status quo* μ_o *then* $WSP(\mu_0)$ *is a solution to* (1).

In particular, if the status quo is the empty matching, our algorithm finds a core stable matching whenever such matching exists, dispensing with the condition of q-substitutability.

Corollary 1 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$ *and let the status-quo* μ_o *be the empty matching. Then, when a core stable matching exists, the output of the WS algorithm is core stable.*

Moreover a solution to (1) cannot be Pareto dominated: let μ and μ' be two matchings such that μ dominates μ'. All blocking coalitions to μ are also blocking coalitions to μ' by transitivity of the preferences. The reverse is not true. Thus, the order of stability of μ is smaller than the order of stability of μ'.

Corollary 2 $WSP(\mu_0)$ *is status-quo stable.*

6 Concluding remarks

Since the WS algorithm selects the firm-best core stable matching whenever it exists, it recovers the same incentive properties of the mechanisms studied in Dubins and Freedman (1981) and Roth (1982, 1984) on the respective settings. Nevertheless, the simplicity of Example 1 and Proposition 1 suggest that the lack of existence of a core stable solution is not pathological. Moreover, manipulating core consistent procedures does not seem to require a sophisticated behavior, as suggests Example 2.

Our motivation to develop Weakened Stability is not primarily applicability. Rather, we are interested in capturing features of matching markets usually ignored by the modeler. Nevertheless, some properties of the solution concept look attractive. First, it is core consistent and core stability has shown to be a remarkable property of enforceability. Second, there is no conflict between Weakened stability and Pareto efficiency: if a matching dominates another in Pareto terms, its order of stability is lower. Third, in some countries, workers of the public sector are associated to an index which takes into account their seniority, professional performance or family situation. This index makes them comparable; in particular it is used to define priority orders. Thus, building up a societal welfare function might be seen as the formalization of a real life practice. Moreover, these functions depend on observable variables, coping partially with the problem of manipulability. Finally, Branch and Bound algorithms are versatile tools able to solve a large scope of variations from problem (1). In particular, we believe it is relevant to compare the performances of BB algorithms and the procedure used by the National Resident Matching Program.

Appendix

We investigate now the sufficient conditions which guarantee the existence of a group stable matching Pareto superior to a status-quo. We recall the following definitions.

Definition 7 A matching μ is *worker quasi-stable* if it is individually rational and for any blocking coalition (S, f), $\mu(w) = \emptyset$, for all $w \in S$.

Definition 8 A matching μ is *firm quasi-stable* if it is individually rational and for any firm f, worker $w \in \mu(f)$ and subset of workers $S \subseteq \mathcal{W}_{f,\mu}$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$.

Definition 9 A matching μ is *quasi-stable* if it is individually rational and for every blocking coalition (S, f), for every $w \in \mu(f)$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$ and $\mu(w) = \emptyset$, for all $w \in S$.

Proposition 1 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$ and a matching μ_0. Assume that the set of matchings Pareto superior to μ_o is non-empty, we know that one of them is core stable when firms have q-substitutable preferences and the input matching is quasi-stable.*

Proof The argument is constructive: if the matching of departure is quasi-stable, in particular it is firm quasi-stable. Since firms have q-substitutable preferences, Proposition 1 in Cantala (2004) shows that applying his modified version of the DA algorithm leads to a core stable matching and that all along the sequence of tentative matchings, workers are never dismissed and all assignments are firm quasi-stable. by quasi-stability, original blocking pairs only involve unmatched workers, therefore resolving them makes no firm worse off and no new blocking coalition appears along the process. Thus, all agents get better assignments, no new blocking pair appears, all tentative matchings are quasi-stable and the resulting matching, say μ, is Pareto superior to the status-quo matching. Finally since μ is stable, it is Pareto efficient.

One cannot dispense with q-substitutability since, then, it might happen that no stable matching exists. The next example shows that quasi-stability is also necessary for Proposition 1 to hold.

Example 5 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and $\succ$ is given by the following profile

$$\begin{array}{cccccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} & \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline \{w_1\} & \{w_3\} & \{w_2\} & \{f_1\} & \{f_1\} & \{f_3\} \\ \{w_2\} & \{w_1\} & \{w_3\} & \{f_2\} & \{f_3\} & \{f_2\} \end{array}.$$

Assume that the worker quasi-stable status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_2 & \emptyset & w_3 & w_1 \end{pmatrix}.$$

The only matching Pareto superior to μ^0 is

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2 & w_1 & w_3 \end{pmatrix},$$

which is blocked by (f_1, w_1).

Proof of Theorem 2 We observe that only arrows representing blocking pairs are drawn on the graph (step 2). Moreover, blocking pairs lead to a Pareto improvement only if dropped mates (if any) are also assigned a blocking mate (by definition preferred to the status-quo). Thus, one needs to identify all the ordered sets of blocking pairs (with the interpretation that $[(f', w')$ or $f']$ follows $[(f, w)$ or $w]$ if (w, f') is the blocking pair to be completed[13]) such that:

(a) if completed simultaneously, the market experiences a Pareto improvement and
(b) if one or some of them is withdrawn from the set, there is no such Pareto improvement.

Obviously Cycles and Paths are such sets; we show that they are the only ones. It is also clear that no blocking pair can appear twice in the sets since agents cannot complete two blocking pairs simultaneously. We adopt the convention that a set begins by a node with a worker (and possibly a firm) pointing towards another node (if there is no "pointing" in the set, neither are there blocking pairs). Since blocking is simultaneous, the order only matters in keeping track of who blocks with whom. Thus, if there is an unmatched worker in the set, there is no loss of generality in shifting all elements, ranking this unmatched worker first and following the original ordering; that is why, if there is an unmatched worker in the set, we put her first in the set.

Case 1 The set starts with an unmatched worker w.

If this worker blocks with an unmatched firm f, $\{w, f\}$ is the Pareto improving set as defined above, it is a path.

If this worker blocks with a matched firm f, the mate of f, w', will have to be assigned a firm f' in the set that is preferred to the status-quo. If this firm is unmatched, the set is $\{w, (f, w'), f'\}$, it is a path. Otherwise a pair (f', w'') has to follow (f, w') so as to assign w' a firm f' that is preferred to her status-quo. One can iterate the argument, until an unmatched firm appears in the sequence. If such unmatched firms did not exist, the blocking pairs specified by the ordered set would not be Pareto improving for the worker of the last pair, who would then remain unmatched. Therefore, the set is a path in any case. If there is more than one unmatched worker in the set, by the previous argument they would generate independent paths, since no pair can appear twice. Hence one of the paths might be withdrawn from the set without altering the Pareto improvement of agents in the other set.

Case 2 The set starts with a pair (f, w).

By our convention, there is no unmatched worker in the set. So as to compensate f for the fact that w blocks with another firm f', the last element of the set in the sequence has to be a couple (f^n, w^n) where w^n blocks with f. We observe that no unmatched firm can be included in the set, since the firm will not point to any other agent, in particular to couples, as required. Thus, the set is a cycle.

Hence, $\mathcal{P}$ contains all sets of blocking pairs such that, if they are completed simultaneously, all agents involved in the set will improve with respect to the status-quo. Of course, it might be that unmatched firms or workers, or matched worker-firm pairs are involved in many paths and cycles and, nevertheless one cannot complete

[13] Both w and f' might be involved in blocking pairs with other agents in the set.

simultaneously many blocking pairs. A composition of $\mathcal{P}$ (Step 4) is a set of compatible cycles and paths such that no other element in $\mathcal{P}$ is compatible with them.

We argue now that there is no matching Pareto superior to the one generated by a composition since the algorithm stops:

- either at step 5.4 when a matching is stable (in which case there is no matching Pareto superior to it, otherwise some agents would block);
- or at 5.5, when μ_1 is selected. Consider step 5.2 that leads to the selection of μ_1. If only one worker is necessary to select μ_1, it means that this worker strictly prefers μ_1 to any other matching in $I(\mu_o, \mathcal{C})$. If many workers are necessary to pick μ_1, notice that each time a matching in $I(\mu_o, \mathcal{C})$ is discarded by a worker, the discarded matching is strictly worse than μ_1 for this worker. Thus, μ_1 is not Pareto dominated by any matching in $I(\mu_o, \mathcal{C})$.

We prove now that the procedure picks the workers' optimal stable matching whenever it exists. We know from the lattice Lemma (Knuth 1976) that in one-to-one markets, if workers do not rank unanimously two stable matchings, by letting them choose their best mate between both matchings, not only does the picking function leads to a matching but leads to a stable one. Of course, if the two matchings are Pareto superior to the status-quo, so is the new matching. Thus, if there are stable matchings Pareto superior to the status-quo, one of them is unanimously preferred by workers. That is why we let workers choose their favorite matchings in $I(\mu_o, \mathcal{C})$ and check if those choices are stable. Specifically, if no offer emitted by firms is accepted by any worker, this is the outcome matching. Else another matching is chosen by new workers until a stable matching is found. If all sets of status-quo matching have been scrutinized and none of the matching is stable, the outcome matching is the first tentative matching. □

The WS algorithm

We keep a record of the following information: in $WSP(\mu_0)$ the best current solution in the process, in i_t its order of weakened stability and in Z_U the value of its objective function.

1. Initial round
   - For all $f \in \mathcal{F}$ define the function $B_f : \mathcal{M} \to 2^{2^{\mathcal{W}}}$ such that $B_f(\mu) = \{S \subseteq 2^{\mathcal{W}} | \, \#S \leq q_f \text{ and } S \succsim_f \mu(f)\}$. Let $A = \times_{f \in \mathcal{F}}(B_f(\mu) \cup \{\emptyset\})$.[Define the subsets of workers preferred by firms to a matching μ.]
   - For all $w \in \mathcal{W}$ define the function $B_w : \mathcal{M} \to 2^{\mathcal{F} \cup \{\emptyset\}}$ such that $B_w(\mu) = \{m \in \mathcal{F} \cup \{\emptyset\} | m \succsim_w \mu(w)\}$. [Define the set of firms preferred by workers to a matching μ.]
   - For all $f \in \mathcal{F}$ define the function $W_f : 2^{2^{\mathcal{W}}} \times 2^{\mathcal{F} \cup \{\emptyset\}} \to 2^{2^{\mathcal{W}}}$ such that $W_f(\mu) = \{W \in \mathcal{W} \mid W \subseteq B_f(\mu) \text{ and for all } w \in W, \, f \in B_w(\mu)\}$. [Define the set of subsets of workers who block μ with f.]
   - Define the function $i_0 : (2^{2^{\mathcal{W}}})^{\#\mathcal{F}} \to \Re$ such that $i(\mu) = \sum_{f \in F} \sum_{W \in W_f(\mu)} u_f(\mu(f)) - u_f(W)$ [$i(\mu)$ is the order of stability of matching μ.]

- For all $R \subset A$, define the function $Z_L : A \to \Re$ such that $Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min\{u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f\}$.
- $WSP(\mu_0) = \mu_o$. [The initial tentative optimal solution is the status-quo.]
- $Z_U = Z_L(\mu_o)$. [The objective value of the initial tentative solution is the one of the status-quo.]
- $i_0 := i(\mu_o)$ [i is the order of stability of the status-quo.]
- $S = \{(\emptyset, \ldots, \emptyset)\}$. [At the beginning, we have to review all possible solutions.]
- $t \equiv 1$.

(a)Iteration

2. Selection within the stack S, of a solution.
If $S = \emptyset$ then stop. [If the stack is empty, there are no more subsets to analyze and the tentative optimal solution is the solution to (1).]
Otherwise, let R be such that $R = \arg\min_{R' \in S} Z_L(R')$, $S := S \setminus \{R\}$. [We select the family of solutions with minimal lower bound.][14]

3. Fathoms. One discards R or checks whether the optimal solution may belong to R.

3.1 If $i_{t-1} = 0$ and $Z_U < Z_L(R)$ then go to 2. [If the tentative optimal solution is core stable and its objective function is smaller than the lower bound of R, solutions in R are discarded.]

3.2 If $\{f_{n+1}, \ldots, f_F, \{\emptyset\}\} \cap B_w(\mu_o) = \emptyset$ for (at least) one $w \in \mathcal{W} \setminus \cup_{f=1}^{n} W_f$, then go to 2. [If all unassigned firms are worse than the status-quo, the solution cannot belong to R for (at least) one unassigned worker, R is discarded.]

3.3 If for some firm $f \in \{f_{n+1}, \ldots, f_F\}$ no $W_f \in B_f(\mu_o)$ is such that $W_f \subseteq \{\mathcal{W} \cup \{\emptyset\}\} \setminus \cup_{f=1}^{n} W_f$, then go to 2. [If one cannot assign a group of workers preferred to the status-quo to each of the unassigned firms, R is discarded.]

3.4 If $n + 1 < F$ go to 4 [If more than one firm is not assigned any subset of workers, the solution is partitioned in subsets of solutions ...].
Else for $f = F$ define $W_F = \{W \subseteq \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$ such that [... else subsets in W_F are the only ones which complete R to form a matching Pareto superior to the status-quo ...]
(a) $W \in B_F(\mu_o)$,
(b) $F \in B_w(\mu_o)$ for all $w \in W$
(c) if $\mu_o(w) \neq \emptyset$ for $w \in \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$, then $w \in W_F\}$. [... in particular matched workers at the status quo have to be included.]

3.4.1 If $W_F = \emptyset$, go to 2.
Else, let $N \equiv \#W_F$ and $l \equiv 1$.

3.4.2.1 If $l \leq N$, select one $W \in W_F$, delete it from W_F and construct $R' = (W_1, \ldots, W)$.

[14] In order to improve the efficiency of the algorithm, one would idealy choose the family of solution with lower bound of stability. Nevertheless this lower bound is not computable, that is why we use as lower bound the value of the objective function.

Else $t = t+1$, go to 2. [One completes R assigning F to an acceptable subset of workers, including a fortiori those who are matched at the status-quo.]

3.4.2.2 If $i(R') < i_{t-1}$ or ($i(R') = i_{t-1}$ and $Z_L(R') \leq Z_U$)
then $i_t = i(R')$, $WSP(\mu_0) = R'$, $Z_U = Z_L(R')$. [A new tentative solution has been detected.]
In any case $l = l+1$, go to 3.4.2.1.

4. Branching: in case we cannot discard R, we break it off in smaller subsets. Notice that only Pareto superior matchings are included in the stack.
$S := S \cup \{(W_1, \ldots, W_{n+1}, \emptyset, \ldots, \emptyset) \subseteq A$ such that
(a) $(W_1, \ldots, W_n, \emptyset, \ldots, \emptyset) = R$, [New solutions in S are subfamilies of $R \ldots$]
(b) $W_{n+1} \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f$, [... obtained by complementing R with subsets of available workers ...]
c $W_{n+1} \in B_{f_{n+1}}(\mu_o)$, $f_{n+1} \in B_w(\mu_o)$ for all $w \in W_{n+1}\}$. [... compatible with the Pareto criterion.]
Then go to 2.

Proof of Theorem 3 We observe that the algorithm is well behaved in the sense that it always ends. To see this, notice first that, when an iteration ends up by a branching, one does not add new solutions to the stack but keeps the subset of solutions selected within a partition of the solution (we consider only the solutions that might be Pareto superior to the status-quo). Since the number of firms is finite, so is the number of iterations which end up by a branching. Furthermore, solutions are deleted from the stack at iterations which do not end up by a branching, hence the stack will end up empty and the algorithm will end.

So as to prove that the output matching of the algorithm is the optimal solution to problem (1), we argue that none of the three following errors occurs.

Error 1. A solution has not been scrutinized that should have been scrutinized.

At the initial round, all possible solutions preferred to the status-quo by firms are included in the stack. Solutions are eliminated from the stack when it is analyzed. Then, either it is discarded, either it is selected as a new tentative solution or one proceeds to branching. In this case, only solutions which are Pareto superior to the status-quo are introduced in the stack (other solutions cannot be optimal for (1)) and, thus, will be analyzed later on.

Error 2. A solution has been discarded which should not have been discarded.

In a given iteration, assume that the tentative optimal solution, $WSP(\mu_0)$, is correct; i.e., it is optimal within the set of solutions already scrutinized. The solution R is discarded at the following steps:

3.1. When the tentative matching is core stable and the lower bound of R is greater than the objective value of the tentative solution, no solution in R can be optimal.

3.2. If for (at least) one worker unassigned at R none of the firms unassigned at R is at least as good as the status-quo, no solution in R can be incentive compatible with μ_o for this worker.

3.3. If for (at least) one firm unassigned at R none of the subsets of workers unassigned at R is at least as good as the status-quo, no solution in R can be incentive compatible with μ_o for this firm.

3.4. Solutions in R which are not Pareto Superior to the status-quo are discarded, they cannot be optimal solutions to (1).

3.4.2 and 3.4.3. All matchings in R which are Pareto superior to the status-quo are compared to the tentative solution and discarded if their order of stability is higher than the one of the tentative solution, or, in case of a tie, when their objective value is higher.

Hence, if the tentative solution is correct, so is the fact to discard families of solutions at 3.1, 3.2, 3.3, 3.4, 3.4.2 and 3.4.3.

Error 3. A solution has been selected as tentative optimal solution which should not have been selected.

In a given iteration, assume that the tentative optimal solution $WSP(\mu_0)$, is correct; i.e., it is optimal within the set of solutions which have been already scrutinized. The solution $\overline{R}$ is selected at 3.4.3. There, all solutions in R which are Pareto superior to the status-quo are compared sequentially to the tentative solution and selected as the new tentative solution if their indices of stability and their values are lower than those of the tentative solution.

Hence, if the tentative solution is correct, so is the fact to select a new tentative solution at 3.4.3. □

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Solutions without dummy axiom for TU cooperative games

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Abstract

In this paper we study an expression for all additive, symmetric and efficient solutions, i.e., the set of axioms that traditionally are used to characterize the Shapley value except for the dummy axiom. Also, we obtain an expression for this kind of solutions by including the self duality axiom. These expressions allow us to give an alternative formula for the consensus value, the generalized consensus value and the solidarity solution. Furthermore, we introduce a new axiom called coalitional independence which replaces the symmetry axiom and use it to get similar results.

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1 Introduction

Since the seminal axiomatic contribution of Shapley in 1953, the variations of his novel axioms have played a key role in the literature of transferable-utility cooperative games, mainly trying to understand different applications. Most variations substitute his original axioms either by weaker versions or by different axioms that fit certain application. However, very few papers consider the raw solutions that emerge by taking out one or two of the original axioms. This paper is one of them; it provides a closed form expression of semivalues without the dummy axiom. It also proposes a weakening of the symmetry axiom, and provides a closed form expression of these solutions.

Semivalues without the dummy axiom have applications in problems incompatible with a subsidy-free scenario. For instance, labor unions require employers to pay some minimum compensation even if an employee does not work: a waiter receives a minimum salary when the restaurant is empty; and, in certain countries, workers enjoy unemployment insurance from the government.

This paper is divided in five parts. In part 2 we give an alternative proof for the general formula for semivalues without the dummy axiom. In part 3 we replace symmetry by a coalition independent axiom. In part 4 we relate the previous results with solutions in the literature. In part 5 we remark the extension of previous formulas to non-additive semivalues. All proofs are written in the appendix.

2 Solutions without a dummy axiom

By a game we mean a pair (N, v) where $N \subset \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a real function such that $v(\emptyset) = 0$. Let $G = G^N$ be the set of games with a fixed set of players N. We consider N fixed and $n = |N|$. Let (N, z) be the zero game, i.e., the game defined by $z(T) = 0$ for every $T \subseteq N$.

Definition 1 *By a solution in G we mean a continuous function $\varphi : G \to \mathbb{R}^N$. Let V be the set of solutions in G.*

A solution is a rule to divide the common gain or cost among the players in N. The requirement of continuity, in the definition of a solution, is necessary to obtain all of our results. Also, it is desirable to have a zero payoff for the zero game because in this case no coalition generates gain or cost.

In this section, we obtain an expression for all additive, symmetric and efficient solutions: that is, all the axioms that traditionally characterize the Shapley value except the dummy axiom. A similar, albeit different expression of linear, symmetric and efficient semivalues can be found in the appendix of Ruiz et al. (1998) and Hernández et al. (2007).

Proposition 1 *The solution φ satisfies additivity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{1}$$

for some $n-1$ *real numbers* $\{\beta_s\}_{s=1}^{n-1}$.

Proof of this result, and the ones that follow, are given in the Appendix.

We denote by φ^β the solution φ with parameters $\{\beta_s\}_{s=1}^{n-1}$ when we need to refer to its parameters. Every additive, symmetric and efficient solution is of the form (1) and for every set of real numbers $\{\beta_s\}_{s=1}^{n-1}$ in (1) we get a solution that satisfies these three axioms. Furthermore, notice that for different sets of real numbers $\{\beta_s\}_{s=1}^{n-1}$ we get different solutions, therefore, the dimension of this (affine) subspace of solutions is $n-1$. Moreover, the Shapley value corresponds to the numbers $\beta_s = \frac{(s-1)!(n-s-1)!}{n!}$, $s = 1, \ldots, n-1$.

We may interpret (1) as follows. With respect to an efficient solution, a game v has the following information: $v(N)$ tells us the total amount to be shared and $v(S)$, $S \neq N$, the amount that coalition S claims for itself. So, we start with the egalitarian solution, i.e., we give $\frac{v(N)}{n}$ to each player. We keep going with one transference from $N\backslash S$ to S for each coalition $S \neq N, \emptyset$: Every player in $N\backslash S$ pays $s\beta_s v(S)$ and every player in S receives $(n-s)\beta_s v(S)$. Notice that we use the same factor β_s for coalitions with equal cardinality s. At the end, player i has an amount $\varphi_i(v)$ given by (1).

3 Solutions satisfying the coalitional independence axiom

In this section we replace the symmetry axiom in the previous proposition with a new one that we call coalitional independence axiom. This axiom looks like the fair ranking axiom of Chun (1989).

Definition 2 *We say that the two games* (N, v) *and* (N, w) *only differ in* S *if and only if* $v(T) = w(T)$ *for every coalition* $T \neq S$.

The coalitional independence axiom requests that the solution changes equally for any two players in S or any two players in $N\backslash S$ for every two games that only differ in S.

Axiom 1 (Coalitional independence) *We say that* φ *satisfies the coalitional independence axiom if*

$$\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w)$$

for every two games (N, v) *and* (N, w) *that only differ in* S *and* $i,\ j \in S$ *or* $i,\ j \in N\backslash S$.

Remark 1 *Notice that additivity and symmetry imply coalitional independence, but additivity and coalitional independence do not imply symmetry. Indeed, let* φ *be the solution given by,*

$$\varphi_i(v) = \sum_{j \in N} jv(\{j\}).$$

Clearly φ *satisfies additivity and coalitional independence but not symmetry. Now, suppose that* φ *is an additive and symmetric solution. Let* v,$w \in G$ *be two games that only differ in*

S, and take either i, $j \in S$ or i, $j \in N\backslash S$. Let θ be the permutation of N that interchanges i and j. Then $\theta^*(v-w) = v-w$, so, $\varphi(v-w) = \varphi(\theta^*(v-w)) = \theta^*\varphi(v-w)$.

Therefore, $\varphi_i(v-w) = \varphi_j(v-w)$. Hence, φ satisfies coalitional independence.

Proposition 2 *The solution φ satisfies additivity, coalitional independence and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N\backslash S} v(N\backslash S)] \tag{2}$$

for some set of $2^n - 2$ real numbers $\{\beta_S\}_{\emptyset \neq S \subsetneq N}$.

We get a similar expression as that of proposition 1, except that now we have one β_S for each non empty coalition $S \neq N$. Again, every solution that satisfies the additive, symmetry and efficiency axioms is of the form (2) and for every set of real numbers $\{\beta_S\}_{\emptyset \neq S \subsetneq N}$ we get a solution that satisfies these axioms. Now, the next corollary includes the self duality axiom as part of its hypotheses **2**. Again, including this axiom roughly halves the dimension of the space of solutions.

Axiom 2 (Self duality) *We say that the solution φ is self dual if $\varphi(v) = \varphi(v^*)$ for every game $v \in G$.*

Corollary 1 *The solution φ satisfies additivity, coalitional independence, efficiency and self duality axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N\backslash S} v(N\backslash S)] \tag{3}$$

for some set of $2^{n-1} - 1$ real numbers $\{\beta_S\}_{S \subseteq N}$ such that $\beta_S = \beta_{N\backslash S}$.

We conclude this section with a characterization of the Shapley value, we replace the symmetry axiom with the coalitional independence axiom.

Proposition 3 *The Shapley value is the unique solution that satisfies additivity, coalitional independence, dummy and efficiency axioms.*

4 Some special cases

In this section we briefly see some special solutions of the form (1) that do not satisfy the dummy axiom (i.e. different from Shapley's value). A first example is the Equal Surplus solution:

$$\varphi_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}$$

which we get when we choose $\beta_1 = \frac{1}{n}$ and $\beta_s = 0$ for $s \neq 1$ in (1). Another self dual solution with a simple expression is

$$\varphi_i(v) = \sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}$$

where $\beta_s = \frac{1}{s(n-s)}$ in (1).

4.1 The Consensus Value

The convex linear combination (rather than the linear combination) of two solutions of the form (1) gives another solution of the same form. Moreover, the parameters of the new solution are just the convex combination of those of the two original solutions. More precisely, for any two solutions φ^β and φ^γ, and a real number $\theta \in [0,1]$:

$$(1-\theta)\varphi^\beta + \theta\varphi^\gamma = \varphi^{(1-\theta)\beta+\theta\gamma}.$$

In this sense, Ju et al. (2007) prove that the consensus value is the middle point between the Equal Surplus solution and the Shapley value. Thus an expression for the consensus value is:

$$\frac{v(N)}{n} + \frac{1}{2}\left[v(\{i\}) - \frac{\sum_{k\neq i} v(\{k\})}{n-1}\right] + \frac{1}{2}\sum_{S\ni i, |S|\neq n, n-1, 1} (n-s)\left[\frac{(s-1)!(n-s-1)!}{n!}(v(S) - v(N\backslash S))\right].$$

In the same way, we could generate an expression for any generalized consensus value, i.e., we would only need to replace $\beta_1 = \frac{1-\theta}{n} + \frac{\theta}{n(n-1)}$ and $\beta_s = \frac{\theta(s-1)!(n-s-1)!}{n!}$ for $s = 2, .., n-1$, in (1).

4.2 Solidarity value

Nowak and Radzik (1997) introduce the solidarity value. They define, for any non-empty coalition T and any game $v \in G$, $A^v(T) = \frac{1}{t}\sum_{k\in T}[v(T) - v(T\backslash\{k\})]$. Then, they define the solidarity value for player i as,

$$\psi_i(v) = \sum_{T\ni i} \frac{(n-t)!(t-1)!}{n!} A^v(T) \tag{4}$$

They characterized this value with the efficiency, additivity, symmetry and A-null player axioms, so the solidarity value must be a special case of (1). Indeed, if we expand (4) we get that the coefficient of $v(S)$, for a coalition T which does not contain i, is $\frac{(n-s-1)!s!}{n!}\frac{1}{s+1}$. Thus, this coefficient corresponds to $s\beta_s$ in (1), and therefore

$$\beta_s = \frac{(n-s-1)!(s-1)!}{(s+1)n!}$$

which gives us an alternative expression for (4):

$$\psi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} \frac{(n-s)!(s-1)!}{n!} \left[\frac{v(S)}{s+1} - \frac{v(N \backslash S)}{n-s+1} \right].$$

Observe that the solidarity value is not self dual since $\beta_s \neq \beta_{n-s}$.

4.3 Least Square Prenucleolus

Lastly, Ruiz et al. (1996) introduce the Least Square Prenucleolus solution,

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \left[\sum_{S \ni i} (n-s)v(S) - \sum_{S \not\ni i} sv(S) \right]$$

This solution is also of the form (1). The corresponding parameters are $\beta_s = \frac{1}{n2^{n-2}}$.

Remark 2 *Current literature does not say much about non-additive semivalues. This paper does not explore this topic; however, the non-additive and non-dummy semivalues can be easily characterized by modifying (1). Indeed, replace the constant β_i by any function $\beta_i^* : G^N \to \mathbb{R}$ that is symmetric in G. This function is clearly efficient and symmetric. It will be additive when β_i^* is constant, and by construction the dummy axiom does not hold. Similar formulas can be given by replacing symmetry by the coalition independence axiom.*

5 Appendix

Lemma 1 *A solution satisfies the additivity axiom if and only if it is linear.*

Before we continue with the proofs, we need to define a game χ_S for every $S \subseteq N$,

$$\chi_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2 *The solution φ satisfies the additivity and symmetry axioms if and only if there exist real numbers $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$ such that*

$$\varphi_i(v) = \sum_{S \ni i} \beta_s v(S) + \sum_{S \not\ni i} \widetilde{\beta}_s v(S) \text{ for every } i \in N.$$

Proof. Let φ be an additive and symmetric solution, from Lemma 1, φ is linear. Clearly $\{\chi_S\}_{S \subset N}$ is a basis for G, and for every game $v \in G$, $v = \sum_{S \subset N} v(S)\chi_S$. Set $\beta_S^i = \phi_i(\chi_S)$, then

$$\varphi_i(v) = \sum_{S \subseteq N} v(S)\beta_S^i$$

for every $i \in N$. On the other hand, let $U, V \subseteq N$ be such that $|U| = |V|$, $k \in U$, $l \in V$ and θ a permutation of N such that $\theta(U) = V$ and $\theta(k) = l$. Since $\theta(\chi_U) = \chi_V$ then $\varphi_k(\chi_U) = \varphi_l(\chi_V)$ by symmetry. Therefore, $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \in U$, $l \in V$. Similarly, we can conclude that $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \notin U$, $l \notin V$. Thus

$$\varphi_i(v) = \sum_{S \ni i} \beta_s v(S) + \sum_{S \not\ni i} \widetilde{\beta}_s v(S)$$

for some constants $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. The proof in the other direction is straightforward. □

Proof of Proposition 1. By Lemma 2, $\varphi_i(v) = \sum_{S \ni i} \beta_s v(S) + \sum_{S \not\ni i} \widetilde{\beta}_s v(S)$ for some numbers $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. Since φ is efficient, we have that $\sum_{i \in N} \varphi_i(\chi_S) = s\beta_s + (n-s)\widetilde{\beta}_s = 0$ for every $S \subsetneq N$, and $\sum_{i \in N} \varphi_i(\chi_N) = n\beta_n = 1$. Therefore, $\widetilde{\beta}_s = -\frac{s}{n-s}\beta_s$ for $s = 1, 2, ..., n-1$ and $\beta_n = \frac{1}{n}$. Thus

$$\begin{aligned} \varphi_i(v) &= \textstyle\sum_{S \ni i} \beta_s v(S) - \sum_{S \not\ni i} \frac{s}{n-s} \beta_s v(S) \\ &= \textstyle\sum_{S \ni i} \beta_s v(S) - \sum_{S \ni i} \frac{n-s}{s} \beta_{n-s} v(N \backslash S) \end{aligned}$$

for some numbers $\{\beta_s\}_{s=1}^{n-1}$. Now, if we replace $q_s = \frac{\beta_s}{n-s}$ we get

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)(q_s v(S) - q_{n-s} v(N \backslash S)). \square$$

Lemma 3 *The solution φ satisfies additivity and coalitional independence axioms if and only if there exist numbers $\{\beta_S\}_{S \subseteq N} \cup \{\widetilde{\beta}_S\}_{S \subsetneq N}$ such that*

$$\varphi_i(v) = \sum_{S \ni i} \beta_S v(S) + \sum_{S \not\ni i} \widetilde{\beta}_S v(S) \text{ for every } i \in N.$$

Proof. Note that, for every S, χ_S and the zero game differ only on S, thus if φ is any linear and coalitional independent solution

$$\varphi_i(\chi_S) = \varphi_j(\chi_S), \text{if } i, j \in S \text{ or if } i, j \in N \setminus S.$$

For every S define $\beta_S := \varphi_i(S)$ for any $i \in S$; similarly, for every $S \neq N$, define $\widetilde{\beta_S} := \varphi_j(S)$ for any $j \notin S$. Then, for every game v

$$\varphi_i(v) = \varphi_i(\sum_S v(S)\chi_S) = \sum_{S \in i} \beta_S v(S) + \sum_{S \not\ni i} \widetilde{\beta_S} v(S).$$

It is straightforward to check the converse. □

Proof of Proposition 2. Keeping the same notation as above, we know that if φ is in addition an efficient solution, then, for every $S \neq N$

$$0 = \chi_S(N) = \sum_i \varphi_i(\chi_S) = s\beta_S + (n-s)\widetilde{\beta_S};$$

thus $\widetilde{\beta_S} = -\frac{s}{n-s}\beta_S$, $S \neq N$. Similarly, $\beta_N = \frac{1}{n}$. Hence,

$$\begin{aligned}\varphi_i(v) &= \sum_{S\ni i}\beta_S v(S) - \sum_{S\not\ni i}\frac{s}{n-s}\beta_S v(S)\\ &= \sum_{S\ni i}\beta_S v(S) - \sum_{S\ni i}\frac{n-s}{s}\beta_{N\setminus S} v(N\setminus S)\end{aligned}$$

Let us take $q_S = \frac{\beta_S}{n-s}$ then

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N}(n-s)\left[q_S v(S) - q_{N\setminus S}v(N\setminus S)\right]$$

for some numbers $\{q_S\}_{S\subsetneq N}$. □

Proof of Corollary 1. Assume $i \in N$, then by Proposition **2** there exists a set of numbers $\{\beta_S\}_{S\subsetneq N}$ such that

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N}(n-s)[\beta_S v(S) - \beta_{N\setminus S}v(N\setminus S)].$$

For every $S \neq N$ we define a game ξ_S as follows

$$\xi_S(T) = \begin{cases} 0 & \text{si } T \neq S \text{ and } T \neq N\setminus S \\ 1 & T = S \text{ or } T = N\setminus S \end{cases}.$$

Therefore

$$(n-s)[\beta_{N\setminus S} - \beta_S] = \varphi(\xi_S^*) = \varphi(\xi_S) = (n-s)[\beta_S - \beta_{N\setminus S}],$$

so, $\beta_S = \beta_{N\setminus S}$. Thus

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N}(n-s)\beta_S[v(S) - v(N\setminus S)].$$

It is easy to show that solutions of this kind satisfy the self duality axiom. □

Proof of proposition 3. We leave to the reader to verify that Shapley's value satisfies the coalitional independence axiom.

Now, let φ be a solution that satisfies additivity, coalitional independence, dummy and efficiency axioms. Fix a player i and consider coalitions $T \subset N \setminus \{i\}$ but $T \neq N \setminus \{i\}$. First of all, notice that i is a dummy player for $\chi_{T\cup\{i\}} + \chi_T$, therefore -using the expression for coalitional independent solutions from Proposition 3-

$$0 = \varphi_i(\chi_{T\cup\{i\}} + \chi_T) = (n-t-1)\cdot\beta_{T\cup\{i\}} - t\cdot\beta_T.$$

Thus, $\beta_T = \frac{n-t-1}{t}\beta_{T\cup\{i\}}$ for each $T \subsetneq N\setminus\{i\}$.

From this, it follows that the solution is determined by the single number $\beta_{N\setminus\{1\}}$. Moreover, from

$$0 = \varphi_1(\chi_N + \chi_{N\setminus\{1\}}) = \frac{1}{n} - (n-1)\beta_{N\setminus\{1\}},$$

we conclude that the solution is unique. Since the Shapley value satisfies these axioms, $\varphi = Sh$. □

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THE CLASS OF EFFICIENT LINEAR SYMMETRIC VALUES FOR GAMES IN PARTITION FUNCTION FORM

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In this paper we study linear symmetric solutions for the space of games in partition function form with n players. In particular, we provide an expression for all linear, symmetric and efficient solutions. Furthermore, adding an additional axiom, we identify a unique value satisfying these properties.

Keywords: Games in partition function form; value; integer partition.

1. Introduction

Several of the models formulated for $n-$person game theory consist of some fundamental concepts. They assume that a game is characterized by giving the worth of each coalition S of the set N of players $1, 2, \ldots, n$. The original von Neumann-Morgenstern theory of solutions and some other models that followed it describe the value of the coalition S by means of a real valued characteristic function which is defined on the set 2^N of all subsets of N. Lucas and Thrall (1963) have formulated a generalization of the von Neumann-Morgenstern theory which describes the worth of a coalition S in terms of a partition function that is defined on the set of all partitions of N.

Due to the richness of the games in partition function form, several solutions have been given for these games. The first paper that proposed a value concept for games in partition function form was Myerson (1977) and then Bolger (1987) derived a class of linear, symmetric and efficient values for games in partition function form. More recently, Ju (2004), Albizuri *et al.* (2005), Macho-Stadler (2006) and Pham Do and Norde (2007) apply the axiomatic approach to find a value.

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This paper provides an expression for all linear, symmetric and efficient solutions (LSE solutions). Once we have a global description of all LSE solutions, we add an additional axiom, which is actually a nullity axiom in terms of an average marginal contribution,[a] and we extend the Solidarity value for cooperative TU games of Nowak and Radzik (1994) to the class of games in partition function form.

The paper is organized as follows. We first recall the main basic features of games in partition function form in the next section. A characterization of all LSE solutions is introduced in Section 3. In Section 4 we discuss a nullity axiom and show that there is one unique value satisfying linearity, symmetry, efficiency and nullity axioms. Finally, we relate previous results with solutions in the literature.

2. Framework and Notation

In this section we give some concepts and notations related to $n-$person games in partition function form, as well as a brief subsection of preliminaries related to integer partitions, since it is a key subjet in subsequent developments.

2.1. *Games in partition function form*

Let $N = \{1, 2, \ldots, n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given N, let CL be the set of all coalitions (nonempty subsets) of $N, CL = \{S \mid S \subseteq N, S \neq \varnothing\} = 2^N \backslash \{\varnothing\}$. Let PT be the set of partitions of N, so

$$\{S_1, S_2, \ldots, S_m\} \in PT \text{ iff } \bigcup_{i=1}^{m} S_i = N,\ S_j \neq \varnothing\ \forall j,\ S_j \cap S_k = \varnothing\ \forall j \neq k$$

Also, let $ECL = \{(S, Q) \mid S \in Q \in PT\}$ be the set of *embedded coalitions*, that is the set of coalitions together with specifications as to how the other players are aligned.

Definition 1. A mapping

$$w : ECL \to \mathbb{R}$$

that assigns a real value, $w(S, Q)$, to each embedded coalition (S, Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by G, i.e.,

$$G = G^{(n)} = \{w \mid w : ECL \to \mathbb{R}\}$$

The value $w(S, Q)$ represents the payoff of coalition S, given the coalition structure Q forms. In this kind of games, the worth of some coalition depends not only on what the players of such coalition can jointly obtain, but also on the way the other players are organized. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N, \{N\})$ to divide among themselves. But we also anticipate that the actual

[a]The precise definitions will be provided in Sec. 4.

allocation of this worth will depend on all the other potential worths $w(S, Q)$, as they influence the relative bargaining strengths of the players.

For any $S \subseteq N$, let $[S]$ denote the partition of S which consists of the singleton elements of S, i.e., $[S] = \{\{j\} \mid j \in S\}$. For $Q \in PT, S \in Q$ and $i \in N$, we define $Q_S = \{S\} \cup [N \backslash S], S_{-k} = S \backslash \{k\}$ and Q^i denotes the member of Q where i belongs. Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance $n = |N|, s = |S|, q = |Q|, q_i = |Q^i|$ and so on.

Given $w_1, w_2 \in G$ and $c \in \mathbb{R}$, we define the sum $w_1 + w_2$ and the product cw_1, in G, in the usual form, i.e.,

$$(w_1 + w_2)(S, Q) = w_1(S, Q) + w_2(S, Q) \quad \text{and} \quad (cw_1)(S, Q) = cw_1(S, Q)$$

respectively. It is easy to verify that G is a vector space with these operations.

A *solution* is a function $\varphi : G \to \mathbb{R}^n$. If φ is a solution and $w \in G$, then we can interpret $\varphi_i(w)$ as the utility payoff which player i should expect from the game w.

Now, the group of permutations of $N, S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$, acts on CL and on ECL in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(S_1, \{S_1, S_2, \ldots, S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), \ldots, \theta(S_l)\})$$

And also, S_n acts on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta(x_1, x_2, \ldots, x_n) = (x_{\theta^{-1}(1)}, x_{\theta^{-1}(2)}, \ldots, x_{\theta^{-1}(n)})$$

Next, we define the usual linearity, symmetry and efficiency axioms which are asked solutions to satisfy in the cooperative game theory frame-work.

Axiom 1 (Linearity). The solution φ is linear if $\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2)$ and $\varphi(cw_1) = c\varphi(w_1)$, for all $w_1, w_2 \in G$ and $c \in \mathbb{R}$.

Axiom 2 (Symmetry). The solution φ is said to be symmetric if and only if $\varphi(\theta \cdot w) = \theta \cdot \varphi(w)$ for every $\theta \in S_n$ and $w \in G$, where the game $\theta \cdot w$ is defined as

$$(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$$

Axiom 3 (Efficiency). The solution φ is efficient if $\sum_{i \in N} \varphi_i(w) = w(N, \{N\})$ for all $w \in G$.

2.2. *Integer partitions*

A partition of a nonnegative integer is a way of expressing it as the unordered sum of other positive integers, and it is often written in tuple notation. Formally,

Definition 2. $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$ is a partition of n (denoted as $\lambda \vdash n$) iff $\lambda_1, \lambda_2, \ldots, \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Two partitions which only differ in the order of their elements are considered to be the same partition.

The set of all partitions of n will be denoted by $\Pi(n)$, and, if $\lambda \vdash n, |\lambda|$ is the number of elements of λ.

For example, the partitions of $n = 4$ are $[1,1,1,1], [2,1,1], [2,2], [3,1]$ and $[4]$. We will abbreviate this notation by dropping the commas, so $[2,1,1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \vdash n$, associated with Q, where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, \ldots, S_m\} \in PT$, then $\lambda_Q = [s_1, s_2, \ldots, s_m]$.

For a given $\lambda \vdash n$, we write $s \rhd \lambda$ if s is an element of the set of numbers determined by the λ_i's. So, if $\lambda = [4,2,2,1,1,1]$, then $s \rhd \lambda$ means $s \in \{1,2,4\}$.

Finally, if $[\lambda_1, \lambda_2, \ldots, \lambda_l] \vdash n$, for $k \geq 1$ we define $[\lambda_1, \lambda_2, \ldots, \lambda_l] - [\lambda_1, \lambda_2, \ldots, \lambda_k] = [\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_l]$. For example, $[4,3,2,1,1,1] - [3,1,1] = [4,2,1]$.

3. LSE Solutions

In this section, we present an expression for all linear, symmetric and efficient solutions, which establishes the main result of this work. First of all, we need to define certain sets which are used in the sequel.

Definition 3. Let A_n be a set of pairs, associated with all partitions $\lambda \vdash n$ and its elements, i.e.,

$$A_n = \{(\lambda, s) \mid s \rhd \lambda \in \Pi(n)\}$$

and similarly, define the set of triples

$$B_n = \{(\lambda, s, t) \mid \lambda \in \Pi(n) \backslash \{[n]\}, s \rhd \lambda, t \rhd \lambda - [s]\}$$

Example 1. If $n = 4$, then

$$A_4 = \{([1111], 1), ([211], 1), ([211], 2), ([22], 2), ([31], 1), ([31], 3), ([4], 4)\}$$

and

$$\begin{aligned} B_4 = \{&([1111], 1, 1), ([211], 1, 1), ([211], 1, 2), ([211], 2, 1), \\ &([22], 2, 2), ([31], 1, 3), ([31], 3, 1)\} \end{aligned}$$

Now, we give a characterization of all linear symmetric solutions in the following

Lemma 1. *If the solution $\varphi : G \to \mathbb{R}^n$ satisfies the linearity and symmetry axioms, then there exist unique real numbers $\{\alpha_{(\lambda,s)} \mid (\lambda, s) \in A_n\} \cup \{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$ such that*

$$\varphi_i(w) = \sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in ECL \\ S \ni i, |S|=s \\ \lambda_Q = \lambda}} \alpha_{(\lambda,s)} w(S,Q) + \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in ECL \\ S \not\ni i, |S|=s \\ \lambda_Q = \lambda, |Q^i|=t}} \beta_{(\lambda,s,t)} w(S,Q) \tag{1}$$

Conversely, for any real numbers $\{\alpha_{(\lambda,s)} \mid (\lambda,s) \in A_n\} \cup \{\beta_{(\lambda,s,t)} \mid (\lambda,s,t) \in B_n\}$, *the solution given by* (1) *is linear and symmetric.*

Proof. Let $\varphi : G \to \mathbb{R}^n$ be a linear and symmetric solution. Define games $\{u_{(S,Q)} \mid (S,Q) \in ECL\}$ by

$$u_{(S,Q)}(T,P) = \begin{cases} 1 & \text{if } (T,P) = (S,Q) \\ 0 & \text{otherwise} \end{cases}$$

Then for every game $w \in G, w = \sum_{(S,Q)\in ECL} w(S,Q)u_{(S,Q)}$.

Thus we can write

$$\begin{aligned} \varphi_i(w) &= \sum_{(S,Q)\in ECL} w(S,Q)\varphi_i(u_{(S,Q)}) \\ &= \sum_{\substack{(S,Q)\in ECL \\ S\ni i}} w(S,Q)\varphi_i(u_{(S,Q)}) + \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i}} w(S,Q)\varphi_i(u_{(S,Q)}) \end{aligned}$$

for every $i \in N$.

Now, observe the following:

- Let Q_1, Q_2 be two partitions such that $\lambda_{Q_1} = \lambda_{Q_2}$. Let $S_1, R_1 \in Q_1, S_2, R_2 \in Q_2$, such that $S_1 \neq R_1, S_2 \neq R_2, |S_1| = |S_2|$ and $|R_1| = |R_2|$. Let $i \in S_1$ and $j \in S_2$. Then, there exists a permutation θ such that $\theta(Q_1) = Q_2, \theta(S_1) = S_2, \theta(R_1) = R_2$ and $\theta(i) = j$.
- If θ is a permutation such that $\theta(i) = j$, then symmetry of φ implies that

$$\varphi_j(u_{(\theta(S),\theta(Q))}) = \varphi_{\theta(i)}(\theta \cdot u_{(S,Q)}) = \varphi_i(u_{(S,Q)})$$

From this observation we conclude that if $(S,Q),(T,P) \in ECL$ are such that $|S| = |T|, \lambda_Q = \lambda_P$ and $i \in S$ and $i \in T$, then $\varphi_i(u_{(S,Q)}) = \varphi_i(u_{(T,P)}) = \alpha_{(\lambda,s)}$ where $\lambda = \lambda_Q = \lambda_P$ and $s = |S| = |T|$. Thus the first summand above can be written as

$$\sum_{\substack{(S,Q)\in ECL \\ S\ni i}} w(S,Q)\varphi_i(u_{(S,Q)}) = \sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in ECL \\ S\ni i, |S|=s \\ \lambda_Q=\lambda}} \alpha_{(\lambda,s)} w(S,Q)$$

Likewise, recall that $Q^i \in Q$ is the set containing i, then the second summand is equal to

$$\sum_{\substack{(S,Q)\in ECL \\ S\not\ni i}} w(S,Q)\varphi_i(u_{(S,Q)}) = \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i, |S|=s \\ \lambda_Q=\lambda, |Q^i|=t}} \beta_{(\lambda,s,t)} w(S,Q)$$

Uniqueness: to check uniqueness it is enough to prove that if

$$0 = \sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in ECL \\ S\ni i, |S|=s \\ \lambda_Q=\lambda}} \alpha_{(\lambda,s)} w(S,Q) + \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i, |S|=s \\ \lambda_Q=\lambda, |Q^i|=t}} \beta_{(\lambda,s,t)} w(S,Q)$$

for every game w and for every player i, then every $\alpha_{(\lambda,s)}$ and $\beta_{(\lambda,s,t)}$ vanish.

Thus, for given $(\lambda, s) \in A_n$ let $S = \{1, \ldots, s\}$ and Q be any partition such that $S \in Q$ and $\lambda_Q = \lambda$. Let $w = u_{(S,Q)}$ and $i = 1$. Then the above sum reduces to

$$0 = \alpha_{(\lambda,s)}$$

Similarly, given $(\lambda, s, t) \in B_n$, choose S and Q as before. Also let $T \in Q$ such that $|T| = t$. Let $w = u_{(S,Q)}$ and pick any $i \in T$. In this case the sum is just

$$0 = \beta_{(\lambda,s,t)}$$

Finally, it is straightforward to check that Eq. (1) defines a linear and symmetric solution for any choice of coefficients. □

Theorem 4. *The solution $\varphi : G \to \mathbb{R}^n$ satisfies linearity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{(\lambda,s,t)\in B_n} \beta_{(\lambda,s,t)} \times \left[\sum_{\substack{(S,Q)\in ECL \\ S\ni i, |S|=s \\ \lambda_Q=\lambda}} \sum_{T\in Q\setminus\{S\} \atop |T|=t} tw(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i, |S|=s \\ \lambda_Q=\lambda, |Q^i|=t}} sw(S,Q) \right] \tag{2}$$

for some real numbers $\{\beta_{(\lambda,s,t)} \mid (\lambda, s, t) \in B_n\}$.

Moreover, such representation is unique.

Proof. By the previous Lemma,

$$\varphi_i(w) = \sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in ECL \\ S\ni i, |S|=s \\ \lambda_Q=\lambda}} \alpha_{(\lambda,s)} w(S,Q) + \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i, |S|=s \\ \lambda_Q=\lambda, |Q^i|=t}} \gamma_{(\lambda,s,t)} w(S,Q)$$

for some constants $\{\alpha_{(\lambda,s)}\} \cup \{\gamma_{(\lambda,s,t)}\}$.

Efficiency implies:

(1) $$0 = \sum_{i\in N} \varphi_i(u_{(S,Q)}) = s\alpha_{(\lambda_Q,s)} + \sum_{T\in Q\setminus\{S\}} t\gamma_{(\lambda_Q,s,t)}$$

for every $(S, Q) \neq (N, \{N\})$.

(2) $$1 = \sum_{i \in N} \varphi_i(u_{N,\{N\}}) = n\alpha_{([n],n)}$$

Hence,

$$\alpha_{(\lambda_Q,s)} = -\frac{1}{s} \sum_{T \in Q \setminus \{S\}} t\gamma_{(\lambda_Q,s,t)}$$

for every pair $(S,Q) \neq (N,\{N\})$, and

$$\alpha_{([n],n)} = \frac{1}{n}$$

Therefore

$$\varphi_i(w) = \frac{w(N,\{N\})}{n} - \sum_{(\lambda,s)\in A_n} \sum_{\substack{(S,Q)\in ECL \\ S\ni i,|S|=s \\ \lambda_Q=\lambda}} \sum_{T\in Q\setminus\{S\}} \frac{t}{s}\gamma_{(\lambda_Q,s,t)} w(S,Q)$$

$$+ \sum_{(\lambda,s,t)\in B_n} \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i,|S|=s \\ \lambda_Q=\lambda,|Q^i|=t}} \gamma_{(\lambda,s,t)} w(S,Q)$$

$$= \frac{w(N,\{N\})}{n} - \sum_{(\lambda,s,t)\in B_n} \gamma_{(\lambda,s,t)}$$

$$\times \left[\sum_{\substack{(S,Q)\in ECL \\ S\ni i,|S|=s \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=t}} \frac{t}{s} w(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i,|S|=s \\ \lambda_Q=\lambda,|Q^i|=t}} w(S,Q) \right]$$

Set $\beta_{(\lambda,s,t)} = -\frac{1}{s}\gamma_{(\lambda,s,t)}$, then

$$\varphi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{(\lambda,s,t)\in B_n} \beta_{(\lambda,s,t)}$$

$$\times \left[\sum_{\substack{(S,Q)\in ECL \\ S\ni i,|S|=s \\ \lambda_Q=\lambda}} \sum_{\substack{T\in Q\setminus\{S\} \\ |T|=t}} t w(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i,|S|=s \\ \lambda_Q=\lambda,|Q^i|=t}} s w(S,Q) \right]$$

The converse is a straightforward computation, and uniqueness follows from the uniqueness part of Lemma 1. □

Let φ^β denote the solution φ with parameters $\{\beta_{(\lambda,s,t)} \mid (\lambda,s,t) \in B_n\}$ when we need to refer to its parameters.

Remark 1. The previous formula can be written as

$$\varphi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{\substack{(S,Q)\in ECL \\ (S,Q)\neq(N,\{N\}) \\ S\ni i}} \sum_{T\in Q\setminus\{S\}} t\beta^{t}_{\lambda_Q,s} w(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ S\not\ni i}} s\beta^{q_i}_{\lambda_Q,s} w(S,Q)$$

Which it can be interpreted as follows. We start by giving $\frac{w(N,\{N\})}{n}$ to each player. For each $(S,Q) \in ECL$, we keep going with one transfer from T to S, for each $T \in Q\setminus\{S\}$: every player in T pays $s\beta_{(\lambda_Q,s,t)}w(S,Q)$ and every player in S receives $t\beta_{(\lambda_Q,s,t)}w(S,Q)$. At the end, the player i has the amount $\varphi_i(w)$ given by the above formula.

Corollary 1. *The space of all linear and symmetric solutions in n players has dimension* $|A_n| + |B_n|$. *Likewise, the* (*affine*) *space of all linear, symmetric and efficient solutions in n players has dimension* $|B_n|$.

Remark 2. The relation

$$|A_n| + |B_n| = |B_{n+1}|$$

holds for every n. It is not clear what is the meaning of the fact that there are as many linear symmetric solutions in n players and linear symmetric and efficient solutions in $n+1$ players.

Example 2. If $N = \{1,2,3\}$, then any linear, symmetric efficient solution is of the form (for player 1):

$$\begin{aligned}\varphi_1(w) = {} & \frac{w(N,\{N\})}{n} \\ & + \beta_{([111],1,1)}[2w(\{1\},(1,2,3)) - w(\{2\},(1,2,3)) - w(\{3\},(1,2,3))] \\ & + \beta_{([21],1,2)}[2w(\{1\},(1,23)) - w(\{2\},(2,13)) - w(\{3\},(3,12))] \\ & + \beta_{([21],2,1)}[w(\{1,2\},(3,12)) + w(\{1,3\},(2,13)) - 2w(\{2,3\},(1,23))]\end{aligned}$$

We should mention that Bolger (1987) shows that there are many LSE solutions for games in partition function form. He gives the following class of LSE solutions:

$$\Phi_i(w) = \sum_{\substack{(S,Q)\in ECL \\ i\in S}} a(s,n,q)w(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ i\notin S}} \frac{s}{n-s} a(s,n,q)w(S,Q)$$

with $a(n,n,1) = \frac{1}{n}$.

These are recovered from our general formula by setting $\beta_{(\lambda,s,t)} = \frac{a(s,n,|\lambda|)}{n-s}$ in (2).

It is also possible to give an expression for all linear, symmetric and efficient solutions for TU games in characteristic function form. Let

$$J = \{v : 2^N \to \mathbb{R} \mid v(\varnothing) = 0\}$$

be the set of all TU games in characteristic function form with player set N.

Corollary 2. *The solution $\varphi : J \to \mathbb{R}^n$ satisfies linearity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N \\ i \in S}} \rho_s \frac{v(S)}{s} - \sum_{\substack{S \subset N \\ i \notin S}} \rho_s \frac{v(S)}{n-s}$$

for some real numbers $\{\rho_s\}_{s=1}^{n-1}$.

Proof. Take $w \in G$ such that $w(S,Q) = v(S)$ for all $(S,Q) \in ECL$ in Eq. (2). □

An equivalent expression of linear, symmetric and efficient solutions for TU games can be found in Ruiz *et al.* (1998).

4. Characterization of a New LSE Solution

The aim of this section is to generalize the Solidarity value to the class of games in partition function form. In order to do so we first define, for any $(S,Q) \in ECL$ such that $S \neq \varnothing$ and any $w \in G$:

$$\overline{C}^w(S,Q) = \frac{1}{s} \sum_{j \in S} [w(S,Q) - w(S_{-j}, \{S_{-j}, \{j\}\} \cup Q \backslash \{S\})]$$

We can interpret $\overline{C}^w(S,Q)$ as an average marginal contribution[b] of a member of S, given the coalition structure Q. By convention, $w(\varnothing, Q) = 0$ for all $Q \in PT$.

Now, we give the following version of a nullity axiom for games in partition function form:

Axiom 4 (Nullity). If $i \in N$ is a null player in $w \in G$, i.e., $\overline{C}^w(S,Q) = 0$ for every $(S,Q) \in ECL$ such that $i \in S$, then $\varphi_i(w) = 0$.

We can now state the main result of this section.

Proposition 1. *The solution $\varphi : G \to \mathbb{R}^n$ given by*

$$\varphi_i(w) = \sum_{\substack{(S,Q_S) \in ECL \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \cdot \overline{C}^w(S, Q_S) \tag{3}$$

for each $i \in N$ and each $w \in G$; is the unique solution satisfying, linearity, symmetry, efficiency and nullity axioms.

Proof. The collection of games $\{e_{(R,P)} \mid (R,P) \in ECL\}$ defined as

$$e_{(R,P)}(S,Q) = \begin{cases} \binom{s}{r}^{-1} & \text{if } S \supseteq R \text{ and } (\forall T' \in Q \backslash \{S\})(\exists T \in P) : T' \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

constitutes a basis of G (see de Clippel and Serrano (2006)).

[b]Just like its counterpart for TU games (c.f. Nowak and Radzik, 1994).

It is easy to check that the solution given by (3) is of the form (2), so it satisfies linearity, symmetry and efficiency axioms; and it is straightforward to prove that it also satisfies the nullity axiom.

We prove uniqueness. Let φ be a solution satisfying the four axioms. For any $w \in G$, there exist $\delta_{(R,P)} \in \mathbb{R}$ such that $w = \sum_{(R,P)\in ECL} \delta_{(R,P)} e_{(R,P)}$. Then, by linearity, $\varphi(w) = \sum_{(R,P)\in ECL} \delta_{(R,P)} \varphi(e_{(R,P)})$. We will show that $\varphi(w)$ is determined for all $w \in G$ and by the previous discussion, it is therefore sufficient to show that $\varphi(e_{(R,P)})$ is determined for all $(R,P) \in ECL$. In this way, φ is unique for each $e_{(R,P)}$ and so for w.

Notice that,

(i) $e_{(R,P)}(N,\{N\}) = \binom{n}{r}^{-1}$

(ii) If $i \in N\backslash R$, then $\overline{C}^{e_{(R,P)}}(S,Q) = 0$ for all $(S,Q) \in ECL$ such that $i \in S$, i.e., i is a null player in $e_{(R,P)}$.

Indeed, it is quite obvious if $S \not\supseteq R$ or (for some $T' \in Q\backslash\{S\}$)($\nexists T \in P$) : $T' \subseteq T$. We just care about the case where $S \supseteq R$ and ($\forall T' \in Q\backslash\{S\}$)($\exists T \in P$) : $T' \subseteq T$:

$$\begin{aligned}\overline{C}^{e_{(R,P)}}(S,Q) &= e_{(R,P)} - \frac{1}{s}\sum_{j\in S} e_{(R,P)}(S\backslash\{j\},\{S\backslash\{j\},\{j\}\}\cup Q\backslash\{S\}) \\ &= \binom{s}{r}^{-1} - \frac{1}{s}\sum_{j\in S\backslash R} e_{(R,P)}(S\backslash\{j\},\{S\backslash\{j\},\{j\}\}\cup Q\backslash\{S\}) \\ &= \binom{s}{r}^{-1} - \frac{1}{s}(s-r)\binom{s-1}{r}^{-1} = 0\end{aligned}$$

(iii) If $i,j \in R$ and $\theta \in S_n$ is such that $\theta(i) = j$ and $\theta(R) = R$, then $\theta \cdot e_{(R,P)} = e_{(R,P)}$ and by symmetry: $\varphi_i(e_{(R,P)}) = \varphi_i(\theta \cdot e_{(R,P)}) = \varphi_{\theta(i)}(e_{(R,P)}) = \varphi_j(e_{(R,P)})$.

Hence, by the efficiency axiom:

$$\varphi_i(e_{(R,P)}) = \begin{cases} \frac{1}{r}\binom{n}{r}^{-1} & \text{if } i \in R \\ 0 & \text{if } i \notin R \end{cases}$$

And so,

$$\varphi_i(w) = \sum_{\substack{(R,P)\in ECL \\ R\ni i}} \frac{\delta_{(R,P)}}{r}\binom{n}{r}^{-1}, \quad \forall i \in N$$

□

Remark 3. The solution we have characterized in the previous Proposition is an extension of the Solidarity value. First, notice that the subset of G formed by the games w such that $w(S,Q) = w(S)\ \forall (S,Q) \in ECL$, can be identified with the set of TU games J. So, taking into account formula (3), it coincides with the Solidarity value for these games.

5. Some Examples of LSE Solutions

In this section we briefly see some LSE solutions of the form (2). As a first example, we take the expected stand-alone value, which tells us how much a player may obtain in a game in partition function form when we focus on the stand-alone side of the game:

$$\psi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{\substack{\varnothing \neq S \subset N \\ i \notin S}} \frac{s!(n-s-1)!}{n!} w(\{i\}, \{S\} \cup [N \backslash (S \cup \{i\})] \cup \{\{i\}\})$$

$$- \sum_{j \in N \backslash \{i\}} \sum_{S \subset N \backslash \{i,j\}} \frac{s!(n-s-2)!}{n!} w(\{j\}, [N \backslash (S \cup \{i\})] \cup \{S \cup \{i\}\})$$

which we obtain when we take in (2):

$$\beta_{(\lambda,s,t)} = \begin{cases} \dfrac{(s-1)!(n-s-2)!}{n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1},\ \ s=1 \ \text{ and } \ r=m. \\ 0 & \text{otherwise} \end{cases}$$

5.1. *Shapley value*

Pham Do and Norde (2007) define an extension of the Shapley (1953) value to the class of games in partition function form as

$$\psi_i(w) = Sh_i(v)$$

for each $i \in N$ and each $w \in G$, where $Sh : J \to \mathbb{R}^n$ is the Shapley value operator for TU games and $v \in J$ is defined as follows:

$$v(S) = w(S, \{S, [N \backslash S]\})$$

for each $S \subseteq N$.

This solution is of the form (2) with parameters

$$\beta_{(\lambda,s,t)} = \begin{cases} \dfrac{(s-1)!(n-s-1)!}{n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1},\ \ s=m \ \text{ and } \ r=1. \\ 0 & \text{otherwise} \end{cases}$$

5.2. *Consensus value*

It is easy to check that the convex linear combination of two solutions of the form (2) gives another solution of the same form. Moreover, the parameters of the new solution are just the convex combination of those of the two original solutions. More precisely, for any two solutions φ^β and φ^γ, and a real number $\delta \in [0,1]$:

$$(1-\delta)\varphi^\beta + \delta\varphi^\gamma = \varphi^{(1-\delta)\beta+\delta\gamma}$$

In this sense, Ju (2004) prove that the consensus value is the middle point between the stand-alone value and the Shapley value of Pham Do and Norde (2007). By the previous discussion, the corresponding parameters for the consensus value are:

$$\beta_{(\lambda,s,t)} = \begin{cases} \dfrac{1}{2n(n-2)} & \text{if } \lambda = [1,1,\ldots,1] \text{ and } s = r = 1 \\ \dfrac{(s-1)!(n-s-2)!}{2n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1},\ s = 1 \text{ and } r = m \\ \dfrac{(s-1)!(n-s-1)!}{2n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1},\ s = m \text{ and } r = 1 \\ 0 & \text{otherwise} \end{cases}$$

Following the same line of reasoning, we could generate an expression for any generalized consensus value, i.e., we would only need to replace in (2):

$$\beta_{(\lambda,s,t)} = \begin{cases} \dfrac{1}{2n(n-2)} & \text{if } \lambda = [1,\ldots,1] \text{ and } s = r = 1 \\ \dfrac{(1-\delta)(s-1)!(n-s-2)!}{2n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1}, \\ & \quad s = 1 \text{ and } r = m \\ \dfrac{\delta(s-1)!(n-s-1)!}{2n!} & \text{if } \lambda \in \{[m,1,\ldots,1]\}_{m=1}^{n-1}, \\ & \quad s = m \text{ and } r = 1 \\ 0 & \text{otherwise} \end{cases} .$$

5.3. *Myerson value*

Myerson (1977) proceeds axiomatically and proposes a value that extends the well-known Shapley value (Shapley, 1953), which is defined for TU games. His proposal satisfies the axioms of linearity, symmetry, efficiency and the "null" player property that states that players which have no effect on the outcome should neither receive nor pay anything.

The Myerson value of a player is given by

$$\psi_i(w) = \sum_{(S,Q)\in ECL} (-1)^{q-1}(q-1)! \left(\frac{1}{n} - \sum_{T\in Q\setminus\{S\}, i\notin T} \frac{1}{(q-1)(n-t)} \right) w(S,Q)$$

which we get with the parameters

$$\beta_{(\lambda,s,t)} = \frac{(-1)^{|\lambda|}(|\lambda|-1)!}{s} \left(\frac{1}{n} - \sum_{t\rhd\lambda-[r,s]} \frac{1}{(|\lambda|-1)(n-t)} \right).$$

5.4. *The value of Albizuri et al.*

Albizuri *et al.* (2005) obtain a unique value characterized by the properties of linearity, symmetry, efficiency, oligarchy and an additional requirement of symmetry with respect to the embedded coalitions. They define the value for a player as

$$\psi_i(w) = \sum_{\substack{(S,Q)\in ECL \\ i\in S}} \frac{(s-1)!(n-s)!}{n!P(S,N)} w(S,Q) - \sum_{\substack{(S,Q)\in ECL \\ i\in S}} \frac{s!(n-s-1)!}{n!P(S,N)} w(S,Q)$$

where $P(S,N) = |\{(T,Q) \in ECL \mid T = S\}|$. In fact, they notice that $P(S,N) = p(n-s)$ where $p(k)$ represents the number of partitions of any set K with cardinality k.

This solution is also of the form (2). The corresponding parameters are $\beta_{(\lambda,s,t)} = \frac{(n-s-1)!(s-1)!}{n!\cdot p(n-s)}$.

5.5. *The Value of Macho-Stadler et al.*

As a final example, Macho-Stadler *et al.* (2006) characterize the value

$$\psi_i(w) = \sum_{\substack{(S,Q)\in ECL \\ i\in S}} \frac{(s-1)! \prod_{T\in Q\setminus\{S\}} (t-1)!}{n!} w(S,Q)$$

$$- \sum_{\substack{(S,Q)\in ECL \\ i\notin S}} \frac{s! \prod_{T\in Q\setminus\{S\}} (t-1)!}{(n-s)n!} w(S,Q)$$

Which we get when we choose $\beta_{(\lambda,s,t)} = \frac{(s-1)! \prod_{t\rhd\lambda-[s]} (t-1)!}{(n-s)n!}$.

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Rankings and values for team games

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Abstract In this article we attack several problems that arise when a group of individuals is organized in several teams with equal number of players in each one (e.g., for company work, in sports leagues, etc). We define a team game as a cooperative game v that can have non-zero values only on coalitions of a given cardinality; it is further shown that, for such games, there is essentially a unique ranking among the players. We also study the way the ranking changes after one or more players retire. Also, we characterize axiomatically different ways of ranking the players that intervene in a cooperative game.

Keywords Cooperative games · Ranking · Bankruptcy games

1 Introduction

Frequently, Shapley's value—as well as other solutions in cooperative game theory—is used to distribute costs or benefits, but it may also be used as a power index to give a ranking of the players. From this point of view it is interesting to determine when two solutions give the same ranking; that is, when both solutions induce the same ordering of the players, for any game considered. We should mention from the start that the article Saari and Sieberg (2001) was a main motivation for the present research.

In Hernández-Lamoneda et al. (2007) the space of games was decomposed as a direct sum of three orthogonal subspaces: the subspace of symmetric games, another subspace which we call U and the common kernel of all linear symmetric solutions. Although U does not have a natural definition in terms of well known game theoretic considerations, it has a simple characterization in terms of vectors in $\mathbb{R}^n$ all of whose entries add up to zero. It arises naturally when taking into account the representation

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theory of the symmetric group S_n and it is an important piece of the space of games. For example, at the beginning of Sect. 3, it is shown (Lemma 3) that two linear symmetric solutions give the same ranking of the players if and only if they restrict to the same map on U—in fact, they agree up to a constant if and only if they agree on U. Two solutions ϕ and ψ are said to agree up to a constant if there is a function k on the space of games such that $\phi_j - \psi_j = k$, for every player j.

In this same section it is also shown that for any given linear symmetric solution ϕ, there exists a unique solution which satisfies the efficiency axiom and agrees up to a constant with ϕ. We give an explicit formula for this solution when the original ϕ is a semi-value in the sense of Dubey et al. (1981); in particular, when ϕ is Banzhaff's value we recover the LS-prenucleolus of Ruiz et al. (1996).

It is clear that two solutions that agree up to a constant induce the same ranking of the players. Conversely, at the end of the section, we show (Theorem 3) that if two linear and symmetric solutions induce the same ranking for all games, then a positive multiple of one agrees up to a constant with the other.

Section 4 deals with "team games". In certain applications it is interesting to consider games that are not necessarily defined for every coalition $S \subset N$ (or equivalently, whose values on certain coalitions are not relevant for the corresponding problem). For example, if we think of N as the set of all players in the NBA and $v(S)$ is the worth of a basketball team "on the floor", then it only makes sense to consider $v(S)$ for coalitions of cardinality 5. With this and similar examples in mind, we define a team game as a game that vanishes on all coalitions but those of a certain fixed cardinality s. We then look at the power index ranking for these games. We show that for team games, there is essentially one possible power index ranking (up to sign). This is the content of Corollary 4, which provides an answer to a problem of Saari and Sieberg ("The Michael Jordan problem"). It is also given a simple formula for computing the ranking (Theorem 6).

Also following ideas of Saari and Sieberg (2001), we study, in this same section, the way the power index ranking of a team game changes when a player "retires". A formula for the index of a player once a subset of players has retired follows immediately from Theorem 6. Nevertheless, the formula alone does not explain the source of the contributions that change the index once a player retires. We give in this section an example (The double sculls example) that shows how dramatically the ranking of the players may change after only one of them retires. Then, Theorem 7 shows how to construct examples where every possible change in ranking can happen. In the appendix to the article we explain in detail where the contributions for the change in ranking are coming from when a player drops out. The basic tools of representation theory exploited in Hernández-Lamoneda et al. (2007) are intensively used in this appendix to understand exactly how the index is modified.

In Sect. 5 we introduce the concept of a natural solution. Briefly, natural solutions are those that take an additive game to itself. We prove a theorem characterizing such solutions (that are also linear and symmetric) and discuss the situation when they are further assumed to satisfy the efficiency axiom. Also, we show that a semi-value is natural if and only if the sum of its marginal contribution weights -over all coalitions not containing player i—is one.

The last section of the article deals with a slight modification of the notion of team game as presented above. We further assume now that there is a given, exogenous amount, c to be distributed among the players. Continuing with the NBA example, we could think that c is a certain amount coming, say, from television rights for games transmission which has been agreed to distribute among the players. We propose several solutions satisfying certain desired properties.

More explicitly, the results obtained in this section are: First, a concrete expression for every linear symmetric solution (in fact, the dimension of all such solutions is found to be three); as a corollary we get also formulas for solutions that further satisfy either the efficiency or the naturalness axioms. Finally, it is shown that there exists a unique linear solution which is symmetric, natural and efficient.

The section ends by studying bankruptcy games as a particular case of these type of games.

We finish this introduction with a comment on the methods employed in the article. Although it is true that every theorem in the article could be proved without any explicit mention to the representation theory of the symmetric group, we feel that by doing that we would be withholding valuable information from the reader. After all, it was this algebraic tool which pointed us not only to the proofs, but to the statements of the theorems themselves. This tool, we believe, sheds new light on the structure of the space of games and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

To make the paper as self contained as possible we have included as the second section of the article some of the background material that we need. This is the place where also some of the notation for the whole paper is introduced.

2 Notation and some preparatory material

We begin by fixing some notation and recalling certain standard facts about games in TU-form.

- $N = \{1, 2, \ldots, n\}$ the set of players. S_n the symmetric group of N, that is, the group of permutations of N

$$S_n = \{\theta : N \to N \mid \theta \text{ is a bijection }\}.$$

- $G = G^N = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$, the space of games.
- Every permutation $\theta \in S_n$ determines a linear bijection $\theta : G \to G$ by setting

$$(\theta \cdot v)(S) = v(\theta^{-1} S)$$

  for every $v \in G$ and every coalition $S \subset N$.
- $1_n \in \mathbb{R}^n$ is the vector with ones in every entry: $1_n = (1, 1, \ldots, 1)$ and $\Delta_n = \mathbb{R} \cdot 1_n$.
- $\Delta_n^{\perp} \subset \mathbb{R}^n$ is the orthogonal complement to 1_n; a vector $z = (z_1, \ldots, z_n)$ is in $\Delta_n^{\perp}$ if and only if $z_1 + \cdots + z_n = 0$.

- Every permutation $\theta \in S_n$ also determines a linear bijection $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ via $\theta \cdot (x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$. Notice that $\theta \cdot 1_n = 1_n$ and $\theta \cdot z \in \Delta_n^{\perp}$, for all $\theta \in S_n$ and every $z \in \Delta_n^{\perp}$.

A solution $\phi : G \to \mathbb{R}^n$ is *symmetric* if $\phi(\theta \cdot v) = \theta \cdot \phi(v)$, for every $\theta \in S_n$ and every $v \in G$.

The point of view we take in this article depends heavily on a certain decompostion of G as a direct sum of subspaces. We briefly recall this decomposition and some of its properties. For the proofs, we refer the reader to Hernández-Lamoneda et al. (2007).

First, we have the natural decomposition

$$G = \bigoplus_{s=1}^{n} G_s$$

into orthogonal subspaces, where

$$G_s = \{v \in G \mid v(T) = 0 \text{ if } |T| \neq s\}.$$

We call G_s the space of team games of size s. Orthogonality is with respect to the natural inner product on G; namely, for any games $v, w \in G$

$$\langle v, w\rangle = \sum_{S \subset N} v(S)w(S)$$

Now, each G_s in turn decomposes as follows:

$$G_s = C_s \oplus U_s \oplus W_s \tag{1}$$

where

- $C_s = \{tc_s \mid t \in \mathbb{R}\}$ is the one dimensional subspace generated by

$$c_s(T) = \begin{cases} 1 & \text{if } |T| = s \\ 0 & \text{if } |T| \neq s \end{cases}.$$

  The space $C = \bigoplus_{s=1}^{n} C_s$ is precisely the space of symmetric games.
- For $s : 1, \ldots, n-1$, $U_s = \{z^s \mid z \in \Delta_n^{\perp}\}$ is an $n-1$ dimensional subspace modeled on $\Delta_n^{\perp}$. Here, for each $z \in \Delta_n^{\perp}$, z^s stands for the game defined as

$$z^s(T) = \begin{cases} \sum_{i \in T} z_i & \text{if } |T| = s \\ \\ 0 & \text{if } |T| \neq s \end{cases}.$$

  Set $U = \bigoplus_{s=1}^{n-1} U_s$.
- $W_s = \{w \in G_s \mid \forall i, \sum_{S \ni i} w(S) = 0\}$. Setting $W = \bigoplus_{s=1}^{n} W_s$, it turns out that W is the common kernel of all linear and symmetric solutions (Hernández-Lamoneda et al. 2007).

- Moreover,
  - $G_n = C_n$; i.e. $U_n = W_n = 0$, and
  - $W_1 = W_{n-1} = 0$.

Summarizing, we have the important decompositon of the space of games

$$G = C \oplus U \oplus W \tag{2}$$

Thus, given a game $v \in G$, we may decompose it relative to the above as $v = c + u + w$, where in turn $c = \sum a_s c_s$, $u = \sum z_s^s$. This decomposition is very well suited to study the image of v under any linear symmetric solution, thanks to the following version of Schur's Lemma (a well-known result of representation theory).

Theorem 1 (Schur's Lemma) *Any linear symmetric solution*

$$\phi : G = C \oplus U \oplus W \to \mathbb{R}^n = \mathbb{R}1_n \oplus \Delta_n^{\perp}$$

satisfies

(1) $\phi(C) \subset \mathbb{R}1_n$,
(2) $\phi(U) \subset \Delta_n^{\perp}$,
(3) $\phi(W) = 0$.

Moreover, for each $s : 1, \ldots, n-1$, there is a constant b_s such that, for every $z^s \in U_s$,

$$\phi(z^s) = b_s z \in \Delta_n^{\perp}.$$

Remark 1 The above theorem follows from the standard Schur's Lemma because, as proved in Hernández-Lamoneda et al. (2007), W contains no copies of either the trivial or the standard representations of S_n.

Lemma 1 *Let $v \in G$. Then*

$$v = \sum_{s=1}^{n} a_s c_s + \sum_{s=1}^{n-1} z_s^s + w$$

where

(1) *a_s is the average of the values of v on cardinality s coalitions:*

$$a_s = \frac{\sum_{|T|=s} v(T)}{\binom{n}{s}}.$$

(2) *For every $s < n$,*

$$z_s = (n-1)Sh(v_s)$$

where Sh denotes Shapley's value and v_s is the G_s-component of v (i.e., $v_s(T) = v(T)$ if $|T| = s$ and $v_s(T) = 0$ otherwise).

(3) *$w \in W$ may be computed as "the rest", i.e., $w = v - c - u$.*

Proof We start by computing the orthogonal projection of v onto C. Notice that $\{c_s\}$ is an orthogonal basis for C, and that $\|c_s\|^2 = \binom{n}{s}$.

Thus, the projection of v onto C is

$$\frac{\langle v, c_1\rangle}{\langle c_1, c_1\rangle}c_1 + \cdots + \frac{\langle v, c_n\rangle}{\langle c_n, c_n\rangle}c_n$$

and so

$$a_s = \frac{\langle v, c_s\rangle}{\langle c_s, c_s\rangle} = \frac{\sum_{|T|=s} v(T)}{\binom{n}{s}}.$$

Now, for $s < n$, let v_s be defined as above. Then, $v_s = a_s c_s + z_s^s + w_s$, where w_s is the component of w in G_s.

Applying Shapley's value we get

$$Sh(v_s) = a_s Sh(c_s) + Sh(z_s^s) + Sh(w_s) = \frac{1}{n-1}z_s$$

since $Sh(c_s) = 0$, for $s < n$, because of efficiency, $Sh(w_s) = 0$ by Schur's Lemma, and $Sh(z_s^s) = b_s z_s$ (again by Schur's lemma) and the precise value of $b_s = \frac{1}{n-1}$ is easy to compute. □

Remark 2 The use of Shapley's value in order to compute z_s is a matter of personal taste. One could use her favorite linear symmetric solution -as long as it is non-zero on each U_s- to compute them. For example, if one were to use Banzhaf's semi-value B, then the formula would be a little bit more complicated, but computable. Namely,

$$z_s = \frac{(n-1)2^{n-2}}{s\binom{n-1}{s}}\left[B(v_s) - a_s\frac{\binom{n}{s}}{2^{n-1}}1_n\right].$$

Example Let $N = \{1, 2, 3, 4\}$, $S = \{1, 2\}$ and

$$v(T) = u_S(T) = \begin{cases} 1 & \text{if } T \supset S \\ 0 & \text{otherwise} \end{cases}.$$

One then computes with the above formula the decomposition of such a game v as

$$v = \frac{1}{6}c_2 + \frac{1}{2}c_3 + c_4 + z_2^2 + z_3^3 + w,$$

where

$$z_2 = \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$$
$$z_3 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

and

$$w(\{1,2\}) = w(\{3,4\}) = \frac{1}{3}$$
$$w(\{1,3\}) = w(\{1,4\}) = w(\{2,3\}) = w(\{2,4\}) = -\frac{1}{6},$$

and all other values of w are zero.

3 Indices

3.1 Power index ranking (PIR)

Every solution $\phi : G \to \mathbb{R}^n$ provides a power index ranking (PIR) for the players with respect to any game v, by ordering increasingly (or decreasingly) the coordinates of $\phi(v)$. For this reason we call ϕ an "index" instead of a solution.

Let ϕ, ψ: $G \to \mathbb{R}^n$ be two indices. $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ give the *same power index ranking* (PIR) if and only if for every game v, and all players i, j,

$$\phi_i(v) > \phi_j(v) \text{ iff } \psi_i(v) > \psi_j(v).$$

We write $\phi \overset{\text{PIR}}{=} \psi$ to indicate that both indices have the same PIR.

One obvious way in which two indices have the same PIR is if, in each coordinate, they differ by the same constant. More precisely,

Let $\phi, \psi : G \to \mathbb{R}^n$ be two indices. We say that ϕ *and* ψ *agree up to a constant* if there exists a function $k : G \to \mathbb{R}$ such that, for every game v, and for every player i

$$\phi_i(v) - \psi_i(v) = k(v).$$

We write $\phi \overset{\text{up-to-}k}{=} \psi$.

Clearly, if ϕ and ψ agree up to a constant, then $\phi_i(v) - \phi_j(v) = \psi_i(v) - \psi_j(v)$ for every v and for every players i and j, and thus $\phi \overset{\text{PIR}}{=} \psi$. In symbols, $\phi \overset{\text{up-to-}k}{=} \psi \Rightarrow \phi \overset{\text{PIR}}{=} \psi$.

Lemma 2 *Let $\phi : G \to \mathbb{R}^n$ be a symmetric linear solution, let $v \in G$. Decompose $v = c + u + w$ with respect to the S_n-decomposition of G, $G = C \oplus U \oplus W$.*

Then, for every i, j,

$$\phi_i(v) - \phi_j(v) = \phi_i(u) - \phi_j(u).$$

Proof From Schur's Lemma (Theorem 1), we have that for every i, j, $\phi_i(w) = 0$ and $\phi_i(c) = \phi_j(c)$. □

Thus for a game $v = c + u + w$, we have $\phi_{i_1}(v) \geq \phi_{i_2}(v) \geq \cdots \geq \phi_{i_n}(v)$ if and only if $\phi_{i_1}(u) \geq \phi_{i_2}(u) \geq \cdots \geq \phi_{i_n}(u)$, and, $\phi_{i_j}(v) - \phi_{i_k}(v) = \phi_{i_j}(u) - \phi_{i_k}(u)$.

The previous lemma (and the results following) show again the relevance of decomposition (2) of the space of games; in particular it tells us that the PIR that a given solution determines for a game v, depends only on the U-component of v.

Set $\omega = (1, 1, \ldots, 1, 1 - n) \in \Delta_n^{\perp} \subset \mathbb{R}^n$ and define, for every $s < n$, for every $x \in \mathbb{R}^n$ and every $T \subset N$,

$$x^s(T) = \begin{cases} \sum_{i \in T} x_i & \text{if } |T| = s \\ 0 & \text{if } |T| \neq s \end{cases};$$

for example, $\omega^s(T) = s$ if $|T| = s$ and $n \notin T$, $\omega^s(T) = s - n$ if $|T| = s$ and $n \in T$, and is zero otherwise.

Lemma 3 *Let ϕ and ψ be two linear symmetric solutions.*

Then $\phi \overset{up\text{-}to\text{-}k}{=} \psi$ if and only if $\phi_n(\omega^s) = \psi_n(\omega^s)$ for every $s : 1, \ldots, n-1$ if and only if $\phi(u) = \psi(u) \forall u \in U$.

Proof That ϕ and ψ agree up to a constant is equivalent (by Lemma 2) to $\phi_i(u) - \phi_j(u) = \psi_i(u) - \psi_j(u)$ for every $u \in U$ and every i, j.

Set $a_s = \phi_n(\omega^s)$ and $b_s = \psi_n(\omega^s)$. This is equivalent to $\phi(\omega^s) = \frac{a_s}{1-n}\omega$ and $\psi(\omega^s) = \frac{b_s}{1-n}\omega$, which in turn is equivalent to $\phi(z^s) = \frac{a_s}{1-n}z$ and $\psi(z^s) = \frac{b_s}{1-n}z$ for all $z \in \Delta_n^{\perp}$ (by Schur's Lemma (Theorem 1)).

Now, $\phi_i(u) - \phi_j(u) = \psi_i(u) - \psi_j(u)$ for every $u \in U$, if and only if $\phi_i(z^s) - \phi_j(z^s) = \psi_i(z^s) - \psi_j(z^s)$ for every $z \in \Delta_n^{\perp}$ and every $s < n$, if and only if $\frac{a_s}{1-n}(z_i - z_j) = \frac{b_s}{1-n}(z_i - z_j)$ for every $z \in \Delta_n^{\perp}$, $s < n$, if and only if $a_s = b_s$ for every $s < n$. □

Example Let Sh and B denote Shapley's and Banzhaf's values. Then, a simple computation gives

$$Sh_n(\omega^s) = -1 \quad \text{and} \quad B_n(\omega^s) = -\frac{s\binom{n-1}{s}}{2^{n-2}} \text{ for every } s : 1, \ldots, n-1.$$

Thus, if $n = 3$ then $B_n(\omega^s) = -1 = Sh_n(\omega^s)$ $(s = 1, 2)$ and hence $B \overset{\text{up-to-}k}{=} Sh$, therefore $B \overset{\text{PIR}}{=} Sh$, as already noticed by Saari and Sieberg (2001). Here, we have used the fact that for every game $v \in G$,

$$B_n(v) = \frac{1}{2^{n-1}} \sum_{S \not\ni n} [v(S \cup \{n\}) - v(S)] .$$

Recall that a solution ϕ is *self-dual* if $\phi(*v) = \phi(v)$, for every game $v \in G$, where $*v(S) = v(N) - v(N \setminus S)$.

Theorem 2 *Let $\psi : G \to \mathbb{R}^n$ be any linear symmetric solution. Then there exists a unique solution $\phi : G \to \mathbb{R}^n$ which is linear, symmetric, efficient and agrees up to a constant with ψ. Moreover, if ψ is self-dual then so is ϕ.*

Proof Namely, set

$$\phi_n(c_s) = \begin{cases} 0 & s < n \\ \frac{1}{n} & s = n \end{cases}$$

and $\phi_n(\omega^s) = \psi_n(\omega^s)$. Symmetry determines $\phi_j(c_s)$ and $\phi_j(\omega^s)$ for every player j. Then, Theorem 1 (Schur's lemma), tells us that this defines a unique linear and symmetric solution ϕ on G. The previous lemma says that $\phi \overset{\text{up-to-}k}{=} \psi$ since $\phi_n(\omega^s) = \psi_n(\omega^s)$. Moreover, it is shown in Hernández-Lamoneda et al. (2007) that efficiency of (the linear and symmetric) ϕ is equivalent to

$$\phi_n(c_s) = \begin{cases} 0 & s < n \\ \frac{1}{n} & s = n \end{cases} .$$

Finally, an efficient solution is self-dual if and only if $\phi_n(\omega^s) = \phi_n(\omega^{n-s})$ (Hernández-Lamoneda et al. 2007), but this follows from the self-duality of ψ. □

In other words, pick your favorite index (e.g., one satisfying the nullity axiom) and you can turn it into an efficient one (losing nullity in the process) without changing the ranking for all games.

Corollary 1 *There exists a unique solution $\phi : G \to \mathbb{R}^n$ which is linear, symmetric, efficient and agrees up to a constant with the Banzhaf semi-value B.*

If we recall the general formula for solutions without the nullity axiom derived in Hernández-Lamoneda et al. (2007), then this efficient solution with the same index ranking as Banzhaf, may be expressed as

$$\phi_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \sum_{S \ni i} (n - s) [v(S) - v(N \setminus S)]$$

where $s = |S|$. One recognizes this solution as the additive efficient normalization of the Banzhaf value (i.e., it is the LS-prenucleolus considered in Ruiz et al. 1996).

More generally,

Corollary 2 *Let*

$$\psi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S) \right]$$

be an arbitrary linear symmetric and null solution (*see* Dubey et al. 1981 *or* Hernández-Lamoneda et al. 2007). *Then the unique linear symmetric efficient solution agreeing up to a constant with* ψ *is given by*

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i} (n-s) \left[t_s v(S) - t_{n-s} v(N \setminus S) \right]$$

where

$$t_s = \frac{r_{s-1} + r_s}{n}.$$

The following theorem tells us that, although agreeing up to a constant looks a priori a much stronger condition than having the same PIR, for linear symmetric solutions, these properties are the same up to a homothety.

Theorem 3 *Let* ϕ *and* ψ *be two linear and symmetric indices. Then* $\phi \overset{\text{PIR}}{=} \psi$ *implies there exists* $\lambda > 0$ *such that* $\phi \overset{up\text{-}to\text{-}k}{=} \lambda\psi$. *The converse is also true.*

In other words, two linear symmetric indices, ϕ and ψ, rank all players, for all games, in the same order, if and only if there exists a function $k : G \to \mathbb{R}^n$ and a positive constant λ, such that

$$\phi_i(v) = \lambda \psi_i(v) + k(v)$$

for every v and every i.

Proof Set $a_s = \phi_n(\omega^s)$, $b_s = \psi_n(\omega^s)$.

We check first that $a_s \neq 0$ if and only if $b_s \neq 0$. Now, $b_s > 0$ if and only if $\psi_n(\omega^s) = b_s > \frac{b_s}{1-n} = \psi_{n-1}(\omega^s)$, if and only if $a_s = \phi_n(\omega^s) > \phi_{n-1}(\omega^s) = \frac{a_s}{1-n}$, if and only if $a_s > 0$. Similarly, $b_s < 0$ if and only if $a_s < 0$.

Thus, if $\psi_n(\omega^s) = 0$, for all s, then also $\phi_n(\omega^s) = 0$ for all s, and we are done (set $\lambda = 1$, for example).

So, let t be such that $b_t \neq 0$. Let s be any other number such that $b_s \neq 0$. We want to prove that $a_t/b_t = a_s/b_s$.

Suppose not, then $a_t b_s - b_t a_s \neq 0$, and hence, we may solve for α and β, the linear system

$$\begin{aligned} \alpha a_t + \beta a_s &= 1 - n \\ \alpha b_t + \beta b_s &= n - 1; \end{aligned}$$

setting $z = \alpha\omega^t + \beta\omega^s$ we see that $\phi(z) = \omega$, whereas $\psi(z) = -\omega$. Therefore, $\phi_1(z) = 1 > 1 - n = \phi_n(z)$ while $\psi_1(z) = -1 < n - 1 = \psi_n(z)$, contradicting $\phi \overset{\text{PIR}}{=} \psi$.

Thus, for every s such that $b_s \neq 0$, the ratio a_s/b_s is a constant $\lambda > 0$, and $\phi(\omega^s) = \lambda\psi(\omega^s)$.

For the converse, recall that we have that $\phi \overset{\text{up-to-}k}{=} \lambda\psi$ implies $\phi \overset{\text{PIR}}{=} \lambda\psi$. But, clearly, $\phi \overset{\text{PIR}}{=} \psi$ if and only if $\phi \overset{\text{PIR}}{=} \lambda\psi$, for every $\lambda > 0$. □

Example revisited. Let $u \in U$, thus $u = \sum_s z_s^s$ with $z_s \in \Delta_n^\perp$. Then

$$Sh(u) = \frac{1}{n-1}\sum_{s=1}^{n-1} z_s$$

$$B(u) = \frac{1}{2^{n-2}(n-1)}\sum_{s=1}^{n-1} s\binom{n-1}{s} z_s.$$

Thus, for example, for $n = 4$, $Sh(u) = \frac{1}{3}[z_1 + z_2 + z_3]$ whereas $B(u) = \frac{1}{4}[z_1 + 2z_2 + z_3]$. This explains the difference in power index rankings both values assign Saari and Sieberg (2001).

We finish the section with a general formula, for an arbitrary semi-value, that computes the ranking weights -as in the above example- in terms of the marginal contribution weights.

Theorem 4 *Let*

$$\psi_i(v) = \sum_{S \not\ni i} r_s\,[v(S \cup \{i\}) - v(S)]$$

be the semi-value corresponding to the marginal contribution weights $r_0, \ldots, r_{n-1}$. *Let* $z \in \Delta_n^\perp$ *arbitrary. For* $s = 1, \ldots, n-1$, *let* $b_s \in \mathbb{R}$ *such that* $\psi(z^s) = b_s z$. *Then*

$$b_s = \binom{n-2}{s-1}(r_{s-1} + r_s).$$

Thus, if $v = c + u + w \in G$ *is any game and* $u = \sum_s z_s^s$ *with* $z_s \in \Delta_n^\perp$, *then the ranking of the players, for* v, *with respect to* ψ *is given by the vector*

$$\psi(u) = \sum_{s=1}^{n-1}\binom{n-2}{s-1}(r_{s-1} + r_s)z_s.$$

4 Team games and the Michael Jordan problem

Donald Saari, at a conference in his honor in Guanajuato, told us about "the Michael Jordan problem". The problem is the following: how is it possible that, even though

everyone agrees that Michael Jordan is the best basketball player in the NBA, different indices when applied to all the players rank them differently and not all of them place Michael Jordan at the top?

In a sense, what one would like is for the game of basketball to have an intrinsic power index ranking of its players; then we could verify whether in fact Michael Jordan is at the top, or whether someone else is there.

In this section, we propose a solution to this problem. The idea is to consider indices for "team games" (after all, basketball is a team game). By a team game we understand a cooperative game in TU-form, whose values on coalitions of every cardinality but one are zero. Formally,

For every $s : 1, \ldots, n$, let $G_s = \{v \in G \mid v(S) = 0 \; if |S| \neq s\}$; we think of the elements in G_s as "team games" with s players.

Basketball example. n is the number of all basketball players, $s = 5$ means that we consider only coalitions of 5 players; the game of basketball may be represented by a game $v \in G_5^{(n)}$. Such a game should assign a higher value, to a team of 5 players, depending on how successful such a coalition is (e.g., one that wins more games when faced -theoretically- against every other possible combination of players).

We will show that team games have an intrinsic index I_0 associated to them. Then, for example, basketball players will be ranked according to $I_0(v)$ where -presumably- Michael Jordan will end up at the top.

Also, following an idea of Saari and Sieberg, we determine how the PIR of a team game changes when a player drops out (retires).

Remark 3 Not to raise false expectations, we should make it perfectly clear that, although we will prove that indeed any team game does determine a unique ranking among the players, we will not give any explicit team game expression that models basketball. We define a "theoretical" team game of five players for basketball which then implies a ranking for the players. Nevertheless, such a game has no chance to be computed in practice since most teams -coalitions of 5 players- will never form.

Thus, the problem of ranking is solved provided we are given the concrete team game itself.

The Poker example. $n = 52$, and one can identify every player with a card of a deck of poker cards. Every coalition with 5 players is thus a hand of poker. Then, the game of poker may be thought of as a game in $G_5^{(52)}$ which assigns to each hand of poker a certain value, where the value is higher for a better hand. Let us define more precisely a team game $v \in G_5^{(52)}$ representing the game of poker.

Thus, let S be a coalition of five cards (i.e. a hand of poker). We wish to define $v(S)$. To simplify the calculations we assume that we confront S with any other possible hand (S included) and that all suits are of equal worth; we further assume that in each confrontation, one coin is at stake. Then we define $v(S)$ as the expected payoff for the hand S; in other words, $v(S)$ is defined as the number of hands strictly worse than S, minus the number of hands strictly better than S, divided by the total number of possible hands. Notice that there will be hands which "draw" with S.

Double sculls example. Double sculls is an olympic sport where a pair of oarsmen row together in a boat on a course of 2 kilometers in length. Consider the following

problem: We have a pool of 5 oarsmen who compete in double sculls. Also, there are three national fellowships (of decreasing amounts) to be assigned to the three "best" rowers.

Have each pair of rowers, $\{i, j\}$, take a time trial. Define a team game $v \in G_2^{\{1,2,3,4,5\}}$ as

$$v(\{i, j\}) = \text{time recorded by } \{i, j\} \text{ on their time trial.}$$

Notice that for this example the rower with lowest index is the best.

4.1 PIR for team games

Recall -see (1)- that G_s is invariant under S_n and that it decomposes as

$$G_s = C_s \oplus U_s \oplus W_s.$$

So, an arbitrary team game with s players will be of the form

$$rc_s + z^s + w$$

where $r \in \mathbb{R}$, $z \in \Delta_n^{\perp}$ and $w \in W_s$.

Let $I_0 : G_s \to \mathbb{R}^n$ be given by $I_0(rc_s + z^s + w_s) = z$. I_0 is a linear, symmetric solution. Moreover, $\frac{1}{n-1} I_0$ agrees up to a constant with the restriction of Shapley's value to G_s.

Theorem 5 *Let $\phi : G_s \to \mathbb{R}^n$ be any linear, symmetric index. Then ϕ has the same PIR as εI_0 where $\varepsilon = -1, 0$ or 1.*

Proof The proof follows from the arguments in the previous section. In any case, for added clarity we give an independent proof here.

Let $v = rc_s + z^s + w_s$ be an arbitrary element of G_s. By Schur's Lemma, $\phi(c_s) = a(1, 1, \ldots, 1) = a1_n$, for some $a \in \mathbb{R}$, that $\phi(z^s) = bz$ for some $b \in \mathbb{R}$, and that $\phi(w_s) = 0$.

Then, for every i, j,

$$\phi_i(v) - \phi_j(v) = b(z_i - z_j) = b((I_0)_i(v) - (I_0)_j(v));$$

therefore, $\phi \overset{\text{up-to-}k}{=} bI_0$, which, according to Theorem 3, gives the three cases depending on whether $b < 0$, $b = 0$ or $b > 0$. □

Corollary 3 *To every team game $v \in G_s$ is associated a unique (up to sign) power ranking of its players.*

Proof Simply take $I_0(v)$. Any linear symmetric index, either will rank equally all the players for all team games (the $\varepsilon = 0$ case), or have the same PIR as $\pm I_0$. □

This solves the Michael Jordan problem.

Now, a natural question comes about: given a team game v, how do we compute its ranking index in the most economical way?

Theorem 6 *For $s < n$, define $I : G_s \to \mathbb{R}^n$ by*

$$I_i(v) = \frac{1}{\binom{n-2}{s-1}} \sum_{S \ni i} v(S).$$

Then I agrees with I_0 up to a constant and thus provides a formula, in terms of v, for computing the intrinsic ranking in v.

Proof Clearly $I = (I_1, \ldots, I_n)$ is a symmetric, linear map. Therefore it agrees with bI_0 up to a constant, for some real b. We need only to check that $b = 1$. Thus, to verify the claim, it is enough to check that $I_i(z^s) = 1$, where $z = e_i - e_1 = (-1, 0, \ldots, 0, 1, 0, \ldots, 0)$, since $I_0(z^s) = z$.

Now,

$$\sum_{\substack{|S|=s \\ S \ni i}} z(S) = \sum_{\substack{|S|=s \\ S \ni i \\ S \not\ni 1}} 1 = \binom{n-2}{s-1}.$$

□

Remark 4 One could argue that $v \mapsto \sum_{S \ni i} v(S)$ gives a more economical index. We have retained the coefficient in front of this expression so that $z^s \mapsto z$, for every $z \in \Delta_n^{\perp}$.

The Poker example again. Recall we have defined a game $v \in G_5^{52}$ that models the game of poker. Thus, the index $I : G_5^{52} \to \mathbb{R}^{52}$ ranks the cards. Recall that all suits are assumed to have the same value, thus we may think that I takes values in $\mathbb{R}^{13}$. We list the rankings of the cards:

j	$I_j(v)$
A	0.217571
K	0.118247
Q	0.061402
J	0.027070
10	0.004912
9	−0.011684
8	−0.025120
7	−0.036887
6	−0.047765
5	−0.058791
4	−0.070952
3	−0.082982
2	−0.095022

It is perhaps surprising to find out that the 10 is the first card to have a positive index coordinate (since the 8 is the middle card).

4.2 Retiring players

In this subsection we answer the following: for a team game, if player n drops out, how does the ranking of the remaining players change? For example -considering the game of poker- if a card (or group of cards) is removed from the deck, how is the power of the remaining cards affected? Similarly, if an oarsmen is injured, how does this affect the importance of the other ones.

The answer, at least numerically, is readily available from the formula of Theorem 6. To explain the source of the contributions that change the rankings of the remaining players, we carry out a detailed analysis in the appendix to this article. Still, we use the insight gained there to give in this section an example and a theorem (Theorem 7). Theorem 7 tells us that the ranking of the players can drastically change when one of them leaves, whereas Theorem 8 gives an interpretation of this possibility in terms of an auxiliary game where the dropped player is always present.

Let $v \in G_s^N$ be an arbitrary game. Then, if player n drops out, we get a new game $Rv \in G_s^{N\setminus\{n\}}$ by restriction: for every $S \subset \{1, 2, \ldots, n-1\}$, let $Rv(S) := v(S)$.

One can compute the intrinsic PIR of the restricted game $Rv \in G^{N\setminus\{n\}}$. In fact, we can, in the same fashion as Theorem 6, state a general formula when we assume that a set P of players retire. Let $v \in G_s^N$ be an arbitrary team game. Suppose all the players in the set $P \subset N$ drop out. Let $p = |P|$. Let $Rv \in G_s^{N\setminus P}$ be the restricted game. Then the intrinsic PIR of Rv may be computed via the formula

$$I_j^{N\setminus P}(Rv) = \frac{1}{\binom{n-p-2}{s-1}} \sum_{\substack{S \ni j \\ S \subset N\setminus P}} v(S),$$

for $j \in N\setminus P$.

The Poker example once again. Suppose three persons are playing a hand of poker. Two of them have already discarded the ace of hearts ($A\heartsuit$), the ace of diamonds ($A\diamondsuit$) and the 3 of clubs ($3\clubsuit$). The third player is deciding whether to keep the king of clubs ($K\clubsuit$) or the ace of spades ($A\spadesuit$).

Computing the new rankings of the cards -for the whole deck minus the three discarded cards, $N \setminus \{A\heartsuit, A\diamondsuit, 3\clubsuit\}$- tells us that

j	$I_j(Rv)$
$A\spadesuit$	0.151666
$K\spadesuit$	0.151627
$K\heartsuit$	0.150746
$K\diamondsuit$	0.150746
$A\clubsuit$	0.150701
$K\clubsuit$	0.150501

so, in particular, the ace of spades is stronger than the king of clubs -in fact, it is stronger than any of the kings- even though there are already two aces out of the game. Curiously, the other ace is weaker than three of the kings. □

We will need the following computation for the proof of the next theorem.

Lemma 4 *Let $k \in N$, $i \neq k$ and $z \in \Delta_n^{\perp}$ then*

$$(1)\ \sum_{\substack{S \ni i \\ |S|=s}} z(S) = \binom{n-2}{s-1} z_i$$

$$(2)\ \sum_{\substack{S \ni i, S \not\ni k \\ |S|=s}} z(S) = \binom{n-3}{s-1} z_i - \binom{n-3}{s-2} z_k$$

$$(3)\ \sum_{\substack{S \ni i, S \ni k \\ |S|=s}} z(S) = \binom{n-3}{s-2} [z_i + z_k]$$

Proof

$$\sum_{\substack{S \ni i \\ |S|=s}} z(S) = \sum_{\substack{S \ni i \\ |S|=s}} z_i + \sum_{\substack{S \ni i, j \\ |S|=s}} z_j = \binom{n-1}{s-1} z_i + \binom{n-2}{s-2} \sum_{j \neq i} z_j$$

$$= \left[\tbinom{n-1}{s-1} - \tbinom{n-2}{s-2} \right] z_i = \tbinom{n-2}{s-1} z_i .$$

For (2),

$$\sum_{\substack{S \ni i \\ S \not\ni k \\ |S|=s}} z(S) = \tbinom{n-2}{s-1} z_i + \sum_{\substack{S \ni i, j \\ S \not\ni k \\ |S|=s}} z_j = \binom{n-2}{s-1} z_i + \binom{n-3}{s-2} \sum_{j \neq i,k} z_j$$

$$= \tbinom{n-2}{s-1} z_i - \tbinom{n-3}{s-2} [z_i + z_k] = \tbinom{n-3}{s-1} z_i - \tbinom{n-3}{s-2} z_k .$$

The last statement is proved in the same fashion □

The analysis on player retirement carried out in the appendix shows exactly how the ranking is modified when one of the players retires. In particular, it tells you how to construct team games where any change in the ranking of the players is possible. More precisely, we have the following:

Theorem 7 *Fix a player $k \in N$. Let $u \in \mathbb{R}^N$ and $x \in \mathbb{R}^{N \setminus \{k\}}$ be arbitrary vectors. If $2 \leq s \leq n-2$, then there exists a team game $v \in G_s^N$ such that:*

- *the ranking of the players for v is given by the vector u; and*
- *if player k drops out, then the ranking of the players for the restricted game, Rv, to $N \setminus \{k\}$ is given by x.*

Proof We will construct a concrete game v with the required properties.

Notice that the vector $u - \frac{1}{n}\left(\sum_{i=1}^{n} u_i\right) 1_n$ determines the same ranking as u and the sum of its coordinates is zero. Thus, we may assume that $u_1 + \cdots + u_n = 0$ and, likewise, that $\sum_{j \neq k} x_j = 0$.

Let $y \in \mathbb{R}^n$ be given by

$$y_j = x_j - u_j - \tfrac{u_k}{n-1}, \quad \text{if } j \neq k, \text{ and } y_k = 0.$$

Notice that also $\sum_j y_j = 0$.

Define a team game $v \in G_s^N$ whose values on cardinality s subsets S is given by

$$v(S) = \begin{cases} u(S) + y(S) & S \not\ni k \\ u(S) + \left(1 - \frac{n-2}{s-1}\right) y(S) & S \ni k \end{cases}.$$

We claim that this v meets the requirements.

We compute $I_l(v)$. Let us assume first that $l \neq k$. Then

$$\begin{aligned} I_l(v) &= \frac{1}{\binom{n-2}{s-1}} \sum_{S \ni l} v(S) \\ &= \frac{1}{\binom{n-2}{s-1}} \sum_{S \ni l} u(S) + \frac{1}{\binom{n-2}{s-1}} \left[\sum_{S \ni l,\, S \not\ni k} y(S) + \left(1 - \frac{n-2}{s-1}\right) \sum_{S \ni l, S \ni k} y(S) \right], \end{aligned}$$

which, by the previous lemma and the fact that $y_k = 0$, equals

$$u_l + \frac{1}{\binom{n-2}{s-1}} \left[\binom{n-3}{s-1} y_l + \left(1 - \frac{n-2}{s-1}\right) \binom{n-3}{s-2} y_l \right] = u_l.$$

Similarly,

$$\begin{aligned} I_k(v) &= u_k + \frac{1}{\binom{n-2}{s-1}} \left(1 - \frac{n-2}{s-1}\right) \sum_{S \ni k} y(S) \\ &= u_k + \left(1 - \frac{n-2}{s-1}\right) y_k = u_k. \end{aligned}$$

This shows that the ranking of the players of v is given by the vector u. It remains to examine the restricted game Rv.

To compute $I(Rv)$, we need to evaluate $Rv(S) = v(S)$ on coalitions S, of cardinality s, such that $S \subset N \setminus \{k\}$. To wit,

$$\begin{aligned} I_l(Rv) &= \frac{1}{\binom{n-3}{s-1}} \sum_{S \ni l,\, S \not\ni k} v(S) = \frac{1}{\binom{n-3}{s-1}} \left[\sum_{S \ni l,\, S \not\ni k} u(S) + \sum_{S \ni l,\, S \not\ni k} y(S) \right] \\ &= \frac{1}{\binom{n-3}{s-1}} \left[\binom{n-3}{s-1} u_l - \binom{n-3}{s-2} u_k + \binom{n-3}{s-1} y_l \right] \\ &= u_l + y_l - \frac{s-1}{n-s-1} u_k = x_l - \frac{s(n-2)}{(n-1)(n-s-1)} u_k \end{aligned}$$

the second summand is a constant independent of l, so the vector $I(Rv)$ gives the same ranking as x. □

Rowing example revisited. Assume the following are the times (in minutes) recorded by each pair of oarsmen during the trials:

$$\begin{gathered} v(\{1,5\}) = 6.26 \\ v(\{2,5\}) = 6.46 \\ v(\{3,5\}) = 6.66 \\ v(\{4,5\}) = 6.86 \\ v(\{1,2\}) = 6.54 \\ v(\{1,3\}) = 6.5 \\ v(\{1,4\}) = 6.46 \\ v(\{2,3\}) = 6.46 \\ v(\{2,4\}) = 6.42 \\ v(\{3,4\}) = 6.38 \end{gathered}$$

Then,

$$I(v) = \frac{1}{3}(25.76, 25.88, 26, 26.12, 26.24),$$

and so,

$$1 \prec 2 \prec 3 \prec 4 \prec 5$$

which means that the fellowships should go to rowers [1, 2, 3] in that order. Unfortunately, a few days later it is discovered that rower # 5 failed the anti-doping test. Thus, all his trial times must be removed (even though he was the worse, by our index measurement, of all rowers) and the index for the restricted game (with the fifth rower "retired") is computed (the restricted game corresponds to the bottom 6 rows on the above table):

$$I(Rv) = \frac{1}{2}(19.5, 19.42, 19.34, 19.26).$$

Thus the new ranking is $1 \succ 2 \succ 3 \succ 4$. And, therefore, the fellowships should be reassigned to [4, 3, 2] in that order.

Remark 5 It is clear now that a player could be dominant for a game v, but not so dominant as to remain on top after other players' retirements. One could say that a player d is "very dominant" for a team game $v \in G_s^N$ if it is dominant for every restriction game. This is equivalent to asking that, for every other player l, and for every coalition $S \subset N\backslash\{d, l\}$, of cardinality $s-1$, we have

$$v(S \cup \{d\}) > v(S \cup \{l\}). \tag{I}$$

Thus, in terms of "The Michael Jordan problem", what one would need to verify for dominance of a player would be to check all the set of inequalities (I).

We conclude this subsection with an interpretation of the different indices that appear once a player has retired.

For a team game $v \in G_s$, define a team game $Pv \in G_{s-1}^{N\backslash\{n\}}$ by

$$Pv(T) = v(T \cup \{n\})$$

for every $T \subset N\backslash\{n\}$. The game Pv is defined by considering all coalitions where n is present; $Rv \in G_s^{N\backslash\{n\}}$, as before, the restricted game. Then

Theorem 8

$$(n-2)I^N(v) = (n-s-1)I^{N\backslash\{n\}}(Rv) + (s-1)I^{N\backslash\{n\}}(Pv).$$

Proof For every $j \in N \setminus \{n\}$

$$\begin{aligned}\sum_{S \ni j} v(S) &= \sum_{S \ni j, S \not\ni n} v(S) + \sum_{S \ni j, S \ni n} v(S) = \sum_{S \ni j, S \not\ni n} v(S) + \sum_{T \ni j, T \not\ni n} v(T \cup \{n\}) \\ &= \sum_{S \ni j} Rv(S) + \sum_{T \ni j} Pv(T);\end{aligned}$$

thus

$$\binom{n-2}{s-1} I^N(v) = \binom{n-3}{s-1} I^{N\backslash\{n\}}(Rv) + \binom{n-3}{s-2} I^{N\backslash\{n\}}(Pv).$$

□

5 Natural solutions

Consider the subspace $A \subset G$ of additive games,

$$A = \{v \in G : v(S \cup T) = v(S) + v(T), \text{ whenever } S \cap T = \emptyset\}.$$

For every vector $x \in \mathbb{R}^n$, define the game $\hat{x}$ by

$$\hat{x}(S) = \sum_{i \in S} x_i;$$

the following is well known:

Lemma 5

$$A = \{\hat{x} : x \in \mathbb{R}^n\}.$$

It is usually accepted that additive games "solve themselves", i.e., that the value a "natural" solution assigns to an additive game v should be $v(\{i\})$. Thus, a solution $\phi : G \to \mathbb{R}^n$ is *natural* if $\phi(\hat{x}) = x, \forall x \in \mathbb{R}^n$.

Remark 6 Additive games are sometimes called, in the literature, "inessential games"; similarly, natural solutions were called "inessential solutions" in Kultti and Salonen (2007).

Given a vector $x \in \mathbb{R}^n$ and an integer $s : 1, \ldots, n$ we have defined a game, x^s, as follows:

$$x^s(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = s; \\ 0 & \text{if } |S| \neq s. \end{cases}$$

Notice that $\hat{x} = \sum_{s=1}^{n} x^s$.

Let $\omega = (1, 1, \ldots, 1, 1 - n) \in \mathbb{R}^n$.

Every linear symmetric solution ϕ is determined by its n-th coordinate ϕ_n. More precisely, from Theorem 1, every linear symmetric solution is uniquely determined by specifying the $2n - 1$ numbers $\phi_n(c_s)$ and $\phi_n(\omega^s)$, $j = 1, \ldots, n$ $(\omega^n = 0)$.

The next theorem gives necessary and sufficient conditions for a solution to be natural.

Theorem 9 *Let $\phi : G \to \mathbb{R}^n$ be a linear symmetric solution, then ϕ is natural if and only if $\phi_n(\widehat{1_n}) = 1$ and $\phi_n(\hat{\omega}) = 1 - n$.*

Proof It is clear that the conditions are necessary. We prove they are sufficient.

Every $x \in \mathbb{R}^n$ can be decomposed as $x = r1_n + z$, with $z \in \Delta_n^{\perp}$.

Now, by Schur's Lemma, $\phi(\hat{\omega}) = b\omega$ for some $b \in \mathbb{R}$, and, in fact, $\phi(\hat{z}) = bz$ for every $z \in \Delta_n^{\perp}$. Thus $\phi_n(\hat{\omega}) = 1 - n$ if and only if $\phi(\hat{\omega}) = \omega$, and hence equivalent to $\phi(\hat{z}) = z$ for every $z \in \Delta_n^{\perp}$.

Thus the hypotheses imply that $\phi(\hat{x}) = r\phi(\widehat{1_n}) + \phi(\hat{z}) = r1_n + z = x$. □

If ϕ is efficient, then automatically $\phi_n(\widehat{1_n}) = \phi_n(\sum sc_s) = n\phi_n(c_n) = 1$. Thus ϕ is natural if and only if $\sum_{s=1}^{n-1} \phi_n(\omega^s) = \phi_n(\hat{\omega}) = 1 - n$. Therefore, there's an $(n - 2)$-dimensional (affine) space of solutions that are both efficient and natural.

Example Shapley (Sh) and Banzhaf (B) values are natural.

Sh is efficient and $Sh_n(\omega^s) = -1$ for all $s : 1, \ldots, n-1$; whereas $B_n(\widehat{1}) = 1$ and $B_n(\omega^s) = -\frac{s\binom{n-1}{s}}{2^{n-2}}$ and thus $B(\hat{\omega}) = \sum_s B_n(\omega^s) = 1 - n$. □

Recall that given an arbitrary semi-value

$$\psi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S)\right],$$

for every $z \in \Delta_n^{\perp}$ and every $s : 1, \ldots, n-1$,

$$\psi(z^s) = b_s z$$

for certain numbers $b_s \in \mathbb{R}$. Moreover, by Theorem 4,

$$b_s = \binom{n-2}{s-1}[r_{s-1} + r_s].$$

It is not unusual to consider the marginal contribution weights r_S as the probability that coalition S forms. For this interpretation to be accurate, one requires that $r_S \geq 0$ and that $\sum_{S \not\ni i} r_S = 1$ for every $i \in N$. In the case of semi-values, these probabilities depend only on the cardinality s of each coalition. Thus, we may define the probability of choosing cardinality s as

$$p_s = \binom{n-1}{s} r_s.$$

Now, a simple computation tells you that

$$b_s = \frac{n-s}{n-1} p_{s-1} + \frac{s}{n-1} p_s.$$

In particular, $\sum_{s=1}^{n-1} b_s = \sum_{s=0}^{n-1} p_s = \sum_{S \not\ni i} r_S$, which gives rise to the next theorem.

Theorem 10 *Let $\psi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S)\right]$ be a given semi-value. Then ψ is natural if and only if $\sum_{S \not\ni i} r_s = 1$ for every $i \in N$.*

Proof

$$\psi_n(\hat{1}) = \sum_{S \not\ni n} r_s[|S \cup \{n\}| - |S|] = \sum_{S \not\ni n} r_s.$$

And, for every $z \in \Delta_n^{\perp}$,

$$\psi(\hat{z}) = \sum_{s=1}^{n-1} \psi(z^s) = \sum_{s=1}^{n-1} b_s z = z \sum_{S \not\ni i} r_s.$$

□

6 $G_s \oplus \mathbb{R}$ games

In this section we take up the problem of dividing a sum c in a "fair" way, among the players in a team game. One wants that each player's share depends on his performance, although we take into account only the teams' performance.

As an example, we could think that a certain company is organized into work teams, and that every project that arrives is assigned to one and only one of these teams. Individual members of the company are assumed to participate in all possible teams of cardinality s.

The problem then is how to divide, among the workers, a certain profit obtained from these projects considering only the performance of the teams.

6.1 General results

We look at pairs $(v, c) \in G_s \oplus \mathbb{R}$ and solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are linear and S_n-equivariant (S_n acting trivially on the $\mathbb{R}$ summand). Here $1 \leq s < n$.

Every $v \in G_s$ can be (uniquely) written as $v = x^s + w$, where $x \in \mathbb{R}^n$ and $w \in W_s$.

Lemma 6 *With the previous notation, the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ may be computed from v using the following formula*

$$x_i = \frac{\sum_S v(S)}{s\binom{n}{s}} + \frac{s(n-s)}{n\binom{n-2}{s-1}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s} \right].$$

Proof Express $x = k1_n + z$ where $1_n = (1, 1, \ldots, 1)$ and $z \in \Delta_n^{\perp}$. Then, $\langle v, {1_n}^s \rangle = k\langle {1_n}^s, {1_n}^s \rangle$ and therefore,

$$k = \frac{\langle v, {1_n}^s \rangle}{\langle {1_n}^s, {1_n}^s \rangle} = \frac{s \sum v(S)}{s^2 \sum 1} = \frac{\sum v(S)}{s\binom{n}{s}};$$

now, setting $f_i(v) = \sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}$ defines a symmetric linear map $f : G_s \to \mathbb{R}^n$. Notice that $f({1_n}^s) = 0$. Thus, the image of f is contained in $\Delta_n^{\perp}$. Schur's lemma implies then that, for every $z \in \Delta_n^{\perp}$, $f(z^s) = \lambda z$ for some λ. Computing $f(z_0^s)$ for $z_0 = (1, -1, 0, 0, \ldots, 0)$, we see that $\lambda = \frac{n}{s(n-s)}\binom{n-2}{s-1}$ □

Remark 7 Using Shapley's value one can give an alternate (equivalent) description of the x^s component of an arbitrary team game $v \in G_s$. Namely, if $v = k1_n{}^s + z^s + w$ is the decomposition of v with

$$k = \frac{\langle v, (1_n)^s \rangle}{\langle (1_n)^s, (1_n)^s \rangle} = \frac{\sum_{S \subset N} v(S)}{s \binom{n}{s}}$$

and $z \in \Delta_n^{\perp}$, then

$$Sh(v) = Sh(z^s) = \frac{1}{n-1} z,$$

since $Sh(w) = 0$ by Schur and $Sh(1_n{}^s) = 0, s < n$, because of efficiency.

Therefore,

$$v = [k(1_n) + (n-1)Sh(v)]^s + w \in (\mathbb{R}^n)^s \oplus W_s$$

is the decomposition of any team game $v \in G_s$.

For $x \in \mathbb{R}^n$ define

$$x(N) := \sum_{i=1}^{n} x_i .$$

Theorem 11 *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is 3-dimensional. Moreover, their general expression is given by*

$$\phi(x^s + w, c) = \left[\alpha \frac{x(N)}{n} + \gamma c \right] 1_n + \beta (x - \frac{x(N)}{n} 1_n) \tag{3}$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$. Equivalently,

$$\phi_i(v, c) = Ac + B \sum_{S \ni i} v(S) - C \sum_{S \not\ni i} v(S)$$

for every $(v, c) \in G_s \oplus \mathbb{R}, i : 1, \ldots, n$, for arbitrary $A, B, C \in \mathbb{R}$.

Proof $x \in \mathbb{R}^n$ is decomposed as

$$x = \frac{x \cdot 1_n}{n} 1_n + \left(x - \frac{x \cdot 1_n}{n} 1_n \right).$$

Thus,

$$\phi(x^s + w, c) = \phi(x^s + w, 0) + c\phi(0, 1) = \phi(x^s, 0) + c\phi(0, 1),$$

where we have used the fact (by Schur's Lemma) that $\phi(w, 0) = 0$ for every $w \in W_s$.

Schur's Lemma also tells us that $\phi(0, 1) = \gamma 1_n$ for some $\gamma \in \mathbb{R}$, $\phi({1_n}^s, 0) = \alpha 1_n$ for some $\alpha \in \mathbb{R}$, and $\phi(\left(x - \frac{x \cdot 1_n}{n} 1_n\right)^s, 0) = \beta \left(x - \frac{x \cdot 1_n}{n} 1_n\right)$.

The result follows. □

The solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is *efficient* if

$$\phi(v, c) \cdot 1_n = c.$$

Theorem 12 *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are also efficient is 1-dimensional. Their general expression is given by*

$$\phi(x^s + w, c) = \frac{c}{n} 1_n + \beta(x - \frac{x(N)}{n} 1_n)$$

for arbitrary $\beta \in \mathbb{R}$.

Proof Since $\phi(1_n, 0) \cdot 1_n = 0$ implies (by Schur) $\phi(1_n, 0) = 0$, then $\alpha = 0$ in formula (1).

Now, $\phi(0, 1) \cdot 1_n = 1$ implies $\phi(0, 1) = \frac{1_n}{n}$, thus $\gamma = \frac{1}{n}$ and we are done (since for $z \in \Delta_n^{\perp}$ the efficiency property is always satisfied for $(z^s, 0)$). □

An equivalent formulation to the above is that every linear symmetric efficient solution is of the form

$$\phi_i(v, c) = \frac{c}{n} + \lambda \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $\lambda \in \mathbb{R}$.

The solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is *natural* if $\phi(x^s, x \cdot 1_n) = x$, for every x.

Theorem 13 *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are also natural is 1-dimensional. Their general expression is given by*

$$\begin{aligned} \phi(x^s + w, c) &= \left[\alpha \frac{x(N)}{n} + \frac{1 - \alpha}{n} c \right] 1_n + \left(x - \frac{x(N)}{n} 1_n \right) \\ &= x + \frac{(1 - \alpha)}{n} (c - x \cdot 1_n) 1_n \end{aligned}$$

for arbitrary $\alpha \in \mathbb{R}$.

Proof For every $z \in \Delta_n^{\perp}$, $\phi(z^s, 0) = z$, implies that $\beta = 1$ in formula (1).

Also, $1_n = \phi((1_n)^s, n) = (\alpha + n\gamma) 1_n$, from which the result follows. □

Corollary 4 *There exists a unique solution $\psi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ which is linear, symmetric, efficient and natural. It is given by*

$$\psi(x^s + w, c) = x + \frac{1}{n} (c - x \cdot 1_n) 1_n.$$

Theorem 14 *The formula for the unique linear, symmetric, efficient and natural solution, ψ, on $G_s \oplus \mathbb{R}$ takes the form:*

$$\psi(v,c) = \frac{c}{n}1_n + (n-1)Sh(v).$$

Proof $\psi(v,c) = \psi(x^s + w, c) = x + \frac{1}{n}(c - x \cdot 1_n)1_n$ where $x = a1_n + (n-1)Sh(v)$. $x \cdot 1_n = na$, and hence

$$\psi(v,c) = x + \frac{1}{n}(c - x \cdot 1_n)1_n = a1_n + (n-1)Sh(v) + \frac{1}{n}(c - na)1_n.$$

□

Remark 8 This unique solution can also be written as

$$\psi_i(v,c) = \frac{c}{n} + \frac{n-1}{\binom{n}{s}}\left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}\right]$$

for every $(v,c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$.

Example When a traveler purchases a Europass he can choose to visit a subset of s countries (where $s = 3, 4$ or 5) from a set N of European countries, to travel for a certain fix number of days d. We could think of a game (v,c) where $v \in G_s$ is such that $v(S) =$ the number of travelers (in a particular summer, say) that chose the set S of countries for his pass, and where c is the total amount collected, at the end of the period, from Europasses purchased.

Then a solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^N$ will assign to (v,c) the vector $\phi(v,c)$ where $\phi_j(v,c)$ is the amount that corresponds to country j.

6.2 Bankruptcy problem

Recall that G_1 consists of games that can only take non zero values on cardinality one subsets of N. Thus, it is naturally identified with $\mathbb{R}^n : x(\{i\}) = x_i$.

Bankruptcy problems were formally introduced by O'Neill (1982), see also the survey article Thomson (2003) for a more recent and complete account of this topic.

A bankruptcy problem is an element $(x,c) \in G_1 \oplus \mathbb{R}$ such that

$$x(N) = \sum_{i=1}^{n} x_i \geq c.$$

We interpret x_i as the amount that the ith creditor demands, whereas c is the liquidation value or estate to be divided among the creditors.

All the results of the previous section apply to bankruptcy problems. Let us summarize them together with an interpretation.

- There is a 3-dimensional space of linear, symmetric, solutions $\phi : G_1 \oplus \mathbb{R} \to \mathbb{R}^n$ for the bankruptcy problem. The general expression for such solutions is given by

$$\phi_i(x, c) = \alpha c + \beta x_i + \gamma x(N), \quad i : 1, \ldots, n;$$

  for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$.
- Efficiency (i.e., $\phi(x, c) \cdot 1_n = c$) means that all of the liquidation value, c, is divided among all creditors. There is a 1-dimensional space of linear, symmetric and efficient solutions given by

$$\phi(x, c) = \frac{c}{n} 1_n + \lambda \left(x - \frac{x(N)}{n} 1_n \right)$$

  for arbitrary $\lambda \in \mathbb{R}$. Equivalently,

$$\phi_i(x, c) = \frac{c}{n} + \lambda \left(x_i - \frac{x(N)}{n} \right), \quad i : 1, \ldots, n;$$

  for arbitrary $\lambda \in \mathbb{R}$.
- Naturalness (i.e., $\phi(x, x \cdot 1_n) = x$) says that if the total amount claimed by the creditors equals the liquidation value, then each creditor receives what he claims. There is a 1-dimensional subspace of natural solutions (that are linear and symmetric) to the bankruptcy problem. They are given by

$$\phi(x, c) = x + \frac{(1 - \alpha)}{n} (c - x \cdot 1_n) 1_n$$

  for arbitrary $\alpha \in \mathbb{R}$.

Finally,

Corollary 5 *There exists a unique bankruptcy solution which is linear, symmetric, efficient and natural. It is given by*

$$\phi(x, c) = x + \frac{1}{n} (c - x \cdot 1_n) 1_n;$$

or equivalently, by

$$\phi_i(x, c) = \frac{c}{n} + x_i - \frac{x(N)}{n}, \quad i : 1, \ldots, n.$$

Appendix: Explaining the change of ranking after retirement of a player

Given a team game v of size s its players are ranked using $I(v)$. Likewise, if a player - say n- drops out, one can easily compute the ranking of the remaining players applying again I to the restricted game Rv. This procedure gives no clue on what the outcome

will be for Rv, nor it can explain a posteriori why some of the players in $N\backslash\{n\}$ might have switched places in the ranking after n is deleted.

Consider the following: there exist team games $w \in W_s \subset G_s^N$ that when restricted to the first $n-1$ players give games outside of $W_s^{N\backslash\{n\}}$. This means that although all players are equally ranked for w (since $I(w) = 0$), they are non-trivially ranked for the restricted game Rw. For example, consider the following team game w in 4 players where teams consist of 2 individuals, i.e., $w \in G_2^{1,2,3,4}$, and its restricted team game $Rw \in G_2^{1,2,3}$:

$$\begin{array}{l} w(\{1,2\}) = -2 = Rw(\{1,2\}) \\ w(\{1,3\}) = \ 0 \ = Rw(\{1,3\}) \\ w(\{2,3\}) = \ 2 \ = Rw(\{2,3\}) \\ w(\{1,4\}) = \ 2 \\ w(\{2,4\}) = \ 0 \\ w(\{3,4\}) = -2\ . \end{array}$$

w is in W_2, since, for every player i, $\sum_{S\ni i} w(S) = 0$. Thus $I(w) = (0,0,0,0)$, and all 4 players are equally ranked. Nevertheless, once player 4 drops out, we have, for the restricted team game Rw,

$$I(Rw) = (-2, 0, 2)$$

which ranks $1 \prec 2 \prec 3$.

A more dramatic example is given in 4.3, and others could be manufactured using Theorem 7.

It is the purpose of this -somewhat technical- appendix to explain in detail the reasons behind such behavior.

Decomposition of G under H_n:

Definition 1 H_n is the subgroup of S_n that fixes n. That is

$$H_n = \{\theta \in S_n \mid \theta(n) = n\}.$$

H_n is isomorphic to S_{n-1}.

Every symmetric solution $\phi : G \to \mathbb{R}^n$ is uniquely determined by its last coordinate $\phi_n : G \to \mathbb{R}$; for every $v \in G$ the value of any other coordinate, say ϕ_j, can be determined via $\phi_j(v) = \phi_n((jn) \cdot v)$, where (jn) denotes the transposition interchanging j with n and fixing every other player.

The function ϕ_n is no longer S_n-symmetric, but satisfies another kind of symmetry: it is H_n-invariant, that is

$$\phi_n(\theta \cdot v) = \phi_n(v)$$

for every $v \in G$ and every $\theta \in H_n$.

Thus, following the principles of representation theory the natural thing to do next is to decompose G into invariant subspaces for the H_n-action. Of these, the only ones that will not be in the common kernel of all ϕ_n's will be the trivial ones.

Since $H_n \subset S_n$, the invariant subspaces for S_n are still invariant for H_n. Thus we need to further decompose each $G_s = C_s \oplus U_s \oplus W_s$ under H_n. The statement is the following

Theorem 15 (H_n-decomposition of G_s)

- *C_s, being a trivial subspace for S_n, remains trivial under H_n.*
- *$U_s \simeq \mathbb{R}^{n-1}$ and thus splits as a sum of a 1-dimensional trivial subspace and its orthogonal complement (which is isomorphic to $\Delta_{n-1}^{\perp}$). Let $\omega = (1, 1, \ldots, 1, 1 - n) \in \Delta_n^{\perp}$, then the trivial summand is the span of ω. Thus*

$$U_s = \mathbb{R}\omega^s \oplus \{z^s \mid z \in \Delta_n^{\perp} \text{ and } z_n = 0\}.$$

- *W_s contains no trivial subspaces for the action of H_n.*

Therefore,

$$G_s = \mathbb{R}c_s \oplus \mathbb{R}\omega^s \oplus V_s,$$

where V_s contains no trivial summands.

Let $V = \bigoplus V_s$, $T = \bigoplus \mathbb{R}\omega^s$. Then

$$G = C \oplus T \oplus V. \tag{4}$$

Notice that $C \oplus T$ is the largest trivial subspace inside G (for the H_n-action) and that V is precisely the common kernel of the last coordinates of all linear symmetric solutions.

Corollary 6 *Let $\phi : G \to \mathbb{R}^n$ be a linear symmetric solution. Then*

(1) *$\phi_n(V_s) = 0$ for every s.*
(2) *ϕ is uniquely determined by the $2n-1$ numbers $\phi_n(c_1), \ldots, \phi_n(c_n), \phi_n(\omega^1), \ldots, \phi_n(\omega^{n-1})$.*

Retiring a player. Here we explain where the contributions come to the changes in ranking of remaining players once a player (say n) has dropped out of the game. We will require intensive use of some of the representation ideas presented in this appendix.

If $X_s = \{S \subset N \mid |S| = s\}$, then $G_s = G_s^N = \mathbb{R}[X_s] = \{v : X_s \to \mathbb{R}\}$.

Note that $H_n (\simeq S_{n-1})$ acts on X_s and that this action has exactly two orbits, $X_s = \emptyset_1 \cup \emptyset_2$. More precisely:

$$X_s = \{S \in X_s \mid S \not\ni n\} \cup \{S \in X_s \mid S \ni n\} \simeq X_s^{N\setminus\{n\}} \cup X_{s-1}^{N\setminus\{n\}}.$$

Thus, this decomposition of X_s into orbits, corresponds to the following decomposition of G_s under H_n:

$$\begin{aligned} G_s^N &= \mathbb{R}[X_s] = \mathbb{R}[\emptyset_1] \oplus \mathbb{R}[\emptyset_2] \\ &= \{v \in G_s^N \mid v(S) = 0 \text{ if } S \ni n\} \oplus \{v \in G_s^N \mid v(S) = 0 \text{ if } S \not\ni n\} \end{aligned}$$

and $G_s^N \simeq G_s^{N\setminus\{n\}} \oplus G_{s-1}^{N\setminus\{n\}}$ as S_{n-1} representations (because the first summand in the above decomposition of G_s^N can be identified with team games of s players in $\{1, 2, \ldots, n-1\}$; the second summand are team games in $s-1$ players in $\{1, 2, \ldots, n-1\}$, since n is already chosen).

Remark 9 Observe that there is an S_n-isomorphism between G_s^N and G_{n-s}^N, given by the equivariant bijection $X_s \to X_{n-s}$ defined by sending S to its complement $N\setminus S$.

So, for $s > 1$, the decomposition found in Hernández-Lamoneda et al. (2007) gives

$$G_s^N \simeq G_s^{N\setminus\{n\}} \oplus G_{s-1}^{N\setminus\{n\}} \simeq 1 \oplus \Delta_{n-1}^{\perp} \oplus W_s^{N\setminus\{n\}} \oplus 1 \oplus \Delta_{n-1}^{\perp} \oplus W_{s-1}^{N\setminus\{n\}},$$

where 1 stands for a 1-dimensional trivial representation, $\Delta_{n-1}^{\perp}$ the standard representation of S_{n-1} and the W_i do not contain copies of 1 nor of $\Delta_{n-1}^{\perp}$; whereas

$$G_1^N \simeq 1 \oplus \Delta_{n-1}^{\perp} \oplus 1.$$

This says that G_s^N contains exactly two copies of $\Delta_{n-1}^{\perp}$ when $n-2 \geq s \geq 2$ and only one when $s = 1$ or $s = n-1$. This will be of essential importance when computing the index of a team game once a player has dropped out.

We proceed now to identify each of these summands. First, we take care of the case $s = 1$.

Lemma 7

$$G_1^N = T_1 \oplus V_1$$

where T_1 is the 2-dimensional trivial H_n-representation spanned by c_1 and ω^1, and where $V_1 = \{z^1 \mid z \in \Delta_n^{\perp} and z_n = 0\} \simeq \Delta_{n-1}^{\perp}$. The decomposition is orthogonal.

Proof $G_1^N = \mathbb{R}[X_1] \simeq \mathbb{R}^n$, thus under S_n decomposes as $G_1^N = \langle c_1 \rangle \oplus U_1$, and $U_1 = \langle \omega^1 \rangle \oplus V_1$ under H_n (see Hernández-Lamoneda et al. (2007) for details). □

Similarly,

$$G_{n-1}^N = \langle c_{n-1}, \omega^{n-1} \rangle \oplus \{z^{n-1} \mid z \in \Delta_n^{\perp} \text{ and } z_n = 0\}.$$

For every $y \in \mathbb{R}^n$ and every $s : 2, \ldots, n-2$, define a game $\widetilde{y^s}$ by

$$\widetilde{y^s} = \begin{cases} y^s(S) & S \not\ni n \\ \left(1 - \frac{n-2}{s-1}\right) y^s(S) & S \ni n \end{cases}.$$

Lemma 8 *For $2 \leq s \leq n-2$, we have an orthogonal decomposition*

$$G_s^N = T_s \oplus V_s \oplus A_s \oplus B_s.$$

- *T_s is the 2-dimensional trivial representation spanned by c_s and ω^s,*
- *$V_s = \{z^s \mid z \in \Delta_n^\perp \text{ and } z_n = 0\} \simeq \Delta_{n-1}^\perp$,*
- *$A_s = \{\widetilde{y^s} \mid y \in \Delta_n^\perp \text{ and } y_n = 0\} \simeq \Delta_{n-1}^\perp$, and*
- *$B_s \subset W_s$ is the orthogonal complement to A_s inside W_s and thus contains no copies of either the trivial representation nor of $\Delta_{n-1}^\perp$.*

Proof We know that up to isomorphism

$$G_s^N \simeq 1 \oplus \Delta_{n-1}^\perp \oplus W_s^{N\setminus\{n\}} \oplus 1 \oplus \Delta_{n-1}^\perp \oplus W_{s-1}^{N\setminus\{n\}}$$

where, neither $W_s^{N\setminus\{n\}}$ nor $W_{s-1}^{N\setminus\{n\}}$ contain any summands isomorphic to 1 or $\Delta_{n-1}^\perp$.

So, one needs only to check directly that T_s is a trivial 2-dimensional representation (which is clear), and that V_s and A_s are isomorphic to $\Delta_{n-1}^\perp$. This is easy noticing that the maps $y \mapsto y^s$ and $y \mapsto \widetilde{y^s}$ are equivariant under H_n.

The only thing left to verify is that V_s and A_s are orthogonal to each other (this will imply that $A_s \subset W_s$). Now, define on $\Delta_{n-1}^\perp$ the following inner product: for $z, y \in \Delta_{n-1}^\perp$ set $\langle\langle z, y\rangle\rangle = \langle z^s, \widetilde{y^s}\rangle$. Is easy to check that $\langle\langle , \rangle\rangle$ is an H_n-invariant inner product, and since $\Delta_{n-1}^\perp$ is irreducible, it is a multiple of the standard inner product on $\mathbb{R}^n$, i.e., there exists an $r \in \mathbb{R}$ such that for every $z, y \in \Delta_{n-1}^\perp$

$$\langle z^s, \widetilde{y^s}\rangle = rz \cdot y.$$

We want to show that $r = 0$.

Let $z = (1, -1, 0, 0, \ldots, 0)$, and $k = \frac{n-2}{s-1}$. Then,

$$\begin{aligned} 2r = rz \cdot z = \langle z^s, \widetilde{z^s}\rangle &= \sum_{\substack{|S|=s \\ S\ni 1 \\ S\not\ni 2 \\ S\not\ni n}} 1 + \sum_{\substack{|S|=s \\ S\ni 1 \\ S\not\ni 2 \\ S\ni n}} (1-k) + \sum_{\substack{|S|=s \\ S\not\ni 1 \\ S\ni 2 \\ S\not\ni n}} 1 + \sum_{\substack{|S|=s \\ S\not\ni 1 \\ S\ni 2 \\ S\ni n}} (1-k) \\ &= 2\binom{n-3}{s-1} + 2(1-k)\binom{n-3}{s-2} \\ &= 2\left[\frac{(n-3)!}{(s-1)!(n-s-2)!} - \frac{n-s-1}{s-1} \cdot \frac{(n-3)!}{(s-2)!(n-s-1)!}\right] = 0. \end{aligned}$$

□

Let $v \in G_s^N$ be an arbitrary game. Then, if player n drops out, we get a new game $R(v) \in G_s^{N\setminus\{n\}}$ by restriction: for $S \subset \{1, 2, \ldots, n-1\}$ let $R(v)(S) := v(S)$. Moreover, this restriction map $R : G_s^N \to G_s^{N\setminus\{n\}}$ is not only linear, but is an S_{n-1}-equivariant map as well ($R(\theta v)(S) = \theta v(S) = v(\theta^{-1}S) = Rv(\theta^{-1}S) = (\theta R(v))(S)$). Hence, we may use the decomposition found above to aid us in computing $R(v)$.

Let $v \in G_s^N$ arbitrary. Then, with respect to the S_n-decomposition of G_s^N, v can be written as $v = rc_s + u^s + w$ with $u \in \Delta_n^\perp$ and $w \in W_s$. This decomposition refines to that under H_n as follows. First, $u = \frac{u_n}{1-n}\omega + z$, where $z \in \Delta_n^\perp$ and $z_n = 0$. Second, $w = \widetilde{y^s} + w'$, for some $y \in \Delta_n^\perp$, $y_n = 0$, and $w' \in B_s$. Thus,

$$v = rc_s + u^s + w = (rc_s + \frac{u_n}{1-n}\omega^s) + z^s + \widetilde{y^s} + w'.$$

Theorem 16 *Let $v \in G_s^N$ as above. Then,*

$$R(v) = \left(r + \frac{su_n}{1-n}\right)c_s + (Z^s + Y^s) + R(w') \in C_s^{N\setminus\{n\}} \oplus U_s^{N\setminus\{n\}} \oplus W_s^{N\setminus\{n\}}$$

where $Z = (z_1, z_2, \ldots, z_{n-1})$ if $z = (z_1, z_2, \ldots, z_{n-1}, 0)$.

Proof First, it is clear that $R(\omega^s) = sc_s$, that $R(z^s) = Z^s$ and that $R(\widetilde{y^s}) = R(y^s) = Y^s$. Finally, by Schur's Lemma $R(w') \in W_s^{N\setminus\{n\}}$. □

Remark 10 The games in A_s are exactly those games v (in n players) where every solution on n players vanishes, but for which there exists a solution in $n-1$ players that takes a non-zero value on their restriction $R(v)$.

Corollary 7 *If $v \in G_s^{(n)}$ is expressed as above, then its index ranking is u, whereas the index ranking of its restriction (after player n has dropped) $R(v)$ is given by $Z+Y$.*

Remark 11 Observe that Z gives the same ranking to $1, \ldots, n-1$ as u; the difference in ranking for the restricted game all comes from Y.

Example Let $n = 4, s = 2$, and consider the game $v \in G_2^{\{1,2,3,4\}}$ given by $v = u^2 - 2\widetilde{y^2}$ for $u = (-3, -1, 1, 3)$ and $y = (-2, 0, 2, 0)$; then $v = -\omega^2 + y^2 - 2\widetilde{y^2}$. Thus, the ranking of the players for game v is given by $u = (-3, -1, 1, 3)$ and so player 4 > player 3 > player 2 > player 1.

Now, for the restriction of v, once player 4 has dropped out, the new ranking is given by $Y - 2Y = -Y = (2, 0, -2)$, i.e., exactly the reverse ranking!

We list the values of v on the 6 coalitions of two players, and the values of $R(v)$ on the remaining coalitions:

$$\begin{aligned}
v(\{1,2\}) &= \ 0 \ = R(v)(\{1,2\})\\
v(\{1,3\}) &= -2 = R(v)(\{1,3\})\\
v(\{2,3\}) &= -4 = R(v)(\{2,3\})\\
v(\{1,4\}) &= -4\\
v(\{2,4\}) &= \ 2\\
v(\{3,4\}) &= \ 8 \ .
\end{aligned}$$

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Cooperative Games with Minimum Payoffs

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Abstract

In this paper we study solutions for several types of cooperative games, satisfying a condition in which every player receive at least a minimum payoff. In particular, we provide minimum-payoff solutions for games in characteristic function form, games with coalitional structures, games in partition function form and games with cooperation graphs.

Mathematics Subject Classification: 91A12, 91A70, 91A80

Keywords: Cooperative games, axiomatic solutions

1 Introduction

Traditionally, cooperative game theory models problems where benefits are allocated. In those situations in which certain benefits are shared sometimes it is desirable to have a minimum payment (e.g., minimum wages). This would help to reduce the income inequality.

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Some of the most important works where the problem of benefit allocation has been modelled (although without including a "minimum payment" option) are: The first paper that proposed a value concept for games in characteristic function form was Shapley (1953), then Aumann and Drèze (1974) gave a solution for games with coalition structures; and Myerson (1977) apply the axiomatic approach to find a value for games in partition function form, and a value for cooperative games with cooperation structures represented as graphs.

In this article we characterize possible extensions of these classical values that include a notion of minimum payoff. For some of these values we have had to modify certain of the usual axioms, e.g., the null player axiom.

The paper is organized as follows. We first introduce the condition of a minimum payoff for a solution for games in characteristic function form and we show that a convex combination between the Shapley value and the egalitarian solution satisfies such condition. An analysis of a minimum-payoff solution for games with coalition structures is discussed in section 3. In section 4 we provide a characterization of a minimum-payoff solution for games in partition function form. Finally, we use the ideas of Myerson (1977) to give a framework within which we can discuss the condition of minimum payoff for cooperation graphs.

2 Games in characteristic function form

By an *n-person game* in characteristic function form (or a TU cooperative game), we mean a pair (N, v), where $N = \{1, ..., n\}$ is a finite set of players and v is a function $v : 2^N \to \mathbb{R}$ with the property that $v(\varnothing) = 0$ (2^N denotes the set of subsets of N). Since N is fixed, the function v itself will also be called a game in characteristic function form. We usually refer to subsets S of N as *coalitions* and to the number $v(S)$ as the *worth* of S. A game v is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$, such that $S \cap T = \varnothing$, and it is subadditive if the inequality holds in the other direction. There are several interpretations for such a v, they depend on what people want to model. For instance, if the game is superadditive, $v(S)$ could mean the maximal amount the players in S can get if they decide to play together. On the other hand, if the game is subadditive, $v(S)$ usually means the joint cost that players in S have to pay to get a service if they together hired the service. Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance $n = |N|$, $s = |S|$, $t = |T|$, and so on.

We denote by G^N the set of all games in characteristic function form with a fixed set of players N, i.e., $G^N = \{v : 2^N \to \mathbb{R} \mid v(\varnothing) = 0\}$. A *solution* is a function $\varphi : G^N \to \mathbb{R}^n$, which we can interpret as a rule to divide the common gain or cost among the players in N.

Given $v, w \in G^N$ and $\lambda \in \mathbb{R}$, we define the sum and the product $v+w$ and λv in G^N in the usual form, i.e.,

$$(v+w)(S) = v(S) + w(S) \quad \text{and} \quad (\lambda v)(S) = \lambda v(S)$$

respectively. G^N is a vector space with these operations.

Next, we define some axioms which are common in cooperative game theory.

Axiom 1 (Additivity) *The solution φ is additive if $\varphi(v+w) = \varphi(v)+\varphi(w)$, for all $v, w \in G^N$.*

The group of permutations of the set of players, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$, acts on $2^N \backslash \{\varnothing\}$ and on the space of payoff vectors, $\mathbb{R}^n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta^{-1}(1)}, x_{\theta^{-1}(2)}, ..., x_{\theta^{-1}(n)})$$

For every $\theta \in S_n$ and $v \in G^N$ we define the game $\theta \cdot v$ as follows

$$\theta \cdot v(S) = v(\theta^{-1}S)$$

Axiom 2 (Symmetry) *The solution φ is said to be symmetric if and only if $\varphi(\theta \cdot v) = \theta \cdot \varphi(v)$ for every $\theta \in S_n$ and $v \in G^N$.*

Axiom 3 (Efficiency) *The solution φ is efficient if*

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

The player i is said to be a null player in the game v if $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N$. The nullity axiom requests that every null player in v gets a zero payoff.

Axiom 4 (Nullity) *If the player i is a null player in v then $\varphi_i(v) = 0$.*

Shapley (1953) developed a unique value from these four simple axioms.

Theorem 5 (Shapley, 1953) *There exists a unique solution $Sh : G^N \to \mathbb{R}^n$ that satisfies additivity, symmetry, efficiency and nullity axioms. Furthermore it is given by*

$$Sh_i(v) = \sum_{\substack{S \subset N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$$

Definition 6 *For a given $\alpha \in \left(0, \frac{1}{n}\right]$, the solution φ is said to satisfy α-minimum payoff in $H \subseteq G^N$ if and only if*

$$\varphi_i(v) \geq \alpha v(N) \tag{1}$$

for all $i \in N$ and $v \in H$.

Remark 7 *If $H \subseteq G^N$ is such that $v, -v \in H$ for some $v \neq 0$, then for all $\alpha \in \left(0, \frac{1}{n}\right]$, there exists a unique solution satisfying additivity, symmetry and efficiency axioms, and α-minimum payoff in H. Furthermore, it is the egalitarian solution*

$$\varphi_i(v) = \frac{v(N)}{n}$$

for all $i \in N$ and $v \in H$.

Let

$$G_+^N = \{v \in G^N \mid v \text{ superadditive}, v(\{i\}) \geq 0 \ \forall i \in N\}$$

be the cone of non-negative superadditive games.

Lemma 8 *If $v \in G_+^N$ then $Sh(v) \geq 0$.*

Proof. Since v is superadditive, for each $S \subseteq N \backslash \{i\}$ it holds,

$$v(S \cup \{i\}) \geq v(S) + v(\{i\})$$

and so,

$$Sh_i(v) = \sum_{\substack{S \subset N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right) \geq \sum_{\substack{S \subset N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} v(\{i\}) = v(\{i\})$$

since $\sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} = 1$. ■

It is well known that the collection $\{u_T \mid \varnothing \neq T \subseteq N\}$, where the games u_T are defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

form a basis for G^N.

Lemma 9 *Let φ be a symmetric solution. If T, $R \subseteq N$ are such that $|T| = |R|$, then it holds*

a) $\varphi_i(u_T) = \varphi_j(u_R)$ *for all* $i \in T$ *and* $j \in R$.

b) $\varphi_i(u_T) = \varphi_j(u_R)$ *for all* $i \notin T$ *and* $j \notin R$.

Proof. Let $T, R \subseteq N$ be such that $|T| = |R|$ and let $\theta \in S_n$ be such that $\theta(R) = T$. Notice that $\theta u_R = u_T$ since

$$(\theta \cdot u_R)(S) = u_R(\theta^{-1}(S)) = \begin{cases} 1 & \text{if } \theta^{-1}(S) \supseteq R \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } S \supseteq \theta(R) \\ 0 & \text{otherwise} \end{cases} = u_T(S)$$

And so, if $\theta(j) = i$ for $j \in R$, then

$$\varphi_i(u_T) = \varphi_i(\theta \cdot u_R) = \theta\varphi_i(u_R) = \varphi_j(u_R)$$

In a similar way, we can conclude that $\varphi_i(u_T) = \varphi_j(u_R)$ if $i \notin T$ and $j \notin R$. ■

Axiom 10 (Quasi-nullity) *If* $i \in N$ *is a null player in* $v \in G^N$. *The solution* φ *is said to be quasi-null with respect to* $\alpha \in \left(0, \frac{1}{n}\right]$ *if and only if* $\varphi_i(v) = \alpha v(N)$.

Theorem 11 *For all* $\alpha \in \left(0, \frac{1}{n}\right]$, *there exists a unique solution satisfying additivity, symmetry, efficiency and quasi-nullity axioms, and it is the value* φ *defined for all* $v \in G^N$ *and all* $i \in N$, *by*

$$\varphi_i(v) = (1 - \alpha n)Sh_i(v) + \alpha v(N) \tag{3}$$

Proof. Let φ be a solution satisfying such axioms. As usually, let

$$\delta_S = v(S) - \sum_{R \subsetneq S} \delta_R$$

for $S \subseteq N$ with $\delta_\varnothing = 0$, i.e.,

$$v(S) = \sum_{R \subseteq S} \delta_R = \sum_{R \subseteq N} \delta_R u_R(S)$$

And so,

$$v = \sum_{R \subseteq N} \delta_R u_R$$

By the additivity of φ,

$$\varphi_i(v) = \sum_{R \subseteq N} \varphi_i(\delta_R u_R)$$

Notice that,

i) $u_R(N) = 1$.

ii) If $i \in N \backslash R$, then i is a null player in u_R.

From this observation, efficiency and quasi-nullity axioms, and Lemma 9; define

$$\varphi_i(\delta_R u_R) = \begin{cases} \delta_R \frac{1-\alpha(n-r)}{r} & \text{if } i \in R \\ \delta_R \alpha & \text{if } i \notin R \end{cases}$$

So,

$$\begin{aligned} \varphi_i(v) &= \sum_{\substack{R \subseteq N \\ R \ni i}} \varphi_i(\delta_R u_R) + \sum_{\substack{R \subseteq N \\ R \not\ni i}} \varphi_i(\delta_R u_R) \\ &= \sum_{\substack{R \subseteq N \\ R \ni i}} \delta_R \frac{1-\alpha(n-r)}{r} + \sum_{\substack{R \subseteq N \\ R \not\ni i}} \alpha \delta_R = \sum_{\substack{R \subseteq N \\ R \ni i}} \delta_R \frac{1-\alpha n}{r} + \sum_{\substack{R \subseteq N \\ R \ni i}} \alpha \delta_R + \sum_{\substack{R \subseteq N \\ R \not\ni i}} \alpha \delta_R \\ &= (1-\alpha n) \sum_{\substack{R \subseteq N \\ R \ni i}} \frac{\delta_R}{r} + \alpha \sum_{R \subseteq N} \delta_R = (1-\alpha n) Sh_i(v) + \alpha v(N) \end{aligned}$$

Finally, it is straightforward to check that the solution φ, given by (3) satisfies the four axioms of the theorem. ■

Corollary 12 *The solution defined for all $i \in N$ by*

$$\varphi_i(v) = (1-\alpha n) Sh_i(v) + \alpha v(N)$$

satisfies α-minimum payoff in G_+^N.

Proof. The result follows from Lemma 9. ■

Remark 13 *Given $v \in G^N$ and $\alpha \in \left(0, \frac{1}{n}\right]$, let*

$$E(v) = \left\{ x \in \mathbb{R}_+^n \mid \sum_{i \in N} x_i = v(N) \right\}$$

be the space of positive and efficient payoff vectors and let

$$P(v, \alpha) = \{ x \in E(v) \mid x_i \geq \alpha v(N) \ \forall i \in N \}$$

be the set of α-minimum payoffs. The solution we have characterized in Theorem 11 is a homothetic transformation $T_{c,r} : E(v) \to P(v, \alpha)$ with center $c = EG(v)$[4] and ratio $r = 1 - \alpha n$. So, it holds

$$\|\varphi(v) - EG(v)\| = (1-\alpha n) \|Sh(v) - EG(v)\|$$

whenever $Sh(v) \in E(v)$.

[4] $EG(v)$ denotes the egalitarian solution for v, i.e., $EG(v) = \left(\frac{v(N)}{n}, ..., \frac{v(N)}{n}\right)$.

Example 14 *Let $N = \{1, 2, 3\}$, and consider the characteristic function game v, where*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	6	8	0	16	8	12	25

We ask every player to get at least a payoff of 5 units. The Shapley value of v is $Sh(v) = (9, 12, 4)$ and so, the α-minimum payoff solution for this game is $\varphi(v) = \left(\frac{43}{5}, \frac{49}{5}, \frac{33}{5}\right) = (8.6, 9.8, 6.6)$.

Remark 15 *Moving α from $\frac{1}{n}$ to 0 (thus reducing the minimum wages) we get a continuous transition from our solution approaching Shapley's value.*

3 Games with coalition structures

We turn next to cooperative games with coalition structures, where we derive a minimum-payoff solution by considering these games.

The study of coalition structures has been seriously explored in Aumann and Drèze (1974). The contribution by these authors represent the main effort at the development of a value for coalition structures.

A coalition structure $\mathfrak{B}$ is a finite partition $\mathfrak{B} = \{B_1, ..., B_m\}$ of the set of players (i.e., $\bigcup_{k=1}^{m} B_k = N$, $S_k \neq \varnothing\ \forall k$ and $B_k \cap B_j = \varnothing\ \forall k \neq j$), the generic element of which will be denoted B_k. The set of all coalition structures of N is denoted by PT. A game with coalition structure $\mathfrak{B}$ is a pair $(\mathfrak{B}, v) \in PT \times G^N$. Additionally, if $x \in \mathbb{R}^n$ and $S \subseteq N$, we write $x(S) = \sum_{i \in S} x_i$.

Given N and $\mathfrak{B}$, Aumann and Drèze (1974) proposed and characterized a "$\mathfrak{B}$-value", which is a function $\phi : PT \times G^N \to \mathbb{R}^n$ obeying the following conditions.

Axiom 16 (Relative efficiency) *For all k, $[\phi(\mathfrak{B}, v)](B_k) = v(B_k)$.*

Axiom 17 (Symmetry) *For all permutations θ of N under which $\mathfrak{B}$ is invariant*[5]*,*

$$[\phi(\mathfrak{B}, \theta \cdot v)](S) = [\phi(\mathfrak{B}, v)](\theta(S))$$

where $(\theta \cdot v)(S) = v(\theta(S))$.

Axiom 18 (Additivity) *$\phi(\mathfrak{B}, v+w) = \phi(\mathfrak{B}, v)+\phi(\mathfrak{B}, w)$ for all $(\mathfrak{B}, v), (\mathfrak{B}, w) \in PT \times G^N$.*

[5] A coalition structure $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$ is invariant under permutation θ if $\theta(B_k) = B_k$ for all k.

Axiom 19 (Nullity) *Let $(\mathfrak{B}, v) \in PT \times G^N$ and $i \in N$. If i is a null player in v, then $\phi_i(\mathfrak{B}, v) = 0$.*

Theorem 20 (Aumann and Drèze, 1974) *Fix N and $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$. Then there is a unique value ϕ satisfying relative efficiency, symmetry, additivity and nullity axioms, and it is given for all $k = 1, 2, ..., m$, and all $i \in B_k$, by*

$$\phi_i(\mathfrak{B}, v) = Sh_i(v|_{B_k})$$

The exposition of the notion of a minimum-payoff solution by means of coalition structures is the main task of this section.

Definition 21 *Let $(\mathfrak{B}, v) \in PT \times G^N$, where $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$, and a vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_k \in \left(0, \frac{1}{b_k}\right]$. We will say that a solution φ satisfies α-minimum payoff in $H \subseteq G^N$ if and only if*

$$\varphi_i(\mathfrak{B}, v) \geq \alpha_k v(B_k) \tag{4}$$

for all $k = 1, 2, ..., m$, $i \in B_k$ and $v \in H$.

In our search for a minimum-payoff solution for games with coalition structures, we shall relax the nullity axiom by requiring instead that every null player gets the minimum payoff.

Axiom 22 (Quasi-nullity) *Let $(\mathfrak{B}, v) \in PT \times G^N$ and $i \in B_k \in \mathfrak{B}$. If i is a null player in v, then the solution φ is said to be quasi-null with respect to $\alpha_k \in \left(0, \frac{1}{b_k}\right]$ if and only if $\varphi_i(\mathfrak{B}, v) = \alpha_k v(B_k)$.*

Theorem 23 *Fix $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_k \in \left(0, \frac{1}{b_k}\right]$. There exists a unique solution $\varphi : PT \times G^N \to \mathbb{R}^n$ satisfying relative efficiency, symmetry, additivity and quasi-nullity axioms, and is given for all $k = 1, 2, ..., m$, and all $i \in B_k$, by*

$$\varphi_i(\mathfrak{B}, v) = (1 - \alpha_k b_k)\phi_i(\mathfrak{B}, v) + \alpha_k v(B_k) \tag{5}$$

Proof. Fix $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_k \in \left(0, \frac{1}{b_k}\right]$. It is not too difficult to see that the solution given by (5) satisfy the four axioms.

We prove uniqueness. Let φ be a solution satisfying such axioms. For any $v \in G^N$, there exist $\delta_R \in \mathbb{R}$ such that $v = \sum_{\varnothing \neq R \subseteq N} \delta_R u_R$, where u_R is defined

as (2). Then, by additivity, $\varphi(v) = \sum_{\varnothing \neq R \subseteq N} \varphi(\delta_R u_R)$. We will show that $\varphi(v)$ is determined for all $v \in G^N$ and by the previous discussion, it is therefore sufficient to show that $\varphi(\delta_R u_R)$ is determined for all $R \subseteq N$. In this way, φ is unique for each u_R and so for v.

Notice that,

i) $u_R(B_k) = 1$ if $B_k \supseteq R$, where $B_k \in \mathfrak{B}$.

ii) If $i \in N \backslash R$, then i is a null player in u_R and assume $i \in B_k \in \mathfrak{B}$ for some k, therefore $\varphi_i(\delta_R u_R) = \delta_R \alpha_k$.

iii) If $i, j \in R$ and $i, j \in B_k \in \mathfrak{B}$ for some k, then by symmetry $\varphi_i(u_R) = \varphi_j(u_R)$.

Hence, by the relative efficiency axiom it follows that if $i \in B_k \in \mathfrak{B}$, then

$$\varphi_i(\delta_R u_R) = \begin{cases} \delta_R \frac{1-\alpha_k(n-r)}{r} & \text{if } i \in R \subseteq B_k \\ \delta_R \alpha_k & \text{if } i \notin R \subseteq B_k \end{cases}$$

This determines $\varphi_i(\delta_R u_R)$, and so completes the proof.

■

Corollary 24 *The solution*

$$\varphi_i(\mathfrak{B}, v) = (1 - \alpha_k b_k)\phi_i(\mathfrak{B}, v) + \alpha_k v(B_k)$$

satisfies α-minimum payoff in $G_+^{B_k}$, for all $k = 1, 2, ..., m$ and all $i \in B_k$.

Proof. If $v \in G_+^N$, then $v|_{B_k} \in G_+^{B_k}$ and by Lemma 9, $\phi_i(\mathfrak{B}, v) = Sh_i(v|_{B_k}) \geq 0$. ■

Example 25 *Let $N = \{1, 2, 3, 4, 5\}$ and $\mathfrak{B} = \{\{1, 3, 5\}, \{2, 4\}\}$. Consider the game $v \in G^N$ such that:*

S	$\{1\}$	$\{3\}$	$\{5\}$	$\{1,3\}$	$\{1,5\}$	$\{3,5\}$	$\{1,3,5\}$
$v(S)$	4	8	6	12	10	16	32

and

S	$\{2\}$	$\{4\}$	$\{2,4\}$
$v(S)$	0	6	10

Every player in $\{1, 3, 5\}$ are required to receive at least a payoff of 9 units, and every player in $\{2, 4\}$ at least a payoff of 4 units.

The Aumman-Drèze value for v is $\phi(v) = (8, 2, 13, 8, 11)$, which it does not satisfy the α-minimum payoff condition. So, the desirable payoff is $\varphi(v) = \left(\frac{41}{4}, \frac{22}{5}, \frac{353}{32}, \frac{28}{5}, \frac{343}{32}\right) = (10.25, 4.4, 11.031, 5.6, 10.718)$.

4 Games in partition function form

In this section we study a minimum-payoff solution for games in partition function form. Due to the richness of these games, several solutions have been given. The first paper that proposed a value concept for games in partition function form was Myerson (1977). This author proceeds axiomatically and proposes a natural extension of the well-known Shapley value. We follow here his exposition.

Given N, let $ECL = \{(S, Q) \mid S \in Q \in PT\}$ be the set of *embedded coalitions*, that is, the set of coalitions together with specifications as to how the other players are aligned.

Definition 26 *A mapping*

$$w : ECL \to \mathbb{R}$$

that assigns a real value, $w(S, Q)$, to each embedded coalition (S, Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by PG, i.e.,

$$PG = \{w \mid w : ECL \to \mathbb{R}\}$$

The value $w(S, Q)$ represents the payoff of coalition S, given the coalition structure Q forms. For any $w \in PG$ and any $Q \in PT$, we adopt the convention that $w(\varnothing, Q) = 0$.

Now, the group S_n acts on ECL in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S_1, \{S_1, S_2, ..., S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), ..., \theta(S_l)\})$$

We will need some additional notation. For any $Q, \overline{Q} \in PT$, define

$$Q \wedge \overline{Q} = \left\{S \cap \overline{S} \mid S \in Q, \overline{S} \in \overline{Q}, S \cap \overline{S} \neq \varnothing\right\}$$

Next, we define some axioms which we ask solutions on PG to satisfy.

Axiom 27 (Additivity) *The solution Φ is additive if $\Phi(w_1 + w_2) = \Phi(w_1) + \Phi(w_2)$, for all $w_1, w_2 \in PG$.*

Axiom 28 (Symmetry) *The solution Φ is said to be symmetric if and only if $\Phi_i(w) = \Phi_{\theta(i)}(\theta \cdot w)$ for every $\theta \in S_n$, every $w \in PG$ and every $i \in N$, where the game $\theta \cdot w$ is defined as*

$$(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$$

Given $w \in PG$ and $\overline{S} \in 2^N \backslash \{\varnothing\}$, we say that $\overline{S}$ is a *carrier* of w if and only if $w(S,Q) = w\left(S \cap \overline{S}, Q \wedge \{\overline{S}, N \backslash \overline{S}\}\right)$. The next axiom suggests that all available wealth should be divided among the members of a carrier.

Axiom 29 (Carrier) *For all $w \in PG$ and all $S \in 2^N \backslash \{\varnothing\}$, if S is a carrier of w, then*

$$\sum_{i \in S} \Phi_i(w) = w(N, \{N\})$$

Myerson (1977) shows that there is a unique value satisfying the first three axioms.

Theorem 30 (Myerson, 1977) *The solution $\Phi : PG \to \mathbb{R}^n$ given by*

$$\Phi_i(w) = \sum_{(S,Q) \in ECL} (-1)^{q-1}(q-1)! \cdot \left(\frac{1}{n} - \sum_{T \in Q \backslash \{S\}, i \notin T} \frac{1}{(q-1)(n-t)} \right) \cdot w(S,Q)$$

for each $i \in N$ and each $w \in PG$; is the unique solution satisfying additivity, symmetry and carrier axioms.

We also give a requirement of minimum payoff for this kind of games.

Definition 31 *For a given $\alpha \in \left(0, \frac{1}{n}\right]$, the solution φ is said to satisfy α-minimum payoff in $H \subseteq PG$ if and only if*

$$\varphi_i(w) \geq \alpha w(N, \{N\}) \tag{6}$$

for all $i \in N$ and $w \in H$.

Although players outside any carrier have no direct influence on the play since they contribute nothing to any coalition, the next axiom states that such players get the amount $\alpha w(N, \{N\})$.

Axiom 32 (Quasi-carrier) *For all $w \in PG$ and all $S \in 2^N \backslash \{\varnothing\}$, if S is a carrier of w, then*

$$\sum_{i \in S} \Phi_i(w) = (1 - \alpha(n-s)) w(N, \{N\})$$

Theorem 33 *For all $\alpha \in \left(0, \frac{1}{n}\right]$, there exists a unique solution satisfying additivity, symmetry and quasi-carrier axioms, and it is the value φ defined for all $w \in PG$ and all $i \in N$, by*

$$\varphi_i(w) = (1 - \alpha n)\Phi_i(w) + \alpha w(N, \{N\}) \tag{7}$$

Proof. Assume φ satisfies the axioms of additivity, symmetry and quasi-carrier; and let

$$\Phi_i(w) = \frac{1}{1-\alpha n}\left[\varphi_i(w) - \alpha w(N,\{N\})\right]$$

then Φ is the Myerson value since it satisfies the additivity, symmetry and carrier axioms. Indeed,

a) Additivity. For all $w_1, w_2 \in PG$ and all $i \in N$,

$$\begin{aligned}\Phi_i(w_1+w_2) &= \frac{1}{1-\alpha n}\left[\varphi_i(w_1+w_2) - \alpha\,(w_1+w_2)\,(N,\{N\})\right] \\ &\quad \frac{1}{1-\alpha n}\left[\varphi_i(w_1) - \alpha w_1(N,\{N\}) + \varphi_i(w_2) - \alpha\,(w_2)\,(N,\{N\})\right] \\ &= \Phi_i(w_1) + \Phi_i(w_2)\end{aligned}$$

b) Symmetry. For every $\theta \in S_n$, every $w \in PG$ and every $i \in N$,

$$\begin{aligned}\Phi_{\theta(i)}(\theta\cdot w) &= \frac{1}{1-\alpha n}\left[\varphi_{\theta(i)}(\theta\cdot w) - \alpha(\theta\cdot w)(N,\{N\})\right] \\ &= \frac{1}{1-\alpha n}\left[\varphi_i(w) - \alpha w(N,\{N\})\right] = \Phi_i(w)\end{aligned}$$

c) Carrier. If $S \in 2^N\backslash\{\varnothing\}$ is a carrier of $w \in PG$,then

$$\begin{aligned}\sum_{i\in S}\Phi_i(w) &= \frac{1}{1-\alpha n}\left[\sum_{i\in S}\varphi_i(w) - \alpha s w(N,\{N\})\right] \\ &= \frac{1}{1-\alpha n}\left[(1-\alpha(n-s))w(N,\{N\}) - \alpha s w(N,\{N\})\right] \\ &= w(N,\{N\})\end{aligned}$$

On the other hand, it is easy to verify that the solution φ, given by (7) satisfies the three properties of the theorem. ■

Corollary 34 *The solution defined for all $i \in N$ by*

$$\varphi_i(w) = (1-\alpha n)\Phi_i(w) + \alpha w(N,\{N\})$$

satisfies α-minimum payoff in $H = \{w \in PG \mid \Phi(w) \geq 0\}$.

Example 35 *Suppose* $N = \{1, 2, 3\}$ *and* $w \in PG$ *defined by*

Partition	*Worth*		
$\{1\}, \{2\}, \{3\}$	50	50	50
$\{1, 2\}, \{3\}$	120	40	
$\{1, 3\}, \{2\}$	125	45	
$\{2, 3\}, \{1\}$	130	10	
$\{1, 2, 3\}$	200		

where every player should have the opportunity to earn at least a decent payoff of 60 *units.*

The Myerson value for this game prescribes $\Phi(w) = \left(\frac{85}{2}, 80, \frac{155}{2}\right) = (42.5, 80, 77.5)$ *and the* α*-minimum payoff solution gives* $\varphi(w) = \left(\frac{257}{4}, 68, \frac{271}{4}\right) = (64.25, 68, 67.75)$.

5 Games with cooperation graphs

Myerson (1977) used the ideas from graph theory to provide a framework within which we can discuss a broad class of partial cooperation structures and study the question of how the outcome of a game should depend on which players cooperate with each other.

Following Myerson (1977), a graph on N is a set of unordered pairs of distinct members of N. We will refer to these unordered pairs as links, and we will denote the link between i and j by $i\colon j$ (so, $i\colon j = j\colon i$, since the link is an unordered pair). Let g^N be the complete graph of all links:

$$g^N = \{i\colon j \mid i, j \in N,\, i \neq j\}$$

Then let GR be the set of all graphs on N, so that $GR = \{g \mid g \subseteq g^N\}$.

Suppose $S \subseteq N$, $g \in GR$, $i \in N$ and $j \in N$ are given. Then we say that i and j are connected in S by g iff there is a path in g which goes from i to j and stays within S. That is, i and j are connected in S by g if $i = j$, or if there is some $k \geq 1$ and a sequence $(a_0, a_1, ..., a_k)$ such that $a_0 = i$, $a_k = j$, and $a_{r-1}\colon a_r \in g$ and $a_r \in S$ for all r from 1 to k.

Given $g \in GR$ and $S \subseteq N$, there is a unique partition of S (denoted as S/g) which groups players together iff they are connected in S by g:

$$S/g = \{\{i \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} \mid j \in S\}$$

Definition 36 *An allocation rule for* $v \in G^N$ *is any function* $Y : GR \to \mathbb{R}^n$ *such that*

$$\sum_{i \in S} Y_i(g) = v(S) \tag{8}$$

for every $g \in GR$ *and every* $S \in N/g$.

We use the symbol \ to denote removal of a member from a set. Thus

$$g\backslash i\colon j = \{a\colon b \mid a\colon b \in g, a\colon b \neq i\colon j\}$$

Definition 37 *An allocation rule is fair if and only if*

$$Y_i(g) - Y_i(g\backslash i\colon j) = Y_j(g) - Y_j(g\backslash i\colon j) \tag{9}$$

for every $g \in GR$ and every $i\colon j \in g$.

Myerson (1977) shows that there is a unique solution satisfying the above properties of component efficiency and fairness; and it is closely related to the Shapley value.

Theorem 38 (Myerson, 1977) *Given a characteristic function game $v \in G^N$, there is a unique fair allocation rule $Y : GR \to \mathbb{R}^n$. This fair allocation rule satisfies, for every $g \in GR$,*

$$Y(g) = Sh(v/g) \tag{10}$$

where the game $(v/g) \in G^N$ is defined for all $S \subseteq N$, as $(v/g)(S) = \sum_{T \in S/g} v(T)$.

Definition 39 *Given $v \in G^N$ and $\alpha \in \left(0, \frac{1}{n}\right]$, we say that a rule $Z : GR \to \mathbb{R}^n$ satisfies minimum payoff in $H \subseteq GR$ if and only if*

$$Z_i(g) \geq \alpha(v/g)(N) \tag{11}$$

for every $g \in GR$ and $i \in N$.

In order to obtain a minimum-payoff solution for games with cooperation structures represented as graphs, we define what we called a "semi-allocation rule".

Definition 40 *A semi-allocation rule with respect to $\alpha \in \left(0, \frac{1}{n}\right]$ for $v \in G^N$, is a function $Z : GR \to \mathbb{R}^n$ such that*

$$\sum_{i \in S} Z_i(g) = v(S) + \alpha\left[s(v/g)(N) - nv(S)\right]$$

for every $g \in GR$ and every $S \in N/g$.

The least profitable components will be supported by the most profitable components, i.e., the component S has to be supported if $\frac{v(S)}{s} < \frac{(v/g)(N)}{n}$, and has to support if the inequality holds in the other direction. According to the previous definition, the amount of this transfer is proportional to $s(v/g)(N) - nv(S)$.

Theorem 41 *Given $v \in G^N$ and $\alpha \in \left(0, \frac{1}{n}\right]$, there is a unique fair semi-allocation rule with respect to α, $Z : GR \to \mathbb{R}^n$. It is given by*

$$Z_i(g) = (1 - \alpha n)Sh_i(v/g) + \alpha(v/g)(N) \tag{12}$$

for every $i \in N$ and every $g \in GR$.

Proof. Suppose Z is a fair semi-allocation rule with respect to α, and let

$$Y_i(g) = \frac{1}{1 - \alpha n}\left[Z_i(g) - \alpha(v/g)(N)\right]$$

Then Y is the allocation rule of Myerson given by (10), since Y is a fair allocation rule. Effectively,

a) If $S \in N/g$, then

$$\sum_{i \in S} Y_i(g) = \frac{1}{1 - \alpha n}\left[\sum_{i \in S} Z_i(g) - \alpha s(v/g)(N)\right] = \frac{1}{1 - \alpha n}\left[(1 - \alpha n)v(S)\right] = v(S)$$

b) If $g \in GR$ and $i\colon j \in g$, then

$$\begin{aligned} Y_i(g) - Y_i(g \backslash i\colon j) &= \frac{1}{1 - \alpha n}[Z_i(g) - Z_i(g \backslash i\colon j)] \\ &= \frac{1}{1 - \alpha n}[Z_j(g) - Z_j(g \backslash i\colon j)] \\ &= Y_j(g) - Y_j(g \backslash i\colon j) \end{aligned}$$

On the other hand, the allocation rule Z given by (12) satisfies the two properties of the theorem. ■

Corollary 42 *Given $v \in G^N$ and $\alpha \in \left(0, \frac{1}{n}\right]$, the semi-allocation rule with respect to α*

$$Z_i(g) = (1 - \alpha n)Sh_i(v/g) + \alpha(v/g)(N)$$

satisfies α-minimum payoff in $H = \{g \in GR \mid v/g \in G_+^N\}$.

Moreover, if $v \in G_+^N$ then the semi-allocation rule with respect to α, Z, satisfies α-minimum payoff in GR.

Proof. Let $v \in G_+^N$ and $S, T \subset N$ be such that $S \cap T = \varnothing$. Notice that every $R \in (S \cup T)/g$ is formed by the union of components in S/g and of components in T/g. Since $(S \cup T)/g$ is a partition of $S \cup T$, then

$$\sum_{R \in (S \cup T)/g} v(R) \geq \sum_{R \in S/g} v(R) + \sum_{R \in T/g} v(R)$$

And so, $v/g \in G_+^N$. The result follows from Lemma 9. ■

Example 43 (Myerson, 1977) *Let* $N = \{1, 2, 3\}$, $g = \{1\colon 2, 1\colon 3\}$ *and* $v \in G^N$ *be given by*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	12	6	6	12

And suppose that every player wants to get at least a payoff of 2 *units .*

The fair allocation rule for this game and this particular graph is $Y(\{1\colon 2, 1\colon 3\}) = (7, 4, 1)$ *and the fair semi-allocation rule with respect to* α *is* $Z(\{1\colon 2, 1\colon 3\}) = \left(\frac{11}{2}, 4, \frac{5}{2}\right) = (5.5, 4, 2.5)$.

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About Incentives Distribution

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Abstract

In this paper we study the problem of distributing a given amount among a set of agents, according to their performance in a set of issues. In contrast with multi-issue allocation situations where each agent claims some of the amount, in this model the distribution must take into account how each agent behaves on some issues. We characterize a family of solutions for this problem using the well-known additivity, equal treatment and efficiency properties, adapting them to our context. If we want a solution that satisfies two additional properties, reduction and proportional, another problem arises: we need to look for a system of prices for a unity of each issue. Also, we apply classical bargaining theory techniques for the problem and some results are shown.

Mathematics Subject Classification: 91A80

Keywords: Bankruptcy, Multi-issue Allocation Situations, Allocation Rules

1 Introduction

In a typical bankruptcy problem (introduced by O'Neill [2]) it is necessary to distribute a given estate amongst a set of agents, where each agent claims a part of the estate. When the total amount claimed exceeds the available estate it is necessary to provide a distributing mechanism of the estate to be as fair as possible, because not all the claims of the agents can be fully satisfied.

In an extension of the original bankruptcy problem, Kaminski [9] considers situations related to a vector of claims for the agents, where each agent assigns

different priorities to each claim. Calleja, et al. [10] introduce the multi-issue allocation situation, another generalization of the bankruptcy problem, that considers multidimensional claims. They suppose that each claim component is originated from a particular issue. An issue constitutes a reason on the basis of which the estate will be divided. In their work, they provide a generalization of the O'Neill solution [2] for bankruptcy problems, which he calls the method of recursive completion, for the multi-issue problem. The multi-issue allocation situations have been studied by several authors such as González-Alcón et al. [3] and Bergantiños et al. [5]. Another multi-dimensional extension of the bankruptcy model is provided by Lerner [1]. In his work, a pie is divided among groups, not necessarily disjoint, rather than users.

In this work, we consider that each agent has a performance in a given set of issues. The problem that we study is how to divide a given amount $r \in \mathbb{R}$ among the agents considering their performance on each issue. Notice that, unlike multi-issue allocation situations, the issues represent several categories or classes where the agents are evaluated. To illustrate our problem consider the next example: A research center has received, because of its productivity, a economical bonus of the government. That productivity is the result of the work of the researchers. So, the board of directors needs to distribute the bonus amongst the researchers. They consider three categories: published papers, knowledge diffusion and teaching. Each researcher has a profile in each activity. This methodology is very frequent in Mexico. Our objective is to provide some process for making the distribution. In the classical bankruptcy problem and in multi-issue allocation situations there is an amount claimed by each agent. In contrast, we consider that agents have a score on each issue that indicates their performance.

In this study, we consider an axiomatic approach, providing a characterization for a family of solutions to our problem. We make this adapting the well-known efficiency, additivity and equal rewards properties to our context. The result obtained could be also adapted to multi-issue allocation situations. Also, we obtain a generalization of the rights-egalitarian rule for bankruptcy problem for our problem. Finally, we propose a bargaining approach, where the set of alternatives is the set of additive, efficient and equal treatment solutions satisfying other properties, reduction and proportional. The disagreement point is the vector zero n-dimensional. We apply several well-known solutions and we show the results obtained.

The paper is organized as follows: in Section 2 we present the notation and the model of the problem. Next, we provide a characterization for a family of solutions for the problem as well as other properties and particular cases of that family. In Section 4 we present a bargaining approach when a certain kind of solutions must satisfy other properties.

2 Notation and preliminaries.

Let $N = \{1, \dots, n\}$ be the set of agents and $M = \{1, \dots, m\}$ the set of issues. A matrix $X \in \mathbb{R}_+^{M \times N}$ represents the performance of the agents in each one of the issues. So, an element $x_{kj} \geq 0$ of X indicates the performance of agent $j \in N$ in the issue $k \in M$. We refer to this matrix as a *performance matrix*.

Given a performance matrix X and $k \in M$, we denote

$$X_k = \sum_{j \in N} x_{kj}, \quad \text{and} \quad \overline{X_k} = \frac{X_k}{n},$$

the total performance and the average performance of the issue k, respectively.

An *incentives distribution problem*, or simply an *incentives problem*, is a pair (X, r), where $X \in \mathbb{R}_+^{M \times N}$ is a performance matrix and $r \in \mathbb{R}$ represents the amount to be distributed among the agents. We define the sum and scalar product of incentive problems as follows: for (X_1, r_1), (X_2, r_2) and $\lambda \in \mathbb{R}$, $(X_1, r_1) + (X_2, r_2) = (X_1 + X_2, r_1 + r_2)$ and $\lambda(X_1, r_1) = (\lambda X_1, \lambda r_1)$.

We define a *solution* for a incentives problem as an operator $\varphi : \mathbb{R}_+^{M \times N} \times \mathbb{R} \to \mathbb{R}^n$, where $\varphi_j(X, r)$ represents the amount assigned to the agent j according to his performance in X when an amount r must be distributed.

3 Characterization.

In this section, we present our main result. We provide a family of solutions for an incentives problem.

Axiom 3.1. (Additivity). *For a pair of incentive matrices $X_1, X_2 \in \mathbb{R}_+^{M \times N}$ and $r_1, r_2, \lambda \in \mathbb{R}$, a solution $\varphi : \mathbb{R}_+^{M \times N} \times \mathbb{R} \to \mathbb{R}^n$ satisfies the additivity axiom if*

$$\varphi(\lambda X_1, \lambda r_1) + \varphi(\lambda X_2, \lambda r_2) = \lambda \varphi(X_1 + X_2, r_1 + r_2).$$

The idea of this property is the following: suppose two incentives problems, each of them from different concepts. The additivity axiom guarantees that the sum of the incentives on each concept must be equal to the incentives of a single problem containing the sum of the performances and the amounts of the individual concepts.

Axiom 3.2. (Equal treatment) *A solution $\varphi : \mathbb{R}_+^{M \times N} \times \mathbb{R} \to \mathbb{R}^n$ satisfies the equal treatment axiom if for each pair of agents $i, j \in N$ such that $x_{ki} = x_{kj}$ for all $k \in M$ in a performance matrix X, $\varphi_i(X, r) = \varphi_j(X, r)$.*

Axiom 3.3. (Efficiency) *A solution* $\varphi : \mathbb{R}_+^{M\times N} \times \mathbb{R} \to \mathbb{R}^n$ *satisfies the efficiency axiom if*

$$\sum_{j\in N} \varphi_j(X, r) = r$$

for every $X \in \mathbb{R}_+^{M\times N}$ *and* $r \in \mathbb{R}$.

According to this axiom, the amount $r \in \mathbb{R}$ must be distributed among all agents.

Theorem 3.4. *A solution for the incentives problem satisfies the additivity, equal treatment and efficiency axioms if and only if it can be written as follows:*

$$\varphi_j(X, r) = \frac{r}{n} + \sum_{k\in M} \beta_k \left(x_{kj} - \overline{X_k}\right) \qquad \forall\ j \in N \tag{1}$$

for a set of constants $\beta_k \in \mathbb{R}$ *for all* $k \in M$, *and for all* $X \in \mathbb{R}_+^{M\times N}$, $r \in \mathbb{R}$.

Proof. First, we prove that every solution satisfying the additivity, equal treatment and efficiency axioms must be written as (1). For all $k \in M$, $i \in N$, we define $E_{ki} \in \mathbb{R}_+^{M\times N}$, the matrix such that its ki-entry is equal to one and zero otherwise. Now, because of the sum and scalar product operations for an incentives problem, we have

$$(X, r) = \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}(E_{ki}, r_{ki})$$

with $r_{ki} \in \mathbb{R}$ such that

$$\sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki} r_{ki} = r$$

for every $X \in \mathbb{R}_+^{M\times N}$ and $r \in \mathbb{R}$. Also, we have

$$(E_{ki}, r_{ki}) = (E_{ki}, 0) + (E_0, r_{ki})$$

where $E_0 \in \mathbb{R}_+^{M\times N}$ is the zero matrix. Applying the additivity axiom

$$\varphi_j(X, r) = \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\left[\varphi_j(E_{ki}, 0) + \varphi_j(E_0, r_{ki})\right] \qquad \forall j \in N. \tag{2}$$

By the equal treatment and efficiency axioms we have

$$\varphi_j(E_0, r_{kl}) = \frac{r_{ki}}{n} \qquad \text{and then} \qquad \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\varphi_j(E_0, r_{ki}) = \frac{r}{n}. \tag{3}$$

For each $k \in M$,$i \in N$ and the equal treatment axiom we have, for all $j \in N$,

$$\varphi_j(E_{ki}, 0) = \begin{cases} \alpha_k, & \text{if } j = i \\ \mu_k, & \text{otherwise} \end{cases}$$

and then

$$\sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\varphi_j(E_{ki}, 0) = \sum_{k=1}^{m}\left(x_{ki}\alpha_k + \sum_{\substack{i=1 \\ i\neq j}}^{n} x_{ki}\mu_k \right).$$

The efficiency axiom implies $\alpha_k + (n-1)\mu_k = 0$ for each $k \in M$. Substituting in the last equation

$$\sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\varphi_j(E_{ki}, 0) = \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\mu_k - n\sum_{k=1}^{m} x_{kj}\mu_k = \sum_{k=1}^{m} X_k\mu_k - n\sum_{k=1}^{m} x_{kj}\mu_k.$$

Making the change of variable $\mu_k = -\beta_k/n$ we obtain

$$\sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\varphi_j(E_{ki}, 0) = \sum_{k=1}^{m} \beta_k(x_{kj} - \overline{X_k}). \tag{4}$$

Substituting (3) and (4) in (2) we obtain the result.

Now we need to prove that (1), for a set of constants $\beta_k \in \mathbb{R}$ for all $k \in M$, satisfies the axioms cited in the theorem. This proof can be done in a direct way, so we omit it. □

The solution (1) indicates how the amount is divided. In a first step, each agent receives the same part of the amount (i.e. r/n). After that, we need to check the performance of an agent $j \in N$ on each issue $k \in M$: if his performance is better than the average performance, a quantity β_k is added to his initial amount for each unit of difference between his performance and the average performance. If his performance is less than the average performance, that quantity is diminished. After doing this for each issue, the resulting amount for each j is (1).

The set of constants $\{\beta_k\}_{k=1}^{M}$ represents a weight for each issue and we suppose that they are given exogenously. That weight can be interpreted as the importance of each issue. For $k, k' \in M$, $k \neq k'$ such that $\beta_k > \beta_{k'}$, the difference between the profile in the issue k and the average profile has more impact in the final distribution of the amount than the issue k'. So, this means it is more convenient to have a profile greater than the average profile in k rather than in k'. The problems related to the set of constants $\{\beta_k\}_{k=1}^{M}$ are considered later.

Remark 1. The three axioms are necessary and independent.

- $\psi(X,r) = 2 \cdot \varphi(X,r)$ satisfies additivity and equal treatment.
- $\psi_j(X,r) = \left(\frac{\sum_{k\in M} x_{kj}}{\sum_{k\in M} X_k}\right) r$, for all $j \in N$, satisfies equal treatment and efficiency.
- $\psi_j(X,r) = \begin{cases} r, & \text{if } j = 1 \\ 0, & \text{otherwise.} \end{cases}$, for $n \geq 2$, satisfies additivity and efficiency.

Considering alternative properties we provide other results. We denote $\mathbb{P}^m$ as the set of $m \times m$ permutation matrices.

Axiom 3.5. (Symmetry of issues) *A solution $\varphi : \mathbb{R}_+^{M\times N} \times \mathbb{R} \to \mathbb{R}^n$ satisfies the symmetry of issues axiom if for every permutation matrix $P \in \mathbb{P}^m$*

$$\varphi(X,r) = \varphi(PX,r)$$

for all $X \in \mathbb{R}_+^{M\times N}$ and $r \in \mathbb{R}$.

According to this axiom, the solution does not take into account the order of the issues, i.e. their external characteristics are not taken into account.

Theorem 3.6. *A solution for the incentives problem satisfies the additivity, equal treatment, symmetry of issues and efficiency axioms if and only if it can be written as*

$$\varphi_j(X,r) = \frac{r}{n} + \beta \sum_{k\in M} \left(x_{kj} - \overline{X_k}\right) \qquad \forall\ j \in N \tag{5}$$

for a constant $\beta \in \mathbb{R}$, for all $X \in \mathbb{R}_+^{M\times N}$ and $r \in \mathbb{R}$.

Proof. The proof follows the same direction as the proof of Theorem 3.4. The main difference is the next: by applying the symmetry of issues axiom we obtain

$$\varphi_j(E_{ki}, 0) = \begin{cases} \alpha, & \text{if } j = i \\ \mu, & \text{otherwise} \end{cases} \qquad \forall j \in N$$

since $E_{ki} = PE_{li}$ for all $i \in N$, $k \in M$, $l \in M\backslash\{k\}$ and some $P \in \mathbb{P}^m$. So, by the efficiency axiom, we have $\alpha + (n-1)\mu = 0$. Using this equation, the proof follows the same steps as Theorem 3.4, so we omit them. In addition, it is necessary to prove that (5) satisfies the symmetry of issues axiom, but this proof can be done in a direct way, so we omit it. □

Axiom 3.7. (Trivial division) *A solution* $\varphi : \mathbb{R}_+^{M\times N} \times \mathbb{R} \to \mathbb{R}^n$ *satisfies the trivial division axiom if*

$$\varphi_j(X, \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}) = \sum_{k=1}^{m} x_{kj} \qquad \forall j \in N$$

for all $X \in \mathbb{R}_+^{M\times N}$.

According to the previous axiom, if the amount to be distributed is equal to the sum of the individual performances, a trivial distribution is done, where each unit of performance is paid with a unit of the amount.

Theorem 3.8. *A solution for the incentives problem satisfies the additivity, equal treatment, symmetry of issues, trivial division and efficiency axioms if and only if it can be written as follows:*

$$\varphi_j(X, r) = \frac{r}{n} + \sum_{k\in M} \left(x_{kj} - \overline{X_k}\right) \qquad \forall\ j \in N \tag{6}$$

for all $X \in \mathbb{R}_+^{M\times N}$ *and* $r \in \mathbb{R}$.

Proof. Formula (5) gives us a family of solutions satisfying the additivity, equal treatment, symmetry of issues and efficiency axioms. Let (X, r) be an incentives problem with $r = \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}$. Now, we have

$$\varphi_j(X, r) = \frac{r}{n} + \beta \sum_{k\in M} x_{kj} - \beta\frac{r}{n} \qquad \forall\ j \in N.$$

Applying the trivial division axiom, we have $\varphi_j(X, r) = \sum_{k=1}^{m} x_{kj}$. Substituting in the last equation, and solving for β we obtain $\beta = 1$ and substituting in (5) we obtain (6). Finally, we need to prove that (6) satisfies the trivial division axiom, but this can be done in a direct way and we omit it. □

Making algebraic manipulation, solution (6) can be written as

$$\varphi_j(X, r) = \sum_{k\in M} x_{kj} + \frac{1}{n}\left(r - \sum_{k=1}^{m}\sum_{i=1}^{n} x_{ki}\right)$$

and then, solution (6) can be considered as a generalization of the *rights-egalitarian rule* for bankruptcy problems.

Example 3.9. *Consider three researchers* $N = \{1, 2, 3\}$ *and two issues, the number of given subjects* ($k = 1$) *and the number of published papers* ($k = 2$).

Their performance on these issues in a period of time is given by the next performance matrix

$$X = \begin{bmatrix} 8 & 6 & 4 \\ 2 & 1 & 3 \end{bmatrix}.$$

There is an amount of 18 (in some units) to be divided among the researchers according to their profile. Clearly, we can see this problem as an incentives problem. Applying (6) we obtain

$$\varphi(X, r) = (8, 5, 5).$$

This example shows us a weakness of solution (6), but really, it is a weakness of the family of solutions given in Theorem 3.6: it assigns to each issue the same importance, in contrast with the family of solutions (1). That is the reason why researchers 2 and 3 obtain the same amount. This could be a controversial property because, in some contexts, the required efforts for performing a single unit of an issue are greater for some issues than others.

For a performance matrix $X \in \mathbb{R}_+^{M\times N}$ we say X' is a *reduced performance matrix of* X if the performance of a set of agents on N, the involved agents, is collapsed into a single agent, the representative. The performance of the representative agent is equal to the sum of the performances of the involved agents.

Axiom 3.10. (Reduction) *Let $X \in \mathbb{R}_+^{M\times N}$ be a performance matrix and X' a reduced performance matrix of X, with representative agent j^* and a set of involved agents $T \subset N$. A solution φ for the incentives problem satisfies the reduction axiom if*

$$\varphi_{j^*}(X', r) = \sum_{h\in T} \varphi_h(X, r).$$

According to the last axiom, amalgamating agents does not represent an advantage for the resulting agent.

Axiom 3.11. (Proportional) *Let $X \in \mathbb{R}_+^{M\times N}$ be a performance matrix such that there exist two agents $i, j \in N$ and $\alpha \in \mathbb{R}_+$ such that $x_{kj} = \alpha x_{ki}$ for every $k \in M$. A solution φ for the incentives problem satisfies the proportional axiom if*

$$\varphi_j(X, r) = \alpha\varphi_i(X, r).$$

This axiom establishes that agents with proportional performance on each issue must have proportional allocations.

Lemma 3.12. *If a solution for the incentives problem satisfies the additivity, equal treatment, efficiency and reduction (proporcional) axioms then*

$$\sum_{k\in M} \beta_k X_k = r \tag{7}$$

for all $X \in \mathbb{R}_+^{M\times N}$, $r \in \mathbb{R}$ *where* $\{\beta_k\}_{k=1}^m$ *are given according to (1).*

Proof. Suppose (1) satisfies the reduction axiom. Let $X \in \mathbb{R}_+^{M\times N}$ be a performance matrix and X' a reduced matrix of X such that $\{i,j\} \in N$ is the set of involved agents and j^* is the representative agent. Notice that $x'_{kj^*} = x_{ki} + x_{kj}$ for all $k \in M$. Then, applying the reduction axiom we have

$$\frac{r}{n-1} + \sum_{k\in M} \beta_k \left(x_{ki} + x_{kj} - \sum_{h\in N} \frac{x_{kh}}{n-1} \right) = \frac{2r}{n} + \sum_{k\in M} \beta_k \left(x_{ki} + x_{kj} - 2\sum_{h\in N} \frac{x_{kh}}{n} \right).$$

Applying algebraic manipulation we obtain

$$\sum_{k\in M} \left(\beta_k \cdot \sum_{h\in N} \frac{x_{kh}(n-2)}{n(n-1)} \right) = \frac{r(n-2)}{n(n-1)}$$

and then we obtain the result.

Let $X \in \mathbb{R}_+^{M\times N}$ be a performance matrix such that there exist $i,j \in N$ with $x_{jk} = \alpha x_{ik}$ for all $k \in M$ and $\alpha \in \mathbb{R}_+$. Now, suppose (1) satisfies the proportion axiom. Then

$$\frac{r}{n} - \sum_{k\in M} \beta_k \overline{X_k} = \frac{\alpha r}{n} - \alpha \sum_{k\in M} \beta_k \overline{X_k}.$$

Rearranging and simplifying terms we obtain

$$\frac{r(\alpha-1)}{n} = \sum_{k\in M} \beta_k \frac{X_k(\alpha-1)}{n}$$

that is equivalent to (7). □

Theorem 3.13. *A solution* $\varphi : \mathbb{R}_+^{M\times N} \times \mathbb{R} \to \mathbb{R}^n$ *satisfies the additivity, equal treatment, efficiency, reduction (proportional) axioms if and only if, there exist a set of constants* $\beta_k \in \mathbb{R}$, $k \in M$, *such that*

$$\varphi_j(X,r) = \sum_{k\in M} \beta_k x_{kj} \qquad \forall j \in N \tag{8}$$

for all $X \in \mathbb{R}_+^{M\times N}$ *and* $r \in \mathbb{R}$.

Proof. Suppose a solution φ that satisfies the additivity, equal treatment, efficiency and reduction (proportional) axioms. By Lemma 3.12, we can substitute (7) in (1) and doing algebraic manipulations we obtain the result. Now, we need to prove that (8) satisfies the axioms mentioned in the theorem. That can be done in a direct way and we omit it. So, the proof ends. □

In equation (7) we can consider β_k as the price which each unit of performance in the issue $k \in M$ will be paid. If we want an additive, efficient and equal treatment solution that satisfies the reduction and proportional axioms we need to find those prices such that the total quantity distributed is equal to r. By equation (7), these prices depend on the performance matrix. Their existence can violate other assumptions, in particular, additivity.

4 A bargaining approach

Formally, a n-person *bargaining problem* is a pair (S, d) where $d \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$ is a convex and compact set, and that there is at least one point of S strictly dominating d. Let $N = \{1, \ldots, n\}$ be the set of agents. Each point of S indicates the utility reached by the agents through the choice of one of the available alternatives. If there is no agreement the solution d is assigned to the problem. In addition, a usual assumption is that the utility is freely disposable: if $x \in X$ and $x_i \geq y_i \geq d_i$ for some $i \in N$ then $y \in S$.

In this section, we use bargaining theory for solving the problem of incentives distribution. Let (X, r) be an incentives problem. We consider $d = (0, \ldots, 0) \in \mathbb{R}^n$ as the disagreement point: if there is no agreement among the agents, nobody obtains incentives. Also, we are only interested in non-negative efficient solutions, i.e. $\varphi : \mathbb{R}_+^{M \times N} \times \mathbb{R} \to \mathbb{R}^n$ where $\varphi_j(X, r) \geq 0$ for all $j \in N$ and $\sum_{j \in N} \varphi_j(X, r) = r$. So, by freely disposable property, we have $S = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leq r, x_i \geq 0, \forall i \in N\}$. This set represents the possible distributions of r.

There are several solutions for the bargaining problem. We work with three of them. The Nash solution [8], $N(S, d) \in \mathbb{R}^n$, is obtained by maximizing the product of utility gains from the disagreement point. The Kalai-Smorodinsky solution [4], $KS(S, d) \in \mathbb{R}^n$, sets utility gains from the disagreement point proportional to each agent's most optimistic expectations. The Yu solution [11], $Y(S, d) \in \mathbb{R}^n$, tries to make an assignment as close as possible to satisfying everyone. For our problem, these solutions assign the same point as the solution no matter what performance matrix is being considered because of the symmetry of the region S. That is, $N(S, d) = KS(S, d) = Y(S, d) = (r/n, \ldots, r/n)$.

As a second approach, we consider an incentives problem with solutions as (1). Also, we want that condition (8) holds with $\beta_k \geq 0$ for all $k \in M$, in order

to interprete β_k as prices per unit. We call this set of problems $\beta-$restricted incentive problems.

Lemma 4.1. *The set of alternatives for a $\beta-$restricted incentive problems is contained in the alternatives for the same problem without restrictions.*

Proof. Let (X, r) be a $\beta-$restricted incentive problem. According to (8), the maximum value for all β_k is r/X_k when $X_k \neq 0$. For all $j \in N$ we define

$$k^j = \underset{k \in M}{\operatorname{argmax}} \left\{ \frac{r}{x_k}(x_{kj} - \overline{X_k}) \right\}.$$

Then, the maximum allocation for an agent $j \in N$, denoted by $\varphi_j^*(X, r)$ occurs when $\beta_{k^j} = r/X_{k^j}$ and $\beta_k = 0$ otherwise. Then, we have

$$\varphi_j^*(X, r) = \frac{r}{n} + \frac{r}{X_{k^j}}(X_{k^j j} - \overline{X_{k^j}}) = \left(\frac{X_{k^j j}}{X_{k^j}} \right) r.$$

Without restrictions, the maximum allocation that an agent can obtain is r. So, as $\varphi_j^*(X, r) \in [0, r]$, the proof ends. □

In Figure 1, we show the set of alternatives for the incentives problem of Example 1 with and without restrictions.

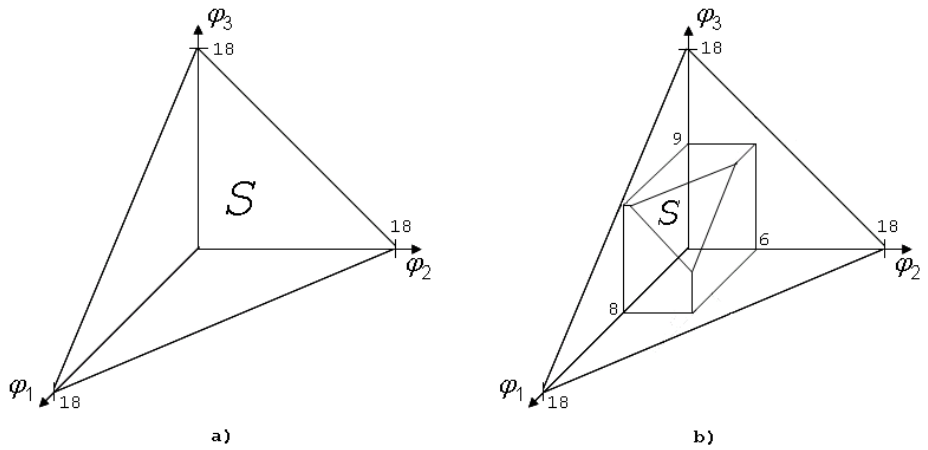


Figure 1: Set of alternatives for Example 3.9. In *a)* the problem has no restrictions. In *b)*, the problem is $\beta-$restricted.

Theorem 4.2. *Let (X, r) be a $\beta-$restricted incentives problem with set of alternatives S and disagreement point $d = (0, \ldots, 0)$. For this bargaining problem (S, d) we have:*

1. *The Nash solution is given by the point in S closest to $(r/n, \ldots, r/n) \in \mathbb{R}^n$.*

2. *The Kalai-Smorodinsky solution is given by*

$$KS_j(S,d) = \left\lfloor \frac{\varphi_j^*(X,r)}{\sum_{h\in N}\varphi_h^*(X,r)} \right\rfloor r \qquad \forall j \in N$$

where $\varphi_j^*(X,r)$ is the maximum amount that player $j \in N$ can obtain in the problem.

3. *The Yu solution is given by*

$$Y_j(S,d) = \varphi_j^*(X,r) + \frac{1}{n}\left(r - \sum_{h\in N}\varphi_h^*(X,r)\right) \qquad \forall j \in N.$$

For the definitions related to $\varphi_j^*(X,r)$, refer to i).

Proof.

1. The Nash solution is given by the maximizer of

$$\prod_{j\in N}\varphi_j(X,r) \qquad \text{s.t. } \sum_{j\in N}\varphi_j(X,r) = r, \varphi_j(X,r) \geq 0 \ \forall j \in N \qquad (9)$$

in the set of alternatives S. The maximizer of (9) is the point $(r/n,\dots,r/n)$. If this point is in S, that is the solution. Otherwise, the closest point in S to that point is the solution.

2. The Kalai-Smorodinsky solution corresponds to the intersection point of the hyperplane $\sum_{j\in N}\varphi_j(X,r) = r$ with the line $(\varphi_1^*(X,r),\dots,\varphi_n^*(X,r))t$. This point is denoted by $KS_j(S,d) = \varphi_j^*(X,r)t$, with t satisfying

$$r = t\sum_{j\in N}\varphi_j^*(X,r) \quad \Rightarrow \quad t = \frac{r}{\sum_{j\in N}\varphi_j^*(X,r)}.$$

Substituting the last equation in $KS_j(S,d)$ we obtain the result.

3. The Yu solution corresponds to the intersection point of the hyperplane $\sum_{j\in N}\varphi_j(X,r) = r$ with the line $(\varphi_1^*(X,r),\dots,\varphi_n^*(X,r)) - (1,\dots,1)t$. This point, $Y(S,d)$, is given by $Y_j(S,d) = \varphi_j^*(X,r) - t$, with t satisfying

$$\sum_{j\in N}(\varphi_j^*(X,r) - t) = r \quad \Rightarrow \quad t = \frac{1}{n}\left(\sum_{j\in N}\varphi_j^*(X,r) - r\right).$$

Substituting this value in the expression for $Y_j(S,d)$ we obtain the result.

□

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VALUES FOR GAMES ON THE CYCLES OF A DIGRAPH

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In this paper we introduce a new class of cooperative games. We define a characteristic function over the cycles of a digraph. We present a mathematical model for this situation and an axiomatic characterization of a solution for this class of cooperative games. This is introduced as a method to measure the importance of the nodes in a digraph, and can be related with the Shapley value of a game in characteristic function form. Also, we extend the modeling by applying coalitional structures for the nodes and r-efficient solutions, where the allocation amount is a real number r, showing axiomatic solutions in both cases.

Keywords: Digraphs; cooperative games; allocation rules.

Mathematics Subject Classification: 05C57, 91A12, 91A43

1. Introduction

There are problems related to walks inside digraphs in which it is necessary to quantify the importance of the nodes in order to measure the contribution of each node in the digraph. With this measurement, it is possible to make comparative studies of the nodes, ranking them or just determining their importance in a process. In this work, we restrict walks to cycles, motivated by the minimal cost-to-time ratio cycle problem formulation due to Dantzig in Lawler [1976]: a merchant ship is free to choose its ports of call and the order in which it calls on them; the voyage from port i to port j earns an established profit a_{ij} and it takes a time t_{ij} to do it; then, it is necessary to find the ports and the visiting order such that the merchant ship optimizes its cost-to-time ratio. Dantzig solves that problem and presents an algorithm for finding the optimal cycle. What we propose is a method to measure the importance of each port for the merchant ship based on techniques for solving cooperative games.

Let us consider a pair of additional problems where having a measure of the importance of the involved agents can help find a solution. We refer to the first problem as the *redistribution in a chain of stores* problem. In some chains of stores (such as bookstores and pharmacies) it can be necessary to make a redistribution

of products among branches. When the requirements reach a certain level, a deliveryman leaves the branch to make the redistribution according to the priorities of the requirements. The redistribution costs need to be allocated among the branches at the end of a certain period of time. We refer to the second problem as the *manufacturing process problem*: A manufacturing industry receives orders continuously and they are processed according to the order of arrival. The orders are processed by batch and they are heterogenous in volume. The machine involved in the process needs to be adjusted before working, and the time spent in this adjustment depends directly on the previous process and the process to be executed. That idle time represents a cost for the industry, and it is necessary to distribute it among the different orders. The industry does not know the arrival sequence of the orders. This problem is a simplified version of a real problem coming from the shoe industry in León, Guanajuato.

Motivated by the model and the two problems presented previously, we introduce a new kind of cooperative game defined on the cycles of a digraph, where a real number is assigned to each cycle on the digraph. This real number represents the worth that the merchant ship obtains when it visits all the nodes in the cycle in the sequential order or, in the redistribution in a chain of stores problem, the cost for visiting the branches according to a given cycle. Since every cycle has an assigned worth, it is possible to use this information and cooperative games techniques to measure how each node contributes to the possible routes of the merchant ship (or how each branch contributes to the total redistribution costs, in the redistribution in a chain of stores problem). We propose an axiomatic characterization based on the set of axioms given in Shapley [1953], but we adapt them for this new class of cooperative games.

A first modification of the original model occurs when a fixed real number must be the allocation amount to be distributed between the nodes. This variation could happen when an external authority or a regulating organism determines the amount to be distributed in the set of nodes. For example, in the redistribution of a chain of stores problem, this amount can be related to operation costs. The fixed amount can be greater or less than the nodes can obtain by themselves. So, it is necessary to provide a distribution mechanism of the possible difference between the amount that nodes can obtain by themselves and the fixed allocation amount. We present this situation in the context of games defined on the cycles of a digraph and we show an axiomatic solution.

Owen (1977) studied situations in which players are partitioned into groups. He considers that players are organized with other players, according to their interests and conveniences, because their union could have a stronger negotiating power than standing alone. In this context, Owen [1977] introduced a value for this kind of situations, which is a generalization of the Shapley value and it is interpreted as an assignment rule in a two-step process. That solution is now known as the *Owen value*. Thus, we consider the organization of the nodes of a digraph into groups before the negotiation process. In the Dantzig problem this situation refers to ports

grouping in trade unions or political organizations such as countries. As a second modification to the original model, we present a characterized solution for this situation, where the importance of each node is assigned in a two-step procedure, applying a similar idea of the Owen value.

The structure of the paper is as follows: in Sec. 2 we present the mathematical model and we provide basic terminology and notation. In Sec. 3 we propose an axiomatic solution for games on the cycles of a digraph and we show its equivalency to the Shapley value for a cooperative game in characteristic function form. Also, we present a family of solutions based on a subset of the previous solution axioms. In Sec. 4, we adapt the model again considering the allocation amount as a real number r, showing a characterized solution. Finally, in Sec. 5, we introduce the nodes organization into coalitional structures and we present an axiomatic solution for this situation.

2. Notation and Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of nodes. The directed complete graph with set of nodes N, or simply a digraph, is the pair (N, E^N) where $E^N = N \times N$ is the set of edges. For $T \subseteq E^N$, the set $n(T)$ represents the set of nodes forming the edges in T. Notice that $n(T) \subseteq N$ for all $T \subseteq E^N$, T is a set of ordered pairs, and $(n(T), T)$ can be interpreted as a subdigraph with $n(T)$ as the set of nodes. We say $T \subseteq E^N$ is a *directed path* if for a sequence of nodes $\{x_i\}_{i=0}^{k}$ $T = \{(x_i, x_{i+1})\}_{i=0}^{k-1}$. Also, we say $T \subseteq E^N$ is a *loop* if $T = \{(i, i)\}$ with $i \in N$. With these concepts, a directed path T is called a *cycle* if $x_0 = x_k$ and $x_i \neq x_j$ otherwise. We consider the empty set of edges as a cycle, the empty cycle.

We denote as $\mathcal{C}(N) := \{T \subseteq E^N \mid T \text{ is a cycle}\}$ the *set of cycles of the complete digraph with set of nodes N*. We define a *game on the cycles of a digraph* as a function $q : \mathcal{C}(N) \to \mathbb{R}$ with $q(\emptyset) = 0$. We use a short notation and we denote q as c-game. For each $T \in \mathcal{C}(N)$, $q(T)$ represents the profit obtained after visiting the nodes in $n(T)$ in the sequential order prescribed by T. The set of c-games $F_g := \{q \mid q : \mathcal{C}(N) \to \mathbb{R} \text{ with } q(\emptyset) = 0\}$ is a real vector space with the sum and scalar product in the usual way: for $q_1, q_2 \in F_g$ and $\lambda \in \mathbb{R}$, $(q_1 + q_2)(T) := q_1(T) + q_2(T)$ and $(\lambda q_1)(T) := \lambda q_1(T)$ for all $T \in \mathcal{C}(N)$. The length of a cycle $T \in \mathcal{C}(N)$ is given by the number of nodes in the cycle, and it is denoted as $|T|$. For $S \subseteq N$, the set $\mathcal{C}_S(N) := \{T \in \mathcal{C}(N) \mid n(T) = S\}$ is the set of cycles that visit exactly the nodes in S.

A solution for c-games is a function $\varphi : F_g \to \mathbb{R}^n$, where $\varphi_i(q)$ represents the importance of the node i in the c-game $q \in F_g$.

In the example related to the redistribution in a chain of stores, each branch can be considered as a node. We can form a complete digraph g that represents the set of communication links between branches. As we are considering that every possible route can be formed, and the routes are cyclical, we can create a c-game q, where $q(C)$ denotes the cost due to the redistribution represented by the cycle

$C \in C(g)$. A solution $\varphi_i(q)$ will indicate how branch i contributes to the total cost.

In the manufacturing process problem, the different kinds of orders can be represented by nodes. In order to be fair, we consider that after the last batch, the machine is adjusted as if the first batch were the next one to be processed. So, we consider that batches are processed in a cyclical way. So, we can define a c-game q where $q(C)$ denotes the cost resulting from the idle time of processing the batches according to the order given by the cycle C. So, a solution $\varphi_i(q)$ indicates the contribution for processing batch i to the total cost for the idle time.

3. Characterization

Now we present a characterization for a solution $\varphi : F_g \to \mathbb{R}^n$.

Axiom 1. (Linearity) A solution φ is a linear solution if

$$\varphi(\lambda_1 q_1 + \lambda_2 q_2) = \lambda_1 \varphi(q_1) + \lambda_2 \varphi(q_2)$$

for $q_1, q_2 \in F_g$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Axiom 2. (Symmetry) A solution φ is a symmetric solution if

$$\varphi_i(q) = \varphi_{\theta(i)}(q \circ \theta), \quad \forall\, i \in N,$$

where $(q \circ \theta)(T) = q(\theta^{-1}(T))$ and $\theta^{-1}(T) = \{(\theta^{-1}(x),\, \theta^{-1}(y)) \mid (x, y) \in T\}$ for all $T \in \mathcal{C}(N)$, $\theta \in S_n$ and $q \in F_g$. S_n is the group of permutations of N.

The previous axioms have the same interpretation for games in characteristic function form.

Let $T \in \mathcal{C}(N)$ be a cycle with length greater or equal to 2, $i, x, y \in n(T)$, where $(x, i), (i, y) \in T$. Then, we denote

$$T_{-i} := T \backslash \{(x, i), (i, y)\} \cup \{(x, y)\}.$$

Moreover, for every $T \in \mathcal{C}(N)$ such that T is a loop and $i \in n(T)$, $T_{-i} = \emptyset$. T_{-i} represents the cycle obtained after removing the node i of the cycle T preserving the visiting order. Also, we denote by T_S the cycle obtained after removing the nodes $n(T) \backslash S$ of T.

We say $i \in N$ is a *null node* if $q(T) = q(T_{-i})$ for all $T \in \mathcal{C}(N)$ such that $i \in n(T)$.

Axiom 3. (Nullity) A solution φ is a null solution if

$$\varphi_i(q) = 0$$

for each null node $i \in N$.

According to Axiom 3, if a node $i \in N$ does not contribute to any cycle which contains it, its importance must be equal to zero.

Given a c-game $q \in F_g$, we define the *average worth on q of the nodes in $S \subseteq N$*, $\eta(q, S)$, as

$$\eta(q, S) := \frac{1}{|\mathcal{C}_S(N)|} \sum_{T \in \mathcal{C}_S(N)} q(T)$$

and it represents the average worth that the subset $S \subseteq N$ can obtain when its nodes are visited in every possible way.

Axiom 4. (Average efficiency) A solution φ is an average efficient solution if

$$\sum_{i \in N} \varphi_i(q) = \eta(q, N)$$

for all $q \in F_g$.

This axiom establishes an allocation amount: it is the average worth of the cycles that visit the complete set of nodes. If all the nodes want to form a cycle, there are $(n-1)!$ ways for do it. On average, they can obtain $\eta(q, N)$. In comparison, for games in characteristic function form there is a unique form where all the players can cooperate: the grand coalition.

Other proposals for allocation amounts arise in a natural way, but several of them have some disadvantages. For example, we can consider that we want to distribute the maximum profit of a complete cycle. That is, we could require a solution $\varphi : F_g \to \mathbb{R}^n$ such that

$$\sum_{i \in N} \varphi_i(q) = q(C^*), \quad \text{with } C^* = \operatorname*{argmax}_{C \in \mathcal{C}_N(N)} \{q(C)\}$$

for all $q \in F_g$, but there is no solution that satisfies that kind of efficiency and the symmetry axiom for all c-game q because

$$\sum_{i \in N} \varphi_i(q \circ \theta) = (q \circ \theta)(C^*) = q(\theta^{-1}(C^*))$$

and

$$\sum_{i \in N} \varphi_i(q \circ \theta) = \sum_{i \in N} \varphi_i(q) = q(C^*).$$

Theorem 1. *There exists a unique linear, symmetric, null and average efficient solution $\phi : F_g \to \mathbb{R}^n$ for c-games. Furthermore, this solution is given by*

$$\phi_i(q) = \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \frac{(n - |T|)!}{n!} [q(T) - q(T_{-i})], \quad \forall\, i \in N. \tag{1}$$

For the proof of the previous theorem, we define, for every $T \in \mathcal{C}(N)$, a standard c-game $\mathcal{U}_T$ as $\mathcal{U}_T(S) = 1$ if $T = S$ and zero otherwise, for each $S \in \mathcal{C}(N)$. In the $\mathcal{U}_T$ c-game there is a unique cycle with nonzero profit, the cycle T.

Lemma 1. *Let S, T be cycles with $|S| = |T|$ and $\phi : F_g \to \mathbb{R}^n$ a symmetric solution; then*

$$\phi_i(\mathcal{U}_S) = \phi_j(\mathcal{U}_T)$$

for all $i \in n(S), j \in n(T)$. Also, this equality holds if $i \notin n(S), j \notin n(T)$.

Proof. Let $i \in n(S)$, $j \in n(T)$ and choose $\theta \in S_n$ such that $\theta(i) = j$ and $\theta(S) = T$. Because $\mathcal{U}_T = \mathcal{U}_{\theta(S)} = \mathcal{U}_S \circ \theta$ and the symmetry property we have

$$\phi_i(\mathcal{U}_S) = \phi_{\theta^{-1}(j)}(\mathcal{U}_S) = \phi_j(\mathcal{U}_S \circ \theta) = \phi_j(\mathcal{U}_T).$$

In a similar way, we can conclude that $\phi_i(\mathcal{U}_S) = \phi_j(\mathcal{U}_T)$ if $|S| = |T|$, $i \notin n(S), j \notin n(T)$. □

Proof of the theorem. For proving the existence of a solution, we need to verify that the solution given in (1) satisfies the mentioned axioms. The verification for linearity, symmetry and nullity axioms can be done in a direct way, so we omit it. Then, we only check that (1) satisfies the average efficiency axiom. We can write

$$\sum_{i \in N} \phi_i(q) = \sum_{i \in N} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \frac{(n - |T|)!}{n!} q(T) - \sum_{i \in N} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \frac{(n - |T|)!}{n!} q(T_{-i}). \tag{2}$$

For the first term of the previous equation we have

$$\sum_{i \in N} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \frac{(n - |T|)!}{n!} q(T) = \sum_{T \in \mathcal{C}(N)} \frac{|T|(n - |T|)!}{n!} q(T)$$

and for the second term we have

$$\begin{aligned} \sum_{i \in N} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \frac{(n - |T|)!}{n!} q(T_{-i}) &= \sum_{i \in N} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \not\ni i}} \frac{|T|(n - |T| - 1)!}{n!} q(T) \\ &= \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \neq N}} \frac{|T|(n - |T|)!}{n!} q(T). \end{aligned}$$

Now, substituting in (2) we can establish

$$\sum_{i \in N} \phi_i(q) = \frac{1}{(n-1)!} \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) = N}} q(T) = \eta(q, N)$$

and this proves that (1) is an average efficient solution.

Now, we need to prove uniqueness. We will adapt the procedure used by Ruiz *et al.* [1988] to our context to obtain a family of linear, symmetric and average efficient solutions for c-games. We consider a solution $\phi : F_g \to \mathbb{R}^n$ which satisfies

the axioms mentioned in the theorem. The standard c-games $\mathcal{U}_T$ form a basis for the vector space of c-games. So, applying the linearity axiom we have

$$\phi_i(q) = \sum_{T \in \mathcal{C}(N)} q(T)\phi_i(\mathcal{U}_T),$$

where

$$q = \sum_{T \in \mathcal{C}(N)} q(T)\mathcal{U}_T$$

is an arbitrary c-game. Applying Lemma 1 we obtain

$$\phi_i(\mathcal{U}_T) = \begin{cases} \lambda_{|T|}, & \text{if } i \in n(T) \\ \mu_{|T|}, & \text{if } i \notin n(T) \end{cases}$$

for every $i \in N$. Because of the average efficiency axiom, we obtain, for $T \in \mathcal{C}(N)$ such that $n(T) = N$

$$\sum_{i \in n(T)} \phi_i(\mathcal{U}_T) = \frac{1}{(n-1)!} = n\lambda_n$$

and

$$\sum_{i \in n(T)} \phi_i(\mathcal{U}_T) = 0 = |T|\lambda_{|T|} + (n - |T|)\mu_{|T|}$$

otherwise. From these equations we obtain $\lambda_n = 1/n!$ and $|T|\lambda_{|T|} = -(n - |T|)\mu_{|T|}$ for $|T| \in \{1, \ldots, n-1\}$. Doing a change of variable $|T|\lambda_{|T|} = \beta_{|T|}$ we have

$$\phi_i(q) = \frac{\eta(q, N)}{n} + \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \neq N \\ n(T) \ni i}} \beta_{|T|} \frac{q(T)}{|T|} - \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \not\ni i}} \beta_{|T|} \frac{q(T)}{n - |T|} \tag{3}$$

with $\{\beta_{|T|}\}_{|T|=1}^{n-1} \in \mathbb{R}$. This is the analog of the Ruiz, Valenciano and Zarzuelo formula mentioned above.

With algebraic manipulation we obtain

$$\sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \not\ni i}} \beta_{|T|} \frac{q(T)}{n - |T|} = \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \beta_{|T|-1} \frac{q(T_{-i})}{(n - |T| + 1)(|T| - 1)}.$$

Solution (3) holds for arbitrary c-games and for all kinds of nodes. Considering a c-game $q \in F_g$ such that the player $i \in N$ is a null node, and the previous expression, we have

$$\varphi_i(q) = \frac{\eta(q, N)}{n} + \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \neq N \\ n(T) \ni i}} \beta_{|T|} \frac{q(T)}{|T|} - \sum_{\substack{T \in \mathcal{C}(N) \\ n(T) \ni i}} \beta_{|T|-1} \frac{q(T)}{(n - |T| + 1)(|T| - 1)} = 0.$$

Again, applying algebraic operations we have

$$\varphi_i(q) = \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)=N}} \frac{q(T)}{n}\left(\frac{1}{(n-1)!} - \beta_n\right) + \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\ni i}} \left(\frac{\beta_{|T|}}{|T|} - \frac{\beta_{|T|-1}}{(n-|T|+1)(|T|-1)}\right) q(T) = 0$$

which implies

$$\frac{1}{(n-1)!} - \beta_n = 0 \text{ and } \frac{\beta_{|T|}}{|T|} - \frac{\beta_{|T|-1}}{(n-|T|+1)(|T|-1)} = 0 \quad \forall |T| \in \{1,\ldots,n-1\}.$$

We suppose $\beta_{|T|} = |T|(n-|T|)!/n!$ and we prove it by induction. For the previous equations, the assumption is true for $|T| = n$. Now we suppose that is true for $|T| = s$, $s \in \{2,\ldots,n\}$ and we prove it for $|T| = s-1$. Again, for the previous equations we have

$$\beta_{|T|-1} = \frac{\beta_{|T|}(n-|T|+1)(|T|-1)}{|T|} = \frac{(n-(|T|-1))!(|T|-1)}{n!}.$$

With this, we have proved the uniqueness. □

In the same way as Ruiz *et al.* [1988] we could obtain a family of values satisfying a subset of the previous axioms.

Proposition 1. *A solution $\varphi : F_g \to \mathbb{R}^n$ for c-games satisfies linearity, symmetry and average efficiency axioms if and only if, for all $i \in N$*

$$\varphi_i(q) = \frac{\eta(q,N)}{n} + \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\neq N\\ n(T)\ni i}} \beta_{|T|}\frac{q(T)}{|T|} - \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\not\ni i}} \beta_{|T|}\frac{q(T)}{n-|T|} \tag{4}$$

with $\{\beta_{|T|}\}_{|T|=1}^{n-1} \in \mathbb{R}$.

Proof. It is explained in the proof of Theorem 1. □

The solution given in (1) is a special case of (4) with

$$\beta_{|T|} = \frac{|T|(n-|T|)!}{n!}.$$

Similar to the Shapley value, the value proposed in (1) has a probabilistic interpretation. Let us choose a cycle $T \in \mathcal{C}(N)$ randomly. First, we choose the length $|T|$ of a cycle with uniform distribution among all possible lengths, excluding the length equal to zero (1 to n). After that, we choose a cycle $T \in \mathcal{C}(N)$ with length $|T|$ which contains node i, again with uniform distribution among all the cycles which satisfy that condition, and we assign as its importance its marginal contribution

$(q(T)-q(T_{-i}))$. Then, the expected value that node i gets with this process is equal to the solution given by the previous theorem for node i.

Also, the solution (1) has an interpretation considering a random cycle's formation process. First, select a permutation of the nodes from a uniform distribution of all possible permutations. In the sequel, we will consider nodes as persons. Suppose that the nodes arrive randomly at a table; the first node sits on any chair. When another node arrives, it sits on the chair to the right of the last node. This process generates a sequence of counterclockwise cycles around the table. That is, given an order $\mathcal{R} \in S_n$ of the nodes, $\mathcal{R} = j_1\ j_2\ \dots\ j_{k-1}\ i\ j_{k+1}\ \dots\ j_{n-1}\ j_n$, we define $\mathcal{P}_i^{\mathcal{R}} = \{(j_1, j_2), \dots, (j_{k-1}, i), (i, j_1)\}$, the cycle made up of the predecessors' nodes of node i, including it. We assign as the importance of the node i its marginal contribution to the cycle $\mathcal{P}_i^{\mathcal{R}}$. The expected value that node i gets according to this process is the solution given by (1). Thus, solution (1) can be written as

$$\phi_i(q) = \frac{1}{n!}\left(\sum_{\mathcal{R}\in S_n} [q(\mathcal{P}_i^{\mathcal{R}}) - q((\mathcal{P}_i^{\mathcal{R}})_{-i})]\right)$$

because $|\{\mathcal{R} \in S_n \mid \mathcal{P}_i^{\mathcal{R}} = T\}| = (n - |T|)!$ for all $T \in \mathcal{C}(N)$.

The solution given in (1) could also be derived as a Shapley value for games in characteristic function form. Let $q \in F_g$ be a c-game; let $v^q : 2^N \to \mathbb{R}$ be a game in characteristic function form defined, for every subset $S \subseteq N$, as follows:

$$v^q(S) = \eta(q, S).$$

This game represents, for each $S \subseteq N$, the average worth obtained by visiting only the nodes in S in every possible way. With this game we obtain the next proposition:

Proposition 2. *Let $\varphi : F_g \to \mathbb{R}^n$ be the linear, symmetric, null and average efficient solution for c-games; then we have that*

$$\varphi_i(q) = \mathrm{Sh}_i(v^q), \quad \forall\, i \in N$$

for all $q \in F_g$, where Sh *is the Shapley value.*

Proof. Because of the definition of v^q and the Shapley value expression, we obtain, for all $S \subseteq N$

$$\mathrm{Sh}_i(v^q) = \sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!}\ [\eta(q, S) - \eta(q, S\backslash\{i\})],$$

where $s = |S|$. If we substitute the definition of η, we obtain

$$\sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!}\ \eta(q, S) = \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\ni i}} \frac{(n-|T|)!}{n!} q(T)$$

and

$$\sum_{S\ni i}\frac{(s-1)!(n-s)!}{n!}\eta(q,S\backslash\{i\}) = \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\not\ni i}}\frac{|T|(n-|T|-1)!}{n!}q(T)$$

$$= \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\ni i}}\frac{(n-|T|)!}{n!}q(T_{-i}),$$

where the last equality is obtained using

$$q(T) = \frac{1}{|T|}\sum_{\substack{C\in\mathcal{C}(N)\\ n(C)\backslash\{i\}=n(T)}} q(C_{-i})$$

for every $T\in\mathcal{C}(N)$ such that $n(T)\not\ni i$ (because of the definition of T_{-i}). Substituting the previous expressions into the Shapley value formula, we obtain the result. □

The last proposition gives us an important property of (1): it is equivalent to an "average worth game" in characteristic function form, in which the worth of each subset $S\subseteq N$ is the average worth that the nodes in S can obtain if a cycle that visits only nodes in S is formed.

For each $T\in\mathcal{C}(N)$ we can form a game $(n(T),v_T)$ in characteristic function form as follows:

$$v_T(S) = q(T_S)\quad \forall\, S\subseteq n(T).$$

This is a game restricted to the cycle T. Now, we have the next result.

Proposition 3. *Let $q\in F_g$ be a c-game. For each $T\in\mathcal{C}(N)$ we form the game in characteristic function form $(n(T),v_T)$ given by $v_T(S)=q(T_S)$ for all $S\subseteq n(T)$. If $\varphi: F_g\to\mathbb{R}^n$ satisfies linearity, symmetry, nullity and average efficiency axioms then*

$$\varphi_i(q) = \frac{1}{(n-1)!}\sum_{T\in\mathcal{C}_N(N)}\mathrm{Sh}_i(v_T)\quad\forall\, i\in N. \tag{5}$$

Proof. We prove the last proposition showing that (5) satisfies the four axioms which characterize (1). First, we prove the average efficiency axiom. The Shapley value is efficient, so we have, for each $T\in\mathcal{C}_N(N)$

$$\sum_{i\in N}\mathrm{Sh}_i(v_T) = q(T)$$

and because $|\mathcal{C}(N)| = (n-1)!$ we show that the expression is average efficient. The others axioms can be proved in a direct way and we omit their proof. □

Proposition 3 shows another probabilistic process for obtaining solution (1): We choose a cycle $T\in\mathcal{C}(N)$ which contains the whole set of nodes with uniform

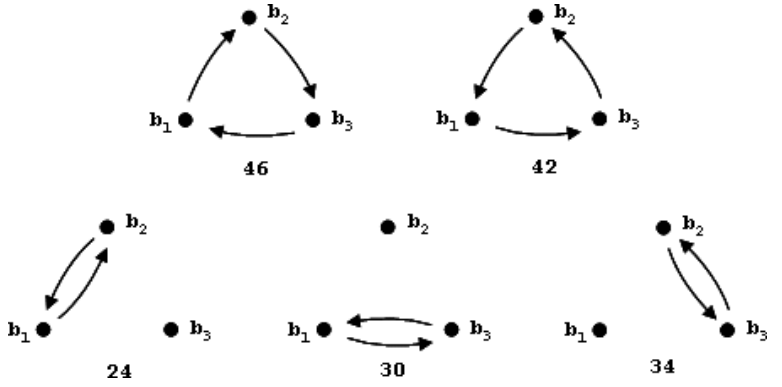


Fig. 1. The set of possible cycles if the branches are the nodes of a complete digraph. Below each cycle, the value $q(C)$ of each cycle C, for Example 1, is indicated.

distribution, and we measure, for each node, its importance in the restricted game $(n(T), v_T)$. The expectation value of its importance under this process is the solution given by (1).

Example 1. Let us consider a redistribution in a chain of stores problem with three branches (b_1, b_2 and b_3). Transportation costs between branch i (a row) and branch j (a column), is depicted in the next table:

	b_1	b_2	b_3
b_1	0	12	14
b_2	12	0	18
b_3	16	16	0

With this information, we can form the corresponding c-game that indicates the cost for visiting the branches in the order given by the cycles. This c-game is depicted in Figure 1.

Now, calculating the solution given by (1) (branch b_i corresponds with node i) we obtain

$$\varphi(q) = (37/3, 43/3, 52/3). \tag{6}$$

According to this solution, the branch b_3 must pay a higher value than the other branches. Also, b_2 must pay more than b_1. The solution (6) denotes a measurement of how each branch contributes to the redistribution costs.

4. r-Efficient Solutions

Now, we study the situation when the problem consists of distributing a real number r among the nodes of a digraph taking into account their importance in a c-game. For example, it could be necessary to distribute the cost due to operation among the branches in the redistribution in a chain of stores situation. So, the problem

is defined as a pair (q, r) where $q \in F_g$ and $r \in \mathbb{R}$, and denotes a c-game q with allocation amount r. Then, it is necessary to modify the original model. In this section we find characterized solutions $\varphi : F_g \times \mathbb{R} \to \mathbb{R}^n$, where $\varphi_i(q, r)$ represents the payoff for the node i where the allocation amount is $r \in \mathbb{R}$.

We present an axiomatic characterization of a solution for this new situation. Most of the presented axioms have a similar interpretation and motivation as the set of axioms presented previously.

For $T \in \mathcal{C}(N), q_1, q_2 \in F_g$ and $\alpha, \beta \in \mathbb{R}$ we define $(q_1, r)(T) = q_1(T)$ and

$$(q_1, \alpha)(T) + (q_2, \beta)(T) = (q_1 + q_2, \alpha + \beta)(T).$$

Axiom 5. (Linearity) A solution φ is a linear solution if for all c-games $q_1, q_2 \in F_g$ and $\alpha, \beta, \lambda_1, \lambda_2 \in \mathbb{R}$

$$\lambda_1 \varphi(q_1, \alpha) + \lambda_2 \varphi(q_2, \beta) = \varphi(\lambda_1 q_1 + \lambda_2 q_2, \lambda_1 \alpha + \lambda_2 \beta).$$

This axiom establishes that the sum of the payoffs for two c-games with different allocation amounts must be equal to the payoff of the the sum c-game with an allocation amount equal to the sum of the allocation amounts of each game.

Axiom 6. (Symmetry) A solution φ is a symmetric solution if

$$\varphi_i(q, r) = \varphi_{\theta(i)}(q \circ \theta, r)$$

for all $q \in F_g$, $r \in \mathbb{R}$ and $\theta \in S_n$.

Axiom 7. (r-efficiency) A solution φ is an r-efficient solution if

$$\sum_{i \in N} \varphi_i(q, r) = r$$

for all $q \in F_g$ and $r \in \mathbb{R}$.

This last axiom represents the main variation in comparison with c-games introduced in Sec. 3: the allocation amount is a real number r. This number can represent the maximum worth of a cycle, the worth of a fixed cycle or an external allocation amount determined by a regulating agent, for example.

Axiom 8. (r-nullity) A solution φ is an r-null solution if, for all $i \in N$ such that i is a null node and $r = \eta(q, N)$

$$\varphi_i(q, r) = 0$$

for all $q \in F_g$ and $r \in \mathbb{R}$.

According to this axiom, a null node receives a zero payoff only if the fixed allocation amount r is exactly the average worth of the cycles which visit the complete set of nodes. If this situation occurs, the problem can be formulated as a standard c-game presented in the previous section. According to the solution presented for these problems, the null nodes receive a zero payoff. Again, we consider that the requirements of the process can change, so it could be necessary to take into account the profit of all complete cycles.

Theorem 2. *There exists a unique solution that satisfies linearity, symmetry, r-efficiency and r-nullity axioms. Furthermore, this solution is given by*

$$\phi_i(q,r) = \frac{r-\eta(q,N)}{n} + \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\ni i}} \frac{(n-|T|)!}{n!}[q(T)-q(T_{-i})] \tag{7}$$

for all $i \in N$, $q \in F_g$ *and* $r \in \mathbb{R}$.

Proof. It is direct to check that (7) satisfies the set of cited axioms and this proves existence of the solution. For proving uniqueness, we consider a solution $\varphi : F_g \times \mathbb{R} \to \mathbb{R}^n$. If φ satisfies the linearity axiom we have

$$\varphi_i(q,r) = \varphi_i(q_0,r) + \varphi_i(q,0) \quad \forall\, i \in N, q \in F_g, r \in \mathbb{R}, \tag{8}$$

where $q_0 : \mathcal{C}(N) \to \mathbb{R}$ is the c-game that assigns a zero worth to each cycle. Thus, applying symmetry and r-efficiency axioms we get

$$\varphi_i(q_0,r) = \frac{r}{n}.$$

For the linearity axiom we have

$$\varphi_i(q,0) = \sum_{T\in\mathcal{C}(N)} q(T)\varphi(\mathcal{U}_T,0).$$

Applying Lemma 1 we obtain

$$\varphi_i(\mathcal{U}_T,0) = \begin{cases} \lambda_{|T|}, & \text{if } i \in n(T) \\ \mu_{|T|}, & \text{if } i \notin n(T). \end{cases}$$

For the r-efficiency axiom we obtain, for $T \in \mathcal{C}(N)$ such that $n(T) = N$

$$\sum_{i\in n(T)} \phi_i(\mathcal{U}_T,0) = n\lambda_n = 0$$

and

$$\sum_{i\in n(T)} \phi_i(\mathcal{U}_T,0) = 0 = |T|\lambda_{|T|} + (n-|T|)\mu_{|T|}$$

otherwise. With these expressions we have $\lambda_n = 0$. This is the main difference with the proof of Theorem 1. In the sequel the proof is the same. So, we obtain

$$\varphi_i(q,0) = \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\neq N\\ n(T)\ni i}} \beta_{|T|}\frac{q(T)}{|T|} - \sum_{\substack{T\in\mathcal{C}(N)\\ n(T)\not\ni i}} \beta_t \frac{q(T)}{n-|T|}$$

with $\{\beta_{|T|}\}_{|T|=1}^{n-1} \in \mathbb{R}$. Moreover, if we substitute the two past expressions in (8) and supposing φ satisfies the r-nullity axiom, we obtain $\beta_{|T|} = |T|(n-|T|)!/n!$, applying the same procedure as Theorem 1. Again, with algebraic manipulation we obtain (7), and so the proof finishes. □

In the previous solution we can appreciate how the allocation process of the payoffs is done: first, we pay each node it importance on the digraph according to (1). After that, it is necessary to distribute the excess or the loss due to the allocation amount r, i.e., $r - \eta(q, N)$. That quantity will be distributed in equal parts among all nodes, in a fair way.

5. Coalitional Structures

Following with the motivation introduced by Dantzig in Lawler [1976], it makes sense that ports are grouped in fixed structures according to their characteristics or interests, for example, into countries or trade unions. So, we extend the model introducing coalitional structure concepts, and it will be necessary to provide a new set of axioms for obtaining a characterization of a solution for a c-game with this modification.

Let $B = \{B_1, B_2, \ldots, B_m\}$ be a partition of the set of nodes in m groups. We say B is a *coalitional structure of* N if $B_i \cap B_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$ and $\bigcup_{i=1}^{m} B_i = N$. We suppose that nodes are organized into a coalitional structure B, and we denote by $\mathfrak{B}_N$ the set of all possible coalitional structures of N. A c-game $q \in F_g$ where the nodes are organized according to $B \in \mathfrak{B}_N$ is represented by a pair (q, B) with $(q, B)(T) = q(T)$ for each $T \in \mathcal{C}(N)$. So, we want to characterize solutions $\varphi : F_g \times \mathfrak{B}_N \to \mathbb{R}^n$, where $\varphi_i(q, B)$ represents the importance of the node $i \in N$ such that the nodes are organized according to $B \in \mathfrak{B}_N$.

The characterization includes a short modification of the set of axioms presented in the last section. Also, we follow the idea of Owen [1977]; let φ be a solution for c-games with coalitional structure $B \in \mathfrak{B}_N$:

Axiom 9. (Linearity in F_g) A solution φ is a linear solution on the space of c-games if

$$\varphi(\alpha q_1 + \beta q_2, B) = \alpha\varphi(q_1, B) + \beta\varphi(q_2, B)$$

for all $q_1, q_2 \in F_g, \alpha, \beta \in \mathbb{R}$.

Axiom 10. (Nullity) A solution φ is a null solution if, for every null node $i \in N$ in $q \in F_g$

$$\varphi_i(q, B) = 0.$$

Given a coalitional structure $B \in \mathfrak{B}_N$ we can form a digraph $(B, B \times B)$ where the nodes are the coalitions in B. For each $T \in \mathcal{C}(N)$ we define

$$\zeta_T := \{(B_i, B_j) \mid \exists\, (x, y) \in T \text{ with } x \in B_i, y \in B_j, B_i \neq B_j\}$$

if $n(T) \not\subseteq B_k$ for some $B_k \in B$ and $\zeta_T := \{(B_k, B_k)\}$ otherwise. Notice that ζ_T is not necessarily a cycle.

We say $T \in \mathcal{C}(N)$ induces a cycle on the coalitional structure B if $\zeta_T \in \mathcal{C}(B)$. The set of cycles on a digraph with set of nodes N which induce cycles on $(B, B \times B)$

is represented by $\mathcal{C}^B(N) := \{T \in \mathcal{C}(N) \mid \zeta_T \in \mathcal{C}(B)\}$. Also, for all $S \subseteq N$, the set $\mathcal{C}^B_S(N) := \{T \in \mathcal{C}^B(N) \mid n(T) = S\}$ represents the subset of cycles induced on $(B, B \times B)$ that contain only nodes in S.

Axiom 11. (Coalitional average efficiency) A solution φ is a coalitional average efficient solution if

$$\sum_{i=1}^{n} \varphi_i(q, B) = \left(\frac{1}{|\mathcal{C}^B_N(N)|}\right) \sum_{T \in \mathcal{C}_N(B)} q(T).$$

According to the previous axiom, the allocation amount will be the average worth of the cycles induced on the coalitional structure B that visit all nodes in every possible way. This axiom has a similar motivation as Axiom 4: if all nodes want to form a cycle that consider each coalition on B as a single node there are $|\mathcal{C}_N(B)|$ forms to do it.

Let $\theta \in S_n$ be a transposition between i and j; i.e., $\theta(i) = j, \theta(j) = i$ and $\theta(h) = h$ for all $h \in N\backslash\{i, j\}$ where $\{i, j\} \subseteq B_k \in B$. We say $i, j \in B_k$ are *substitute nodes* in a c-game q if for all $T \in \mathcal{C}^B(N)$ such that $i \in n(T)$ and $j \notin n(T)$, $q(T) = q(\theta(T))$.

Axiom 12. (Substitute nodes) A solution φ satisfies the substitute nodes axiom if for each pair of substitute nodes $\{i, j\} \subseteq B_k \in B$ in the c-game q

$$\varphi_i(q, B) = \varphi_j(q, B).$$

This axiom can be interpreted as follows: for $i, j \in B_k$, if the worth of each cycle $T \in \mathcal{C}^B(N)$ which contains node i is equal to the worth of the cycle when the node j substitutes to node i, then the nodes i, j have the same importance.

Let $\{B_p, B_r\}$ be coalitions on B. If for all $S \subseteq B\backslash\{B_p, B_r\}$ the expression

$$\frac{1}{|\mathcal{C}^B_{N^p_S}(N)|} \sum_{T \in \mathcal{C}^B_{N^r_S}(N)} q(T) = \frac{1}{|\mathcal{C}^B_{N^r_S}(N)|} \sum_{T \in \mathcal{C}^B_{N^p_S}(N)} q(T) \quad \text{with } A^S = \bigcup_{B_k \in S} B_k$$

and $N^p_S = A_S \cup B_p, N^r_S = A_S \cup B_r$ holds for a c-game q, then B_p, B_r are *substitute coalitions* on q.

Axiom 13. (Substitute coalitions) A solution φ satisfies the substitute coalitions axiom if for all $\{B_p, B_r\} \subset B$ substitute coalitions in a c-game q

$$\sum_{i \in B_p} \varphi_i(q, B) = \sum_{i \in B_r} \varphi_i(q, B)$$

for all $B \in \mathfrak{B}_N$.

The previous axiom has a similar interpretation as Axiom 12, but we apply it considering each group in the coalitional structure as a single node.

For our next result, we introduce new notation. Let $B \in \mathfrak{B}_N$ be a coalitional structure and $T \in \mathcal{C}(N)$.

- $B_T := \{B_k \in B \mid B_k \cap n(T) \neq \emptyset\}$, the set of coalitions in B visited by a cycle T. The cardinality of B_T is denoted by $|B_T|$.
- $\pi(B_T) := \prod_{B_k \in B_T} |B_k|!$, the number of cycles in $\mathcal{C}(N)$ which visit all the nodes in each visited coalition and which induce a cycle $\zeta_T \in \mathcal{C}(B)$.
- For all $S \subseteq B_k \in B$, $B(S) := \{B_1, \ldots, B_{k-1}, S, B_{k+1}, \ldots, B_m\}$ represents the coalitional structure when the subset S represents to B_k.
- $\mathcal{C}_B^S(N) := \{T \in \mathcal{C}(N) \mid \zeta_T \in \mathcal{C}(B(S)) \text{ and } B_k \subseteq n(T)\ \forall B_k \in B_T\}$ the subset of cycles on $(N, N \times N)$ which induce a cycle on the coalitional structure $B(S)$ and that visit all the nodes of each coalition in B_T.

Theorem 3. *There exists a unique solution that satisfies linearity in F_g, nullity, coalitional average efficiency, substitute nodes and substitute coalitions axioms. Furthermore, this solution is given by*

$$\phi_i(q, B) = \sum_{\substack{S \subset B_k \\ S \ni i}} \sum_{T \in \mathcal{C}_B^S(N)} \frac{(|B_k| - s)!(s-1)!}{|B_k|!} \cdot \frac{(m - |B(S)_T|)!}{m!} \times \left(\frac{1}{\pi(B(S)_T)}\right) [q(T) - q(T_{-i})] \tag{9}$$

for all $i \in N$ with $i \in B_k \in B$ and $q \in F_g$.

Proof. First, we prove uniqueness. For $S, T \in \mathcal{C}(N)$, we say S is a reduced cycle of T, and we denote it as $S \sqsubseteq T$, if there exist $P \subseteq n(T)$ such that $S = T_{-P}$. So, given the coalitional structure B, for $T \in \mathcal{C}(N)$ we define the unanimity c-game $(\mathcal{W}_T, B)$ as $(\mathcal{W}_T, B)(S) = 1$ if $T \sqsubseteq S$ and $(\mathcal{W}_T, B)(S) = 0$ for the other cases, for each $S \in \mathcal{C}(N)$. This game forms a basis of the vector space of c-games. Let $\varphi : F_g \times \mathfrak{B}_N \to \mathbb{R}^n$ be a solution for c-games with coalitional structure; applying the linearity axiom we obtain, for each $i \in N$

$$\varphi_i(q, B) = \sum_{T \in \mathcal{C}(N)} \lambda_T \varphi_i(\mathcal{W}_T, B)$$

with $\lambda_T \in \mathbb{R}$ for each $T \in \mathcal{C}(N)$. With this expression, it is necessary to prove uniqueness of the solution for each unanimity c-game. For $T \in \mathcal{C}^B(N)$, if $i \notin n(T)$ then i is a null node and $\varphi_i(\mathcal{W}_T, B) = 0$; applying the substitute node axiom we obtain $\varphi_i(\mathcal{W}_T, B) = \varphi_j(\mathcal{W}_T, B)$ for all $i, j \in B_k \in B$ with $i, j \in n(T)$ because $\{i, j\}$ are substitute nodes in $\mathcal{W}_T$; with substitute coalition axioms we obtain

$$\sum_{i \in B_r} \varphi_i(\mathcal{W}_T, B) = \sum_{i \in B_S} \varphi_i(\mathcal{W}_T, B) \quad \forall B_r, B_s \in B_T.$$

Finally, by the coalitional average efficiency axiom we obtain $\varphi_i(\mathcal{W}_T, B) = 0$ if $T \notin \mathcal{C}^B(N)$ and

$$\varphi_i(\mathcal{W}_T, B) = \begin{cases} \dfrac{1}{|B_T| \cdot (n(T) \cap B_k)} & \text{if } i \in n(T) \\ 0 & \text{in another case,} \end{cases}$$

where $i \in B_k \in B$. Solution (9) can be described in a two-step process. For each $S \subseteq B_k \in B$ we can define a digraph $(B(S), B(S) \times B(S))$. With this new structure we define a c-game on its cycles $q_S^B(T)$, where for all $T \in \mathcal{C}(B(S))$

$$q_S^B(T) = \frac{1}{|\mathcal{C}_T(N, B(S))|} \sum_{C \in \mathcal{C}_T(N, B(S))} q(C),$$

where $\mathcal{C}_T(g, B(S)) := \{C \in \mathcal{C}_B^S(N) \mid \zeta_C = T\}$ represents the subset of cycles that induce the cycle $T \in \mathcal{C}(B(S))$. Now, we form a game for each group, $v_B : 2^{B_k} \to \mathbb{R}$, as follows:

$$v_B(S) = \sum_{i \in S} \phi_i(q_S^B) \quad \forall\, S \subseteq B_k \in B,$$

where ϕ corresponds to (1). Now, we apply the Shapley value to v_B, and with algebraic manipulation we obtain

$$\mathrm{Sh}(v_B) = \phi(q, B).$$

With this last expression we can check, because of the properties of (1) and the properties of the Shapley value, that (9) satisfies the set of axioms required by the theorem. So, this proves the existence of the solution and we finish the proof. □

The solution (9) has a similar interpretation as the Owen value has for games in characteristic function form with coalitional structures as a two-step process. First, we measure the importance of each subset $S \subseteq B_k \in B$ for each group in the coalitional structure. We replace the coalition B_k in B by S obtaining a new cooperation structure $B(S)$. With that, we form a c-game on the groups of $B(S)$ considering each group as a single node. Then, we apply the solution given in (1). That first step give us a value for each subset in every group in the coalitional structure. So, in the second step, we can apply a solution over each $B_k \in B$ considering a game in characteristic function form over each group. In the solution given by (9) we apply the Shapley value of those games.

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Regresar

EL CONCEPTO DE SENSIBILIDAD PARA EL CASO DISCRETO DEL PROBLEMA DE REPARTO DE COSTOS DE PRODUCCIÓN*

SENSIBILITY CONCEPT FOR THE DISCRETE CASE IN THE COST SHARING PROBLEM

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Resumen

Supongamos que se produce un vector en forma conjunta. El problema de reparto de costos es el como dividir el costo de producción de este vector entre sus componentes. En este artículo se caracteriza axiomáticamente una solución para este tipo de problemas basados en dos axiomas: s-sensibilidad y separabilidad. El axioma de s-sensibilidad establece como debe cambiar la solución cuando ocurren pequeños cambios en la función de costos y el axioma de separabilidad tiene la conotación tradiconal. Se estudia el caso discreto y se obtiene una solución de manera axiomática.

Palabras clave: distribución de costos, soluciones axiomáticas, soluciones inducidas por caminos, juegos cooperativos, sensibilidad.

Abstract

Suppose that a vector is produced jointly. We will consider the problem of dividing the cost of production among a group of components. In this paper we characterized a solution through two axioms, s-sensibility and separability. The sensibility is a concept related to the variations of a solution due to slight changes of the cost function and the separability is standard. We study the discrete cost sharing problem. We provide a axiomatic solution.

Keywords: cost sharing, axiomatic solutions, path-induced solutions, cooperative games, sensibility.

Mathematics Subject Classification: 91A12, 91A40.

1 Introducción

En este trabajo estudiamos el problema de reparto de costos de producción en el caso discreto; entendiéndose dicho problema como aquel donde un conjunto de agentes se unen para producir, ya sea porque hacerlo conjuntamente puede abaratar los costos o porque exista la necesidad de producir así. Cada agente tiene una demanda de producción y el tipo de bienes que se generan pueden ser distintos entre ellos. Aunque se puede pensar que se tienen rendimientos crecientes a escala (lo cual motivaría la producción conjunto) no se supone, ni tampoco lo opuesto.

Nosotros solucionamos el problema de manera axiómatica, es decir, pedimos que las soluciones satisfagan un conjunto de *supuestos generales* que nos permitan rechazar o aceptar soluciones hasta obtener unívocamente *la solución.*

Una herramienta apropiada para modelar los problemas de repartición de costos son los llamados juegos de elección multiple, un ejemplo de esto lo tenemos en [3].

La aportación más importante en este artículo es la de relacionar soluciones entre problemas *parecidos*; así surge el concepto de *sensibilidad*.

En [1] el lector encontrará soluciones para nuestro problema pero en el caso especial de dos jugadores, ahí se define un juego bipersonal y se obtienen soluciones que son equilibrios de Nash, es trivial verificar que la solución del presente trabajo también es un equilibrio de Nash en aquel contexto.

Por otra parte, en [4] se describen algunos métodos de distribución de costos que se relacionan con nuestro trabajo: *los métodos por caminos*.

En la siguiente sección, presentamos el modelo matemático y en la Sección 3, mostramos el principal resultado de este trabajo; una caracterización con dos axiomas. En la Sección 4, tenemos las conclusiones.

2 Preliminares

2.1 Definiciones y notación

En esta sección describimos el modelo matemático de nuestro problema. Sea $N = \{1, \ldots, n\}$ un conjunto finito de agentes, y $m = (m_1, \ldots, m_n)$ el vector de demandas, donde cada m_i es un entero mayor que cero, que representa la cantidad que requiere producir el agente i. Además sea,

$$M = \{0, 1, \ldots, m_1\} \times \cdots \times \{0, 1, \ldots, m_n\}$$

y $M^o = M \backslash \{\overline{0}, m\}$ donde $\overline{0} = (0, \ldots, 0) \in \mathbb{R}^n$.

Dada la pareja (N, m) la cual supondremos fija a lo largo del trabajo, definimos el espacio de problemas

$$G_{N,m} = \{c : M \to \mathbb{R} | c(\overline{0}) = 0\}.$$

$G_{N,m}$ es el espacio de problemas[1] para n agentes con vector de demanda m. Definiendo la suma y el producto por escalar de manera usual tenemos que $G_{N,m}$ es un espacio vectorial. Una solución es una aplicación

$$\varphi : G_{N,m} \to \mathbb{R}^n,$$

[1]De hecho, a cada función c también se le conoce como función de costos pero durante el desarrollo de este trabajo les llamaremos problemas.

de manera que la componente $\varphi_i(c)$ significa el pago asignado al agente i. A cada $r = (r_1, \ldots, r_n) \in M$ le llamaremos *nodo* o *vector de producción parcial* si $r \neq m$ y *vector de producción total* si $r = m$.

Ejemplo 2.1 *Supongamos que en una empresa editorial, se tienen que producir de manera conjunta un tiraje de* 3000 *unidades de un nuevo libro de texto y* 4000 *catálogos, el agente* 1 *es el responsable de la producción de libros mientras que el agente* 2*, de los catálogos; las producciones se dan en lotes de millares, es decir* $N = \{1, 2\}$ *y* $m = (3, 4)$*. Los costos de producción por millar en cierta unidad monetaria se listan a continuación*

$$c = \begin{bmatrix} 0 & 8 & 14 & 15 \\ 12 & 19 & 20 & 25 \\ 20 & 22 & 26 & 28 \\ 21 & 23 & 30 & 33 \\ 24 & 29 & 35 & 40 \end{bmatrix}$$

donde $c(i, j)$ *es el costo conjunto de producir* i *millares de libros para el agente* 1 *y* j *millares de catálogos para el agente 2; el costo total de producción conjunta es* $c(3, 4) = 40$*. Note que tanto las columnas, como los renglones, se indexan desde cero.*

3 Una solución obtenida con dos axiomas

El resultado principal de este trabajo es una solución obtenida, pidiendo que esta tenga dos propiedades – $\widehat{s}$*-sensibilidad* y *separabilidad* – que a continuación definimos y explicamos.

Primero, presentamos una base para el espacio de problemas. Para cada $r \in M \setminus \{\overline{0}\}$ definimos el problema $E^r : M \to \mathbb{R}$ como sigue:

$$E^r(y) := \begin{cases} 1, & \text{si } y = r \\ 0, & \text{si } y \neq r. \end{cases}$$

Es claro que $\{E^r\}_{r \in M \setminus \{\overline{0}\}}$ es una base para el espacio de problemas.

3.1 Sensibilidad.

A continuación introducimos el principal concepto en este trabajo.

Sea $s : M^0 \to \mathbb{R}^n$ una función dada por $s_k((r_1, \ldots, r_n)) := \frac{r_k}{m_k}$ con $k \in N$. Es decir para cada nodo $r = (r_1, \ldots, r_n)$ de producción parcial, s_k asigna un valor para el $k - ésimo$ agente que es creciente con respecto

a su producción parcial, por ejemplo si $r_k = m_k$ entonces el agente k ya cumplió con su demanda total de producción por lo que r sería un nodo muy importante para él, y así s_k asignaría el valor de 1.A su vez si existe j tal que $r_j = 0$ entonces el agente j podría tener poco interés por el nodo r, en este caso el valor de s_k sería de cero.

Se pueden definir más funciones como la anterior que modelen criterios de "asignación de importancia" a los nodos por parte de los agentes, por ejemplo,si se consideran ahora la proporción de unidades producidos en el nodo r con respecto a la demanda total $\sum_{i\in N} m_i$ o a la producción parcial conjunta $\sum_{i\in N} r_i$ tendríamos respectivamente las funciones s^1 y s^2 definidas por

$$s_i^1(r) = \frac{r_i}{m_1+\ldots+m_n} \quad y \quad s_i^2(r) = \frac{r_i}{r_1+\ldots+r_n}.$$

Al conjunto de todas las funciones con dominio M^0 y contradominio $\mathbb{R}^n$ lo denotaremos con S y a cada $s \in S$ le llamaremos una función de sensibilidad y $s_k(r)$ se interpreta como la valoración que el k-ésimo agente le está dando al nodo r.

Ahora estamos en condiciones de definir soluciones s-sensibles:

Definición 3.1 *Sea $\varphi : G_{N,m} \to \mathbb{R}^n$ una solución. Dada $s \in S$, decimos que φ es s-sensible, si para todo $c \in G_{N,m}$,*

$$\frac{\partial\varphi(c)}{\partial E^r} = s(r)\ \forall r \in M^o \text{ donde } \frac{\partial\varphi(c)}{\partial E^r} := \lim_{h\to 0} \frac{\varphi(c+hE^r)-\varphi(c)}{h}.$$

Esta propiedad indica que la variación de φ en cada nodo distinto de m queda determinado por s. En particular, si φ es una solución lineal, entonces $\varphi(E^r) = s(r)\ \forall r \in M^o$.

3.1.1 Soluciones s-sensibles

Dadas $s \in S$ y $c \in G_{N,m}$, definimos:

$$\varphi^{h,s}(c) := h(c(m)) + \sum_{r\in M^o} c(r)s(r)$$

donde h es una función arbitraria.

Teorema 3.1 *Dada $s \in S$, φ es s-sensible sí y sólo sí $\varphi = \varphi^{h,s}$, para alguna $h : \mathbb{R} \to \mathbb{R}^n$.*

Prueba: Si $\varphi(c) = h(c(m)) + \sum_{x \in M^0} c(x)s(x)$ para alguna h, entonces, si $r \in M^0$

$$\frac{\partial \varphi}{\partial E^r}(c) = s(r).$$

Para el recíproco, supongamos que $\varphi = (\varphi_1, \ldots, \varphi_n)$ es s-sensible.

En primer lugar, observe que si $f : \mathbb{R}^k \to \mathbb{R}$ es una función diferenciable, tal que para toda $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ se tiene que $\frac{\partial f}{\partial x_r}(x) = s_r$, $r : 1, \ldots, k-1$, donde las s_r son constantes, entonces -por el teorema del valor medio de cálculo-

$$f(x_1, \ldots, x_k) = s_1 x_1 + \cdots + s_{k-1} x_{k-1} + g(x_k)$$

para alguna función diferenciable g. Ahora, $\frac{\partial \varphi}{\partial E^r}(c) = s(r)$, para toda c y para toda $r \in M^0$, se traduce en que para cada $i : 1, \ldots, n$

$$\frac{\partial \varphi_i}{\partial E^r}(c) = s_i(r)$$

y por lo tanto, según lo anterior

$$\varphi_i(c) = \sum_{r \in M^0} c(r)s_i(r) + h_i(c(m))$$

para alguna función diferenciable h_i. ■

Corolario 3.1 *φ es lineal y s-sensible si, y sólo si $\varphi = \varphi^{h,s}$ con h lineal.*

3.2 Separabilidad

Existen problemas en los que los agentes no se benefician, ni se perjudican, con la producción conjunta, es decir, el costo de la producción conjunta sólo es la suma de los costos individuales cuando producen por separado. Si este es el caso, pediremos que la solución para el caso de la producción conjunta, sea que cada quien pague lo suyo.

Definición 3.2 *Se dirá que $c \in G_{N,m}$ es un problema con costos independientes entre agentes si existen funciones $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ con $f_i(0) = 0$, tales que*

$$c(r) = \sum_{i \in N} f_i(r_i)$$

para todo $r = (r_1, \ldots, r_n) \in M$.

Definición 3.3 *Se dirá que φ es separable si para todo problema con costos independientes c (como en la definición anterior), se cumple que*

$$\varphi(c) = (f_1(m_1), \ldots, f_n(m_n)).$$

Si los costos son independientes entre los agentes, entonces cada agente paga el costo de su producción.

3.3 Sensibilidad inducida por caminos

Sea $m = (m_1, \ldots, m_n)$, como antes, el vector de demandas y denotemos por $m(N) = m_1 + \ldots + m_n$. Supongamos que se tiene una secuencia de producción la cual consiste de $m(N)$ etapas. En cada etapa se produce uno de los artículos de alguno de los agentes. Un pago "natural" asociado a dicha secuencia, sería que cada agente pagara la suma de los costos marginales de sus productos. A continuación explicamos con más detalle lo que significa.

Definimos primero el concepto de camino: un camino es una función

$$\pi : \{0, 1, \ldots, m(N)\} \to M$$

con $\pi(0) = \overline{0}$, $\pi(m(N)) = m$ y tal que para todo $t \in \{1, 2, \ldots, m(N)\}$ se cumpla que: $\pi(t) - \pi(t-1) = e^i$ para algún $i \in N$ donde e^i es el i-ésimo vector de la base canónica en $\mathbb{R}^n$. Debemos ver a un camino como una secuencia de producción; en el Ejemplo 2.1 un camino es el siguiente (ver Figura 1):

$$\pi(0) = (0,0) \quad \pi(1) = (0,1) \quad \pi(2) = (1,1) \quad \pi(3) = (2,1)$$

$$\pi(4) = (3,1) \quad \pi(5) = (3,2) \quad \pi(6) = (3,3) \quad \pi(7) = (3,4)$$

lo que significa que el agente 2 hace el primer producto, luego el agente 1 hace consecutivamente tres productos y finalmente, el agente 2 hace tres productos más, para completar el total.

Al conjunto de caminos lo denotamos con Π. Nótese que la cardinalidad de Π es $|\Pi| = \frac{m(N)!}{m_1! \cdots m_n!}$.

Dado un camino π, tenemos que el artículo $t-ésimo$ en ese camino es producido por el agente k si $\pi(t) - \pi(t-1) = e^k$ luego entonces decimos

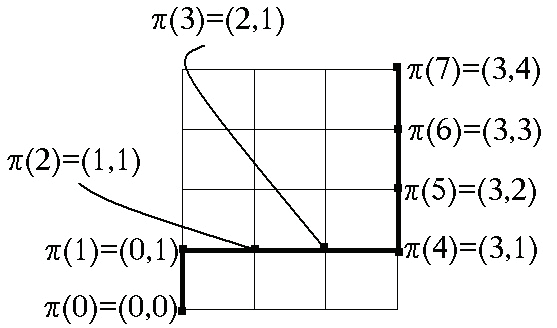


Figura 1: Ejemplo de secuencia de producción

que el costo marginal del $t-$*ésimo* producto es $c(\pi(t))-c(\pi(t-1))$, si asignamos este costo al agente k y luego sumamos todos los costos marginales por agente estaremos definiendo una solución $\varphi^\pi : G_{N,m} \to \mathbb{R}^n$ dada por:

$$\varphi^\pi(c) = \sum_{t=1}^{m(N)} \left[c(\pi(t)) - c(\pi(t-1))\right]\left[\pi(t) - \pi(t-1)\right];$$

Observe que φ^π es lineal. En el caso del camino π de la figura 1 y para el problema c del Ejemplo 2.1, se tiene que $\varphi^\pi(c) = [12-0]\,(0,1) + [19-12]\,(1,0) + [20-19]\,(1,0) + [25-20]\,(1,0) + [28-25]\,(0,1) + [33-28]\,(0,1) + [40-33]\,(0,1) = (13,27)$.

Sea r un nodo y π un camino que pasa por r al tiempo t_0, esto es, $\pi(t_0) = r$; entonces, si $0 < t_0 < m(N)$ tenemos la siguiente situación (ver Figuras 2 y 3):

$$\pi(t_0) - \pi(t_0 - 1) = \pi(t_0 + 1) - \pi(t_0)$$

o

$$\pi(t_0) - \pi(t_0 - 1) \neq \pi(t_0 + 1) - \pi(t_0).$$

En el primer caso tenemos que un mismo agente esta produciendo dos artículos de manera consecutiva mientras que en el segundo hay dos agentes distintos (digamos l y j).

En particular, para el problema E^r, tenemos que $\varphi^\pi(E^r) = 0$ en el primer caso mientras que para el segundo, $\varphi^\pi(E^r) = e^l - e^j$ donde hemos supuesto que $\pi(t_0) - \pi(t-1) = e^l$ y $\pi(t_0+1) - \pi(t_0) = e^j$. Luego, en este

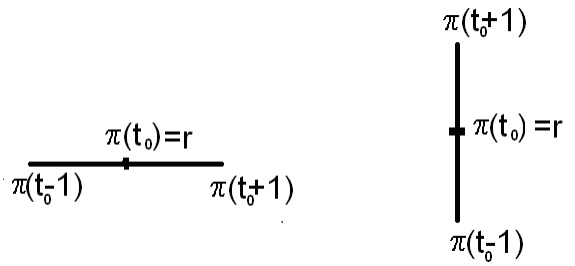


Figura 2: Primera situación posible

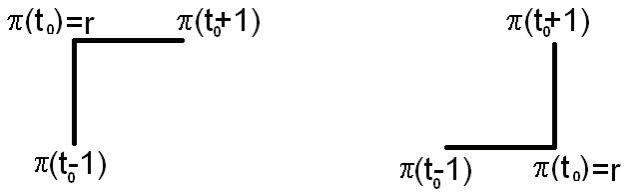


Figura 3: Segunda situación posible

ultimo caso, el agente l paga 1 mientras que el agente j recibe 1 (paga -1). Así para cada $k \in N$ y $r \in M^o$ definimos los conjuntos:

$$D_k(r) = \{\pi \in \Pi \mid \varphi_k^{\pi}(E^r) = 1\}$$

y

$$A_k(r) = \{\pi \in \Pi \mid \varphi_k^{\pi}(E^r) = -1\}.$$

Obsérvese que $\pi \in D_k(r)$ significa que el agente k paga 1 en el problema E^rusando el camino π mientras que $\pi \in A_k(r)$ implicaría que el mismo agente recibiría 1 para el mismo problema.

Ahora, definimos la función $\widehat{s}$ como sigue: $\widehat{s} : M^o \to \mathbb{R}^n$, tal que:

$$\widehat{s_k}(r) = \frac{1}{|\Pi|}\left[\mid D_k(r)\mid - \mid A_k(r)\mid\right]$$

a $\widehat{s}$ le llamaremos sensibilidad inducida por caminos.

$\widehat{s_k}(r)$ esta contando para el problema E^r la proporción de la diferencia (para el agente k) de caminos *desfavorables* (pago de $+1$) vs caminos *favorables* (pago de -1) con respecto al total de caminos. Así $\widehat{s}$ esta valorando (para todos los caminos) que implicaciones tiene pasar por cada uno de los nodos (el nodo r se analiza en el problema básico E^r); dependiendo del

nodo, cada agente identifica caminos *buenos*, *malos* e *indiferentes*. A continuación veremos como esta función de sensibilidad $\widehat{s}$ induce una solución que *promedia* a lo largo de todos los caminos.

3.4 Una solución

El principal resultado de este artículo es un valor caracterizado con dos axiomas: $\widehat{s}$-sensibilidad y separabilidad. El teorema más importante del trabajo es el siguiente:

Teorema 3.2 *Sea $\widehat{s}$ la función de sensibilidad definida anteriormente entonces existe una única solución que satisface $\widehat{s}$-sensibilidad y separabilidad. Dicha solución esta dada por:*

$$\psi(c) := \frac{c(m)}{m(N)}m + \sum_{r\in M^o} c(r)\widehat{s}(r).$$

De hecho, ψ coincide con la solución que se obtiene de promediar los costos marginales inducidos por los caminos. A saber,

$$\psi(c) = \frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi^{\pi}(c).$$

Antes de ver la demostración, notemos que el término $\frac{c(m)}{m(N)}m$ se puede interpretar como un reparto de costos proporcional a la demanda y la suma $\sum_{r\in M^o} c(r)\widehat{s}(r)$ como el término de "corrección"[2] que depende de $\widehat{s}$ y de los nodos.

Prueba: Mostraremos que

$$\psi(c) := \frac{c(m)}{m(N)}m + \sum_{r\in M^o} c(r)\widehat{s}(r)$$

es $\widehat{s}$-sensible y separable, que es única y que

$$\frac{c(m)}{m(N)}m + \sum_{r\in M^o} c(r)\widehat{s}(r) = \frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi^{\pi}(c).$$

Primero mostramos la última aserción; Para cualquier $\pi \in \Pi$ y $x \in M^o$, ya vimos que $\varphi^{\pi}(E^x) = 0$ o $\varphi^{\pi}(E^x) = e^l - e^j$ para algún l y j en N con

[2]Se esta compensando (y acumulando con la suma) a cada jugador, en cada nodo, de manera proporcional al costo de producción parcial, el factor de proporcionalidad es precisamente la importancia del nodo (definida por la función de sensibilidad $\widehat{s}$)

$l \neq j$. Así, $\sum_{\pi\in\Pi} \varphi_i^\pi(E^x) = \mid D_i(x)\mid - \mid A_i(x)\mid$, para todo $i \in N$. Luego entonces

$$\begin{aligned}\psi_i(E^x) &= \frac{E^x(m)}{m(N)}m_i + \sum_{r\in M^o} E^x(r)\widehat{s_i}(r) = \widehat{s_i}(x) = \\ &= \frac{\mid D_i(x)\mid - \mid A_i(x)\mid}{|\Pi|} = \frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi_i^\pi(E^x)\end{aligned}$$

Si $x = m$ entonces

$$\psi_i(E^m) = \frac{E^m(m)}{m(N)}m_i + \sum_{r\in M^o} E^m(r)\widehat{s_i}(r) = \frac{1}{m(N)}m_i = \frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi_i^\pi(E^m).$$

La ultima desigualdad se sigue del hecho de que $\varphi_i^\pi(E^m) = 1$ si π es un camino que pasa por el nodo $m - e^i$ (para el resto de los caminos el valor es cero) y en total hay $\frac{[m(N)-1]!}{m_1!\cdots(m_i-1)!\cdots m_n!}$ de estos caminos, si multiplicamos por $\frac{1}{|\Pi|}$ obtenemos $\frac{1}{m(N)}m_i$.

Como ψ y $\frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi_i^\pi$ son ambas lineales y coinciden en la base, son iguales.

A continuación vamos a probar que ψ satisface separabilidad. Para ello usaremos la igualdad $\psi(c) = \frac{1}{|\Pi|}\sum_{\pi\in\Pi}\varphi^\pi(c)$ que se probo anteriormente.

Sea, entonces, c un problema con costos independientes entre agentes, dado por $c(y) = \sum_{i\in N} f_i(y_i)$, entonces, para cada $\pi \in \Pi$,

$$\begin{aligned}\varphi^\pi(c) &= \sum_{t=1}^{m(N)} [c(\pi(t)) - c(\pi(t-1))]\,[\pi(t) - \pi(t-1)] = \\ &= \sum_{t=1}^{m(N)} \left[\sum_{i\in N} [f_i(\pi_i(t)) - f_i(\pi_i(t-1))]\right][\pi(t) - \pi(t-1)]\,.\end{aligned}$$

Para cada $k \in N$, sea $T_k = \left\{t \mid \pi(t) - \pi(t-1) = e^k\right\}$ entonces $\varphi_k^\pi(c) = \sum_{t\in T_k}\left[\sum_{i\in N}[f_i(\pi_i(t)) - f_i(\pi_i(t-1))]\right]$ obsérvese que $i \neq k$ implica que
$f_i(\pi_i(t)) = f_i(\pi_i(t-1))$, luego:

$$\varphi_k^\pi(c) = \sum_{t\in T_k}[f_k(\pi_k(t)) - f_k(\pi_k(t-1))] = f_k(m_k) - f_k(0) = f_k(m_k)$$

así que

$$\varphi^{\pi}(c) = (f_1(m_1), \ldots, f_n(m_n))$$

y como esta igualdad es válida para todo $\pi \in \Pi$, entonces:

$$\psi(c) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \varphi^{\pi}(c) = (f_1(m_1), \ldots, f_n(m_n)).$$

Observe que el Teorema 3.1 implica que ψ es $\widehat{s}$-sensible. La unicidad se sigue también del Teorema 3.1: Sea $\varphi^{h,\widehat{s}}$ una solución $\widehat{s}$-sensible y supongamos que también es separable. Sea $t \in \mathbb{R}$ arbitrario, entonces para el problema de costos independientes c, con

$$f_1(k) = \begin{cases} 0, & \text{si } k < m_1 \\ t, & \text{si } k = m_1 \end{cases}$$

y $f_2 = \ldots = f_n = 0$.entonces por separabilidad

$$(t, 0, \ldots, 0) = \varphi^{h,\widehat{s}}(c) = h(t) + \sum_{r \neq m | r_1 = m_1} t\widehat{s}(r)$$

lo cual implica que a lo más existe una forma de elegir la función h. ■

Observe que si definimos las soluciones θ y ω por

$$\theta(c) = \sum_{r \in M^o} c(r)\widehat{s}(r) \text{ y } \quad \omega(c) = (\omega_1(c), \ldots, \omega_n(c))$$

donde $\omega_i(c) = c(0, \ldots, m_i, \ldots 0)$; trivialmente se sigue que θ es una solución $\widehat{s}$-sensible pero no separable mientras que ω es separable pero no $\widehat{s}$-sensible. Así que hay independencia entre los axiomas.

4 Conclusiones

Comenzando con la noción de sensibilidad y separabilidad, propusimos una función de sensibilidad $\widehat{s}$ y obtuvimos la solución de los promedios de los caminos, que denotamos con ψ, la cual corrresponde a la solución del Método de Aumman-Shapley ([2]) para el problema general discreto de reparto de costos y también corresponde a la solución de Nouweland en el contexto de juegos de elección multiple.

Es importante señalar que en este trabajo no estamos suponiendo que la función de costos sea creciente. Otro resultado útil que obtuvimos es la expresión de las soluciones s-sensibles para cualquier función de sensibilidad s. En particular, debemos observar que a lo más existe una solución

s-sensible y separable para cada s. Quizás, en algunos casos, dada s,no exista una solución s-sensible y separable.

La función de sensibilidad $\widehat{s}$ nos indujo la solución de los promedios, lo que puede significar que tenemos una *adecuada* función de sensibilidad, sin embargo, un nuevo problema de investigación es buscar más funciones de sensibilidad, incluso dar una axiomatización en segundo grado que determine una única función de sensibilidad. Además a partir de este trabajo podemos identificar las funciones de sensibilidad para las soluciones existentes en la literatura, por ejemplo, las soluciones igualitaria ($\varepsilon_i(c) := \frac{c(m)}{N}$) y proporcional ($\eta_i(c) := \frac{c(m)m_i}{m_1+\ldots m_n}$) son *cero-sensibles* es decir $s \equiv 0$.

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ORIGINAL PAPER

An algorithm based on graphs for solving a fair division problem

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Abstract In this paper we propose an algorithm for solving a fair division problem: We want to find a distribution of a set of divisible items among a set of agents such that the agent who is getting the lowest amount obtains his maximum possible value. Even more, according to the found distribution, each agent obtains the same amount, and this is as high as possible. We model this situation as a linear programming problem and we use its dual problem for solving it. For doing this, we associate a bipartite graph with each set of dual variables. The algorithm presented uses the intrinsic relations in this kind of graphs for searching the optimal solution for the primal and dual problems. Finally, we show the convergence of the proposed algorithm.

Keywords Fair division · Linear programming · Algorithms · Bipartite graphs

Mathematics Subject Classification (2000) 90B15 · 90C05 · 90C35

1 Introduction

The *fair division* problem is related to finding ways to divide a set of m objects among a set of n agents in a *fair way*. There exist many possible ideas about what is fair or not. For example, to be fair may simply mean that each agent obtains the same proportion of the goods or, that the division satisfies requirements as envy-freeness. Also, the problem considers that each agent has a positive valuation of each object. These valuations are generally considered in the criteria of a fair

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division. Therefore, the main goal of the theory of fair division is to provide procedures to achieve a distribution of the objects (according to some definition of fairness) and study the properties of such divisions.

We can recognize two classes of fair division problems according to the kind of objects to be divided: indivisible (discrete) and divisible (continuous) goods. In this paper we study the problem of distributing a set of divisible objects. This problem had been studied by several authors: Austin (1982), Brams and Taylor (1996), Neyman (1946), Steinhaus (1948) and Stromquist (1980). In general, they propose procedures such that each agent obtains $1/n$ of his total valuation of the set of objects. We are proposing an algorithm for finding a distribution of the objects such that the agent who is getting the lowest amount obtains his maximum possible value. In our modeling, each agent can obtain any fraction of any item; so, the value that each of them obtains of each item is proportional to the fraction obtained, and the total value that he obtains is the sum of the value of his fractions. Thus, it is possible that each agent gets a set of fractions of the goods such that the value that he is obtaining is, at least $1/n$ of the total amount. That situation occurs when we divide the goods in n equal parts and we give one of these parts to each agent.

So, for obtaining the division in which we are interested, we solve a linear programming problem using its dual problem and a bipartite graph for modeling the intrinsic relations given by their Kuhn-Tucker conditions. We associate a bipartite graph with each set of dual variables. We show that the graph associated with the optimal dual solution is a tree. Then, we solve the problem by focusing on finding that tree. For doing this, we construct a graph that represents a feasible solution for the dual problem. If this initial tree does not represent an optimal solution, we change it appropriately such that the represented solution keeps dual feasibility. Basically, we repeat this step until we obtain a graph which represents a set of optimal dual variables. Also, the structure of the graph allows us to find the values of the primal variables in a sequential order.

The paper is organized as follows: in Sect. 2 we introduce some basic definitions related to our problem and the notation that we are using. In Sect. 3 we present the linear programming problem that could solve our problem and we study some of its properties. In Sect. 4 we show the ideas behind the algorithm that we are proposing: we present the dual problem and we introduce the concept of a graph associated with a set of dual variables. After that, we show a general version of our algorithm. In the next subsections we provide more explanation about the different steps involved in the algorithm. Then, we show its convergence to the optimal solution for the primal and dual problems. In the last section, we provide an interpretation of the dual problem that we use in the algorithm.

2 Definitions and notation

We are working on how to distribute a set of continuously divisible goods among a set of agents. As a result of that distribution, each agent receives a portion of each good, and the value obtained of each good is proportional to the portion received.

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The total value obtained for each agent is considered the sum of the values obtained according to each portion received of each good.

Let $M = \{1, 2, \ldots, m\}$ be the set of divisible goods (or, for simplicity, *items*) and $N = \{1, 2, \ldots, n\}$ the set of agents. Let $\mathbb{M}_{mn}$ be the set of real matrices with strict positive entries such that $\sum_{i \in M} a_{ij} = y$ for every $j \in N$, for every $A = (a_{ij})_{i \in M, j \in N} \in \mathbb{M}_{mn}$ and some $y \in \mathbb{R}$.

Each matrix $A \in \mathbb{M}_{mn}$ represents a *distribution problem*. The entry a_{ij} indicates the valuation that agent j gives to item i. In this problem, all the agents have a positive valuation for each item.

A *solution* for a distribution problem $A \in \mathbb{M}_{mn}$ is a real matrix $X = (x_{ij})_{i \in M, j \in N}$ such that $\sum_{j \in N} x_{ij} = 1$ for every $i \in M$. The entry x_{ij} indicates the fraction of the item i that is assigned to agent j.

Example 1 Let us consider $m = 2$, $n = 3$ and the next distribution problem:

$$A = \begin{bmatrix} 15 & 16 & 4 \\ 3 & 2 & 14 \end{bmatrix}.$$

A solution for this problem could be

$$X = \begin{bmatrix} 0.5 & 0.5 & 0.0 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}.$$

According to the previous solution, half of item 1 is assigned to agent 1; the other half is assigned to agent 2. Agent 3 does not obtain any fraction of this item. Also, we can notice that all the items are distributed in their entirety. Agent 1 obtains items that he valuates as 9; agent 2 obtains a total amount of 26/3, and so on.

Definition 1 We say that a solution X for a distribution problem $A \in \mathbb{M}_{mn}$ is *even* if the total value obtained according to X for each player is the same. That is, X is even if

$$\sum_{i \in M} a_{ij} x_{ij} = c \qquad \forall j \in N \text{ and some } c \in \mathbb{R}.$$

The distribution matrix showed in Example 1 is not even.

3 Finding the highest even solution

A distribution problem $A \in \mathbb{M}_{mn}$ could have several even solutions. For example, a natural solution could be the matrix X with $x_{ij} = 1/n$ for every $i \in M, j \in N$. According to this distribution, each item is divided into equal parts (n parts) and one of them is given to each agent. Therefore, each agent obtains $1/n$ of his total valuation and, because of the assumption that the valuation of the whole set of goods is the same for each agent, they obtain the same amount. In this section we show how to find an even solution such that the agent who is getting the lowest amount obtains the highest possible amount.

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For doing this, we solve the next linear programming problem

$$\text{Max} \quad z \quad \text{subject to} \tag{1}$$

$$\sum_{i \in M} a_{ij} x_{ij} \geq z \qquad \forall j \in N \tag{2}$$

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall i \in M \tag{3}$$

$$x_{ij} \geq 0 \qquad \forall i \in M, \quad \forall j \in N. \tag{4}$$

According to restrictions (4), the fraction that each agent obtains of each item must be a non-negative real number. Constraints (3) indicate that each item is being divided completely. Inequalities (2) establish that the agent who is getting the lowest amount obtains the maximum possible value. If those inequalities are satisfied with the equal sign, then the solution given by this linear programming problem is an even solution, because of the assumption that the total valuation of the whole set of goods is the same. Notice that this assumption is not established in the linear programming problem.

Proposition 1 *Let $A \in \mathbb{M}_{mn}$ be a distribution problem, X^* an optimal solution of its associated problem (1–4) and z^* the associated value of the objective function. Then*

$$\sum_{i \in M} a_{ij} x^*_{ij} = z^* \qquad \forall j \in N.$$

Proof We prove it by contradiction. We suppose that Proposition 1 is not true; that is, there exists an agent $k \in N$ such that

$$\sum_{i \in M} a_{ik} x^*_{ik} = z_k > z^*.$$

So, there exists a subset of items $I \subseteq M$ such that

$$\sum_{i \in I} a_{ik}(x^*_{ik} - \epsilon_i) + \sum_{i \in M \setminus I} a_{ik} x_{ik} = z^*$$

for some $\{\epsilon_i > 0\}_{i \in I}$. Now, we propose, for every $j \in N$ and $i \in M$

$$\widetilde{x}_{ij} = \begin{cases} x^*_{ij} + \epsilon_i/n, & \text{if } i \in I, j \neq k; \\ x^*_{ij} - \epsilon_i + \epsilon_i/n, & \text{if } i \in I, j = k; \\ x^*_{ij}, & \text{otherwise.} \end{cases}$$

Then, we have

$$\widetilde{z} = \min_{j \in N} \left\{ \sum_{i \in M} a_{ij} \widetilde{x}_{ij} \right\} > z^*.$$

So, X^* cannot be an optimal solution of (1–4). So, there is a contradiction and the proof ends.

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Due to the previous result, a solution for the problem (1–4) is always an even solution. Even more, because (1–4) is a maximization problem, it is as high as possible.

Our proposed solution is not the result of a procedure that the agents must follow for obtaining a division of the goods, as in *cake-cutting* procedures. In order to generate a distribution problem $A \in \mathbb{M}_{mn}$, a judge can ask the agents to allocate a fixed number of points $c \in \mathbb{R}$ among the goods according to his/her preferences. Each agent performs this task independently and without knowing the allocation of other agents. Then, the judge performs the distribution according to the obtained solution.

4 An algorithm

In this section we show an algorithm based on bipartite graphs for finding an optimal solution to (1–4). First, we must identify the number of basic variables (in the sense of simplex method) in an optimal solution.

Proposition 2 *The optimal solution of the linear programming problem given by* (1–4) *has* $m + n - 1$ *basic variables.*

Proof In the set of inequalities (2), we can replace the value of z with the left side of any inequality. So, we can remove that inequality of (2) because of its redundancy. So, the set of n inequalities can be written as $n - 1$ equations, plus the m equations given by (3). Thus, we obtain the number of needed basic variables for an optimal solution and the proof ends.

The corresponding dual problem of (1–4) is

$$\text{Min} \quad w = \sum_{i \in M} v_i \quad \text{subject to} \tag{5}$$

$$v_i \geq a_{ij} u_j \quad \forall i \in M, \quad \forall j \in N \tag{6}$$

$$\sum_{j \in N} u_j = 1 \tag{7}$$

$$u_j \geq 0 \quad \forall j \in N \tag{8}$$

where we are taking the negative values of the dual variables $u_j, j \in N$ corresponding to (2) for obtaining a more convenient problem.

We define *dual vectors* $v \in \mathbb{R}^M$ and $u \in \mathbb{R}^N$, where their coordinates are indexed according to the sets M and N, respectively. Given a distribution problem $A \in \mathbb{M}_{mn}$ and the dual vectors u and v, we define a bipartite graph $g = (M, N, E)$ where the set of edges is given by $E = \{(i, j) | i \in M, j \in N \text{ and } v_i = a_{ij} u_j\}$. So, for a distribution problem, we have a bipartite graph associated with each pair of dual vectors u and v and viceversa. We refer to this graph as an a-graph of u and v. Henceforth, we consider that the problem $A \in \mathbb{M}_{mn}$ is given and fixed.

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We say that the dual vectors u and v are *feasible* if they satisfy (6–8). Also, we say that their a-graph is feasible. Analogously, we define *optimal* dual vectors and *optimal* a-graphs.

For proving the next result we use the following procedures:

Procedure 1 Updating a dual variable u_k and its relationships

Input: Dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$; a real number α; an integer $k \in N$ and $S \subseteq \{(i,j)|i \in M, j \in N\}$.

Output: Updated dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$.

1. Update $u_k := \alpha \cdot u_k$.
2. For each $(l,k) \in S$
   Call Procedure 2 with input $(v, u, \alpha, l, S\backslash\{(l,k)\})$.

Procedure 2 Updating a dual variable v_h and its relationships

Input: Dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$; a real number α; an integer $h \in M$ and $S \subseteq \{(i,j)|i \in M, j \in N\}$.

Output: Updated dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$.

1. Update $v_h := \alpha \cdot v_h$.
2. For each $(h,l) \in S$
   Call Procedure 1 with input $(v, u, \alpha, l, S\backslash\{(h,l)\})$.

In both procedures there exists a relationship, indicated by S, between a pair of dual vectors u and v. So, in Procedure 1, the value of u_k is updated multiplying it by α and then, all the variables in v related to u_k, according to S, are updated too. In Procedure 2 the process is similar, but updating first a dual variable v_k and after that, all the variables in u related to v_k.

Proposition 3 *There exists at least one optimal solution of the dual problem (5–8), where $m + n - 1$ inequalities on (6) are satisfied as equalities.*

Proof According to Proposition 2, the problem (1–4) has $m + n - 1$ optimal basic variables x_{ij}. The Kuhn-Tucker conditions corresponding to (1–4) and (5–8) are $(v_i - a_{ij}u_j)x_{ij} = 0$. Let B be the subset of variables x_{ij} in the optimal basis of (1–4) such that $v_{ij} - a_{ij}u_j = 0$ for the corresponding optimal dual solution $v \in \mathbb{R}^M$ and $u \in \mathbb{R}^N$. If B contains $m + n - 1$ elements, the proof ends. Otherwise, we can construct a new optimal basis, based on B, where for each variable x_{hk} that we add to B, the equation $v_h - a_{hk}u_k = 0$ holds for a (possibly) new set of optimal dual variables u and v. So, after completing the basis, we have the result of the proposition.

For doing this, we calculate

$$x_{hk} = \underset{x_{ij} \notin B}{\operatorname{argmin}} \left\{ \frac{v_i}{a_{ij}u_j} \right\} \text{ and then } \alpha = \frac{v_h}{a_{hk}u_k}.$$

Thus, if we update $u_k := \alpha \cdot u_k$ then we have $v_h - a_{hk}u_k = 0$. But we need to assure that the equation $v_i - a_{ij}u_j = 0$ holds for every $x_{ij} \in B$ after this updating. So, what

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we do is calling Procedure 1 with input (v, u, α, k, S) where $S = \{(i,j)|x_{ij} \in B\}$. After that, we calculate

$$\widehat{v_i} = \frac{v_i}{S_U} \quad \forall i \in M; \qquad \widehat{u_j} = \frac{u_j}{S_U} \quad \forall j \in N; \quad \text{with} \quad S_U = \sum_{j=1}^{n} u_j \tag{9}$$

and we add x_{hk} to B. Then, we have a new set of dual variables $\widehat{u}, \widehat{v}$ such that for every $x_{ij} \in B$, the equation $\widehat{v_i} - a_{ij}\widehat{u_j} = 0$ holds. Because of (9), these new dual variables satisfy constraint (7). We chose the variable x_{hk} with minimal α for entering the base to assure that constraints (6) hold for the new dual variables. So, they are feasible. We continue adding variables to B applying the same process until it has $m + n - 1$ elements. We eliminated null variables of the original optimal basis and we include other null variables to B to obtain a new optimal basis. So, by construction, after completing the new optimal basis of (1–4) the dual variables $\widehat{u}$, $\widehat{v}$ are optimal and $\widehat{v_i} - a_{ij}\widehat{u_j} = 0$ holds for every x_{ij} in the new optimal basis. As Proposition 2 establishes, the number of elements on the basis is equal to $m + n - 1$. Thus, we have the result and the proof ends.

Definition 2 An a-graph is *basic* if it is a tree with $m + n - 1$ edges.

Proposition 4 There exists at least one optimal a-graph $g = (M, N, E)$ *that is basic*.

Proof According to Proposition 3, there exists at least one optimal a-graph g with $m + n - 1$ edges. We will show that at least one of these optimal a-graphs is a tree. Suppose that this is not true. Thus, every g consists of at least two components. Also, one of these components must not be a tree. So, that component must have a cycle. Let us denote this component as $g' = (M' \subseteq M, N' \subseteq N, E')$. If we restrict the problem to M' and N' we notice that, because of Propositions 2 and 3, we can find an a-graph in g' with $m' + n' - 1$ edges that still represent a subset of optimal primal variables and then, we are breaking the original cycle. It is possible to form new cycles (of smaller length) in this process but we can apply the same methodology on them. We repeat this procedure for each component on g that contains a cycle. After these operations, we have the same situation as in the proof of Proposition 3: a subset of optimal basic variables B (with less than $m + n - 1$ elements) where $v_i - a_{ij}u_j = 0$ for each $x_{ij} \in B$. So, we can repeat the procedure applied in the proof of Proposition 3 for completing the set B until we obtain a new primal basis, but now we add the null variable x_{hk} with minimal value α that connects two components. So, we have constructed an optimal a-graph that is basic. That is a contradiction with our initial supposition.

We are interested in finding optimal dual vectors u and v for a distribution problem. If we obtain these vectors, we could find an optimal solution X for (1–4). We will work with basic a-graphs. For all the aforementioned, the problem of finding optimal solutions for (1–4) and (5–8) is equivalent to finding an optimal a-graph. By Propositions 3 and 4, there exists at least one optimal a-graph that is basic. We are interested in one of these a-graphs.

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Now, in Algorithm 1, we present the general scheme of the algorithm that we are proposing.

Algorithm 1 Finding the highest even solution for a distribution problem
Input: A distribution problem $A \in \mathbb{M}_{mn}$
Output: The highest even solution X

1. Find a feasible basic a-graph for A with associated dual vectors u, v.
2. Assume $z = w = \sum_{i \in M} v_i$
3. Find the value of the primal variables X. If X is a feasible primal solution, then the algorithm ends.
4. Otherwise, find a new feasible basic a-graph and return to Step 2.

The general idea of Algorithm 1 is as follows: First, we find a basic a-graph such that its associated dual vectors u and v are feasible (Step 1). We assume that the values of the objective function for the primal and dual problems match (Step 2). Thus, the corresponding primal variables must be feasible. In Step 3 we find the value of these variables. If they are not feasible, u and v are not optimal dual vectors. In this case, we find another pair of dual vectors u' and v' with the same requested characteristics according to Step 1 modifying the a-graph found previously (Step 4). Then, we return to Step 2, and so on. We stop when the primal variables are feasible.

In the next subsections, we provide more explanation about the steps involved in Algorithm 1. Finally, we prove the convergence of the algorithm.

For analyzing the computational tractability of the algorithm, we consider the instance as *the updating of a dual variable*. As we say afterwards, this operation involves the multiplication of a dual variable by a real number (α), which is the result of a quotient, where the numerator is a real number and the denominator is a multiplication. We refer to this instance as an *update*. At the end of each subsection, we talk about the computational tractability of each step of the algorithm.

4.1 Finding a feasible basic a-graph

In this step we construct a basic a-graph for a distribution problem such that the associated dual vectors represent a feasible dual solution. We start with an a-graph without edges and we connect two components until we have a basic a-graph. We show a way to do this in Procedure 3.

For a bipartite graph $g = (M, N, E)$, we denote as $C(g)$ the set of components of g. For a component $C \in C(g)$, we denote as $N(C)$ the set of nodes in C. For a node k, C_k denotes the component $C \in C(g)$ such that $k \in N(C)$.

Procedure 3 Finding a feasible basic a-graph
Input: A distribution problem $A \in \mathbb{M}_{mn}$
Output: Dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$ for A and their a-graph $g = (M, N, E)$

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1. Initialize $E := \varnothing$ and then,

$$j^* = \underset{j \in N}{\operatorname{argmax}}\{a_{ij}\}, \quad v_i = a_{ij^*}, \quad E := E \cup (i, j^*) \quad \forall i \in M \tag{10}$$

$$u_j = 1 \quad \forall j \in N \tag{11}$$

2. Calculate

$$(h, k) := \underset{\substack{(i,j) \notin E \\ i \notin N(C_j)}}{\operatorname{argmin}} \left\{ \frac{v_i}{a_{ij} u_j} \right\}; \quad \alpha = \frac{v_h}{a_{hk} u_k}.$$

3. Call Procedure 1 with input (v, u, α, k, E) and then E: $= E \cup (h, k)$.
4. If the number of edges in g is different from $m + n - 1$, return to Step 2.
5. Apply (9) to the dual variables obtained previously.

Proposition 5 *The output of Procedure 3 is a pair of feasible dual vectors u and v and their a -graph is basic.*

Proof By construction, the dual vectors u and v initialized in (10–11) satisfy $v_i \geq a_{ij} u_j$ for every $i \in M, j \in N$. In particular, $v_i = a_{ij} u_j$ for each initialized edge. So, after the initialization, $g = (M, N, E)$ is an a-graph with m edges and n components (each isolated node is considered as a component) and without cycles. The idea of the following steps is to add an edge between two components of g such that the associated dual vectors satisfy (6). To add the edge (h, k) means that the inequality $v_h > a_{hk} u_k$ must be converted to an equality. So, we have to update the value of u_k multiplying it by $\alpha = v_h/(a_{hk} u_k)$. Also, we need to preserve the equalities represented by edges in the component of g that contains the node k. To perform these updates is exactly what Procedure 1 does (Step 3) in a recursive way. Also, we need to assure that the other inequalities are satisfied. Thus, we have to choose the edge (h, k) that connects two components with minimal value α (Step 2). Finally, we add this edge (Step 3). When we connect two components, we guarantee that a cycle is not formed. We repeat the previous steps until g has just one component. Then, after Step 4, we have an a-graph with $m + n - 1$ edges without cycles (a tree). After that, we normalize the dual vectors according to (9) for satisfying (7) (Step 5). Finally, we obtain a basic a-graph and its associated dual vectors are feasible.

Example 2 We work with the distribution problem A given in Example 1. Applying Procedure 3 to this problem, we obtain the dual vectors $u = (1, 1, 1)$ and $v = (16, 14)$ and a a-graph with edges (1, 2) and (2, 3) after initialization. Also, their associated a-graph has three components. We need to know which edge to add, connecting two components, such that the set of dual constraints (6) remains feasible. The edge (1, 1) gives the minimal value for α, $\alpha = 16/15$ (when there are several edges with the same value for α, break ties arbitrarily). We update the value of the dual variables associated with nodes in $C_1, 1 \in N$ multiplying them by α. After that, we add the edge (1,1). We obtain the a-graph of Fig. 1.

We do not have an a-graph with $m + n - 1$ edges yet. If we redo Step 3, then we obtain $(h, k) = (1, 3)$ with $\alpha = 4$. We update the value of the dual variables

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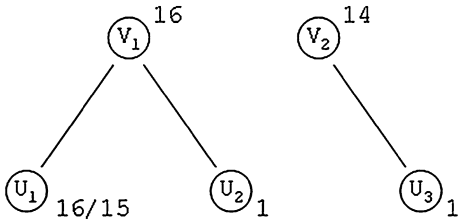


Fig. 1 The first step of Procedure 3 applied to the problem given in Example 1

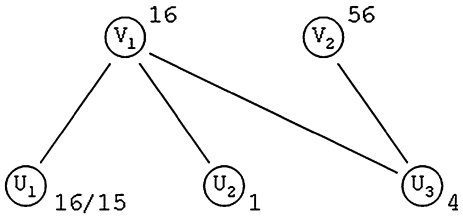


Fig. 2 The second step of Procedure 3 applied to the a-graph of Fig. 1

associated with nodes in $C_3, 3 \in N$ and we add this edge. So, we get the a-graph of Fig. 2.

We have an a-graph with $m + n - 1$ edges now. So, Procedure 3 has finished. Figure 2 depicts the final basic a-graph. Applying equations (9), we obtain the dual vectors $\widehat{v} = \frac{1}{91}(240, 840)$ and $\widehat{u} = \frac{1}{91}(16, 15, 90)$. Calculating the objective function value for these dual variables, we obtain $w = 1080/91$.

Now we analyze the maximum number of updates in this procedure. After performing the first step, we have an a-graph with m edges. So, each node in M is linked with some node in N and we need to add $n - 1$ edges. First, we check the case when $m \leq n$ and we will analyze the case when the number of updates is maximum. This case occurs when there are m components and, while performing the third step, the other $n - m$ nodes are linked to the same component. For doing this, we need $n - m$ updates. So, we have a graph made of one component with $n - m + 2$ nodes and $m - 1$ components with two nodes. Now, the worst situation occurs when we link the components in this way: we connect the node in M of each component with two nodes with a node in N of the component with $n - m + 2$ nodes (at the beginning). Thus, the number of updates is given by

$$\sum_{i=1}^{m-1}[(n-m) + 2i] = n(m-1).$$

So, with these updates plus the first $n - m$ explained in the previous paragraph, we obtain that the maximum number of updates is $m(n - 1)$.

The analysis when $m > n$ is similar. In this case, the maximum number of updates occurs when, after the first step, we have a graph with $n - 1$ components with two nodes and one component with $m - n + 2$ nodes. So, we connect the node in M of each component with two nodes with a node on N of the component with $m - n + 2$ nodes (at the beginning). So, we need to make

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$$\sum_{i=1}^{n-1}[(m-n)+2i] = m(n-1).$$

updates.

Then, Procedure 3 constructs a dual feasible a-graph performing $m(n-1)$ updates as maximum.

4.2 Finding values for the primal variables X

In this section we assume that u and v are feasible dual vectors of a distribution problem $A \in \mathbb{M}_{mn}$ and g their corresponding basic a-graph. According to Algorithm 1, we will assume that the optimal value of the objective function of the dual problem matches with the optimal value of the objective function of the primal problem. In the next procedure, and under this assumption, we propose a method for finding the primal solution associated with u and v such that satisfies the set of constraints (2) and (3), possibly not satisfying restrictions (4). This method is explained in Procedure 4.

We make this assumption because we want to verify if a set of feasible dual vectors is optimal. If the associated primal variables are feasible too, then, the dual and primal variables are optimal.

Given a bipartite graph $g = (M, N, E)$ and a node i, $\deg(i, g)$ denotes the number of edges connecting i with another node in g.

Procedure 4 Finding primal variables associated with a pair of feasible dual vectors

Input: Feasible dual vectors $v \in \mathbb{R}^M, u \in \mathbb{R}^N$ for a distribution problem $A \in \mathbb{M}_{mn}$ and their basic a-graph $g = (M, N, E)$

Output: Primal variables X satisfying (2) and (3).

1. Calculate the value of the objective function w for the dual problem.
2. For every $(i, j) \in E$ such that $\deg(i, g) = 1$, set $x_{ij} := 1$
3. For every $(i, j) \in E$ such that $\deg(j, g) = 1$, set $x_{ij} := w/a_{ij}$
4. Find the other variables using equation (3) and

$$\sum_{i \in M} a_{ij}x_{ij} = w \qquad \forall j \in N. \tag{12}$$

in a suitable sequential order.

Proposition 6 *The primal variables X, given as output of Procedure 4, satisfy constraints* (2) *and* (3).

Proof The proof is straightforward. By construction, Steps 2, 3 and 4 of Procedure 4 assure that the primal variables satisfy constraints (2) and (3). Because the input a-graph is basic, the procedure guarantees to find the values of all the basic primal variables due to the initialization given on Steps 2 and 3.

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Example 3 Let us consider the dual vectors $\widehat{u}$ and $\widehat{v}$ associated with the distribution problem A of Example 1 given in the final part of Example 2 and their a-graph given in Fig. 2. Applying Steps 2 and 3 of Procedure 4, we obtain

$$x_{23} = 1, \qquad x_{11} = 72/91 \qquad \text{and} \qquad x_{12} = 135/182.$$

With this information, we can find the values for the other basic variables. We know that $x_{11} + x_{12} + x_{13} = 1$. Thus, we obtain

$$x_{13} = -97/182,$$

and $x_{ij} = 0$ otherwise. Also, this value for x_{13} verifies $14x_{23} + 4x_{13} = 1080/91$. Clearly, this is not a feasible solution for the primal problem (1–4) because $x_{13} < 0$ (it does not satisfy (4)). So, the dual vectors $\widehat{u}, \widehat{v}$ are not optimal.

If we consider that, before the procedure, the primal variables are initialized to zero, it is necessary to solve $m + n - 1$ linear equations as maximum for finding the corresponding values of the basic variables associated with a basic a-graph.

4.3 Finding new feasible dual vectors u and v

It is possible that the dual vectors u and v for a distribution problem $A \in \mathbb{M}_{mn}$ resulting of Procedure 3 are not optimal. That is, if u and v and their basic a-graph are the input of Procedure 4, it could be that some entries in the output X of this procedure are negative numbers. This fact indicates that X does not represent a feasible solution of (1–4). Then, u and v are not optimal. In Procedure 5, we show how to modify a basic a-graph g such that is associated with other feasible dual vectors.

Procedure 5 Calculating new feasible dual vectors

Input: Non-optimal dual vectors u and v for a problem $A \in \mathbb{M}_{mn}$, their basic a-graph $g = (M, N, E)$ and the corresponding primal variables X resulting from Procedure 4

Output: Dual vectors $\widehat{u}, \widehat{v}$ for A and their basic a-graph $g' = (M, N, E')$

1. Let $(h, k) \in E$ be such that $x_{hk} < 0$. If there exist several negative primal variables, we choose the variable with the minimum value.
2. Set $E' := E \backslash (h, k)$.
3. Calculate

$$(i,j) := \underset{\substack{(i,j)\notin E' \\ i\in N(C_j), j\in N(C_h)}}{\operatorname{argmin}} \left\{ \frac{v_i}{a_{ij}u_j} \right\}; \quad \alpha = \frac{v_i}{a_{ij}u_j}$$

4. Call Procedure 1 with input (v, u, α, j, E') and then $E' := E' \cup (i, j)$.
5. Apply the normalization given by (9) to u and v for obtaining $\widehat{u}$ and $\widehat{v}$.

Proposition 7 *The dual vectors $\widehat{u}$ and $\widehat{v}$ and their basic a-graph resulting from Procedure 5 are feasible and they are different from the input dual vectors.*

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Proof Let us consider the dual vectors u and v of a problem $A \in \mathbb{M}_{mn}$, their basic a-graph $g = (M, N, E)$ and X as the input of Procedure 5. In Step 1 we are removing an edge (h, k) such that $x_{hk} < 0$. This edge exists because g is basic but not optimal. This disconnection causes g to be separated into two components: the one containing node h and the other one containing node k. We define $g' = (M, N, E \backslash (h, k))$. We want that g' to be a basic a-graph, so we need to add an edge connecting the two components. For adding an edge, we use the same idea as Procedure 3. It is not possible to add an edge (i, j) $i \in C_h$ and $j \in C_k$ because, for keeping feasibility in the associated dual vectors, the edge with minimal α ($\alpha = 1$) is (h, k): If we add this edge, the new a-graph will be the same as the input. So, the unique way for connecting the two components and obtaining a different basic a-graph is adding an edge (i, j), $i \in C_k, j \in C_h$. We need to assure the feasibility in the dual vectors. So, we need to add the edge (i, j) with minimal α (Step 3). The ties are broken arbitrarily. Also, we need to multiply all the dual variables represented by nodes in C_h by this minimal α (Step 4). After this, we add the edge. Then, we obtain a new basic a-graph and new dual vectors. For making them feasible, we need to normalize them using (9) (Step 5).

Example 4 Let us consider the dual vectors u and v, their basic a-graph g resulting for Example 2 and the distribution matrix X resulting for Example 3 as input for Procedure 5. In this case, there is only one negative primal variable, x_{13}. Then, we remove the edge (1, 3) (Steps 1 and 2). This removal produces the disconnection of g in two components, $C_1, 1 \in M$ and C_3, as we show in Fig. 3.

We need to add an edge for obtaining a basic a-graph. When we do Step 3, we notice that we must check only the edges (2, 1) and (2, 2). The edge with minimal value α is (2, 1), $\alpha = 35/2$. Now, we update the value for the dual variables in $C_1, 1 \in M$, multiplying them by α. After that we add the edge. Thus, we obtain the a-graph of Fig. 4.

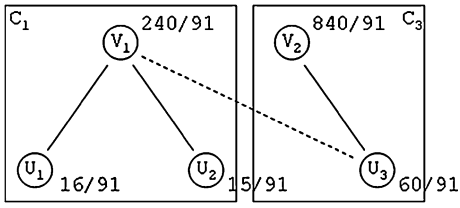


Fig. 3 The disconnection of g in two components, C_1 and C_3, after removing the edge (1, 3)

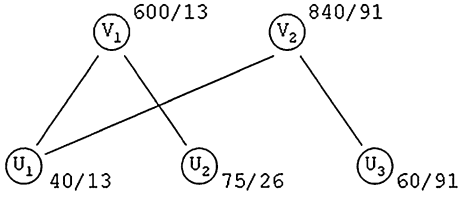


Fig. 4 The a-graph resulting for adding the edge (2, 1) to the graph of Fig. 3

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Then, we normalize the dual variables for obtaining $\widehat{v} = \frac{1}{241}(1680, 336)$ and $\widehat{u} = \frac{1}{241}(112, 105, 24)$.

Now, we analyze the number of updates in this procedure. After removing the edge (i, j), the worst possible situation occurs when $|C_j| = 2$. That indicates that we need to link the node $i \in M$, $i \in C_j$ with a node in $N \backslash \{j\}$.

Then, Procedure 5 constructs a dual feasible a-graph performing $m + n - 2$ updates as maximum.

Now, after explaining the steps involved in the algorithm, we prove its convergence.

Theorem 1 *Algorithm 1 produces, as output, an even solution.*

Proof According to Proposition 3, Procedure 3 provides feasible dual vectors u and v and a basic a-graph g for a distribution problem A. We calculate the value of the primal variables according to Procedure 4 (Step 3). If these primal variables are feasible, then the algorithm ends, and we have an optimal solution of (5–8) and, consecutively, of (1–4). If the primal variables X, resulting from Procedure 4, are not feasible, then we apply Procedure 5 (Step 4) with inputs X, u, v and g. Before performing this procedure, we have a feasible dual solution (represented by the dual vectors u and v) and a set of non-feasible primal variables X. Moreover, some of these primal variables have negative values. According to Procedure 5, we choose the basic primal variable which is negative and has the highest absolute value for leaving the basis. As we show in the proof of Proposition 7, there is only one primal variable that should enter the basis to maintain dual feasibility (or, possibly, a set of primal variables. Anyone of them entering to the basis produces the same results). So, Procedure 5 is performing a step of the simplex-dual method. Thus, from Step 4, we are doing the simplex-dual method but using the information related to the basic a-graph. This ensures that the algorithm converges to some optimal dual vectors u and v for A and some optimal primal variables X. Then, the output of Algorithm 1 is an optimal solution of (1–4) and consecutively, an even solution for A.

Now we are going to talk about the total complexity of the algorithm. In comparison, our algorithm performs a lower number of updates than the simplex-dual method because this method requires updating a higher number of entries in the simplex-dual tableau. We compare our algorithm with the dual simplex method because, as we said in the proof of Theorem 1, we are performing simplex-dual steps. An open question is to estimate the average number of iterations of the algorithm. The number of spanning trees in a bipartite graph with sets of nodes M and N is $m^{n-1}n^{m-1}$. In theory, this number represents the maximum number of a-graphs (and therefore, iterations) that our algorithm can construct until reaching an optimal one. For verifying the computational functionality of the algorithm, it was implemented on Maple 13 (© Maplesoft, Waterloo Maple Inc.). We consider four kinds of 30 randomized problems. The maximum, minimum, average, median and mode of iterations of each case is depicted in Table 1. So, we can conclude that our algorithm has a good behavior on practice.

Also, we make a comparison between the performance of our algorithm versus the linear programming solver of Maple (*LPSolver*). We consider four kinds of 30

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Table 1 Minimum (Min), maximum (Max), average (Avg), median (Mdn) and mode (Mod) number of iterations of Algorithm 1 of 30 randomized problems with different sizes as indicated in column 1

	Min	Max	Avg	Mdn	Mod
$m = n = 10$	3	10	6.6	6	6
$m = n = 20$	12	31	18.6	18	17
$m = n = 30$	17	52	34.7	33.5	36
$m = n = 50$	43	97	69	68	72

Table 2 Minimum (Min), maximum (Max) and average (Avg) time (in seconds) of our algorithm (ABG) and the Maple's command for solving linear programming problems (LPS)

	Min		Max		Avg	
	ABG	LPS	ABG	LPS	ABG	LPS
$m = n = 30$	0.391	0.562	0.750	0.703	0.673	0.610
$m = n = 50$	2.969	3.799	3.782	4.219	3.492	3.995
$m = n = 100$	26.219	48.063	36.075	61.234	31.147	52.959
$m = n = 150$	98.734	265.078	153.110	276.796	123.880	270.312

The routines execute the same 30 randomized problems with different sizes as indicated in column 1

randomized problems and we measure their execution time. In Table 2, we show the minimum, maximum and average time (in seconds) of each case for our algorithm (ABG) and the Maple's solver (LPS). In the case when $m = n = 30$, our algorithm had a better performance than the Maple's solver in only 12 problems. For this case, in the problem with its best performance our algorithm lasted 0.391 s and the Maple's solver, 0.688 s. In addition, in the problem with its worst performance, our algorithm lasted 0.750 s and the Maple's solver, 0.625 s. On the other hand, our algorithm performed faster than the Maple's solver in all the 30 problems when $m = n = 50$, $m = n = 100$ and $m = n = 150$. For example, when $m = n = 100$, in the problem with its best performance, our algorithm lasted 26.219 s and the Maple's solver, 50.379 s. In its worst performance, the Maple's solver lasted 61.234 s and, in the same problem, our algorithm lasted just 28.781 s. After all the results, we conclude that the proposed algorithm has a very stable performance, and it is a very good tool for big problems.

Example 5 We continue applying Algorithm 1 for the dual vectors $\widehat{u}$ and $\widehat{v}$ for the distribution problem given in Example 1 and their a-graph obtained in Example 4. So, we need to apply Procedure 4 for finding the corresponding primal variables X. After doing this, we notice that X is a feasible primal solution. Then, the algorithm stops. Figure 4 depicts an optimal a-graph and the associated dual vectors are optimal variables of (5–8), with $w = 2016/41 \approx 8.3651$. Then, an even and efficient solution for Example 1 is as follows:

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$$X = \frac{1}{241}\begin{bmatrix} 115 & 126 & 0 \\ 97 & 0 & 144 \end{bmatrix}.$$

Notice that if we assign $1/n$ of each item to each agent, each one of them obtains just 6 units.

5 An interpretation of the primal and dual problems

In Sect. 3 we introduce the linear programming problem (1–4) that allows us to find the highest even solution. We can establish a general form of this problem as follows:

$$\text{Max} \quad cz \quad \text{subject to} \tag{13}$$

$$\sum_{i\in M} a_{ij}x_{ij} - z \geq b_j \quad \forall j \in N \tag{14}$$

$$\sum_{j\in N} x_{ij} = d_i \quad \forall i \in M \tag{15}$$

$$x_{ij} \geq 0 \quad \forall i \in M, \quad \forall j \in N. \tag{16}$$

with $c > 0$ and $b_j \geq 0$, $j \in N$ and $d_i > 0$, $i \in M$. This is a generalized formulation for a problem of how to distribute a set of finite divisible items among a set of finite number of agents. As in the model presented in (1–4), a_{ij} represents the valuation of item i for agent j and x_{ij} indicates the fraction of item i that will be assigned to agent j. In the same way, z represents a minimal amount that each agent must obtain because of the distribution of the goods, but, in fact, they are obtaining c times that amount, with $c \in \mathbb{R}, c > 0$. So, c is the valuation of one unit of the value obtained by the distribution. In (1–4), we have $c = 1$. b_j indicates an additional amount to z that must be assigned to agent j. In our model, $b_j = 0$ for every $j \in N$; so, every agent is obtaining only the minimal amount z and, because of Proposition 1, the obtained solution is even. Additionally, d_i represents the number of items of type i available for distribution. In (1–4), we have $d_i = 1$ for all $i \in M$; thus, there is only one item of each type i.

The corresponding dual problem of the previous generalized formulation is

$$\text{Min} \quad \sum_{i\in M} d_i v_i + \sum_{j\in N} b_j u_j \quad \text{subject to}$$

$$a_{ij}u_j \leq v_i \quad \forall i \in M, \quad \forall j \in N \tag{17}$$

$$\sum_{j\in N} u_j = c \tag{18}$$

$$u_j \geq 0 \quad \forall j \in N. \tag{19}$$

The dual variables u_j tell us how the even value z decreases if player j is given one unit of utility. The dual variables v_i tell us how the even value z increases if we add one extra unit of item i. Suppose now that an external observer can modify the

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problem by giving one unit of utility: he will give it to the player with lower variable u_j. Similarly, if the observer can add an item in the division, he will pick the one with higher variable v_i.

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Research Article

Bankruptcy Problem Allocations and the Core of Convex Games

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A well-known result related to bankruptcy problems establishes that a vector is a bankruptcy allocation if and only if it belongs to the core of the associated O'Neill's bankruptcy game. In this paper we show that this game is precisely the unique TU-game based on convex functions that satisfies the previous result. In addition, given a bankruptcy problem, we show a way for constructing bankruptcy games such that the set of bankruptcy allocations is a subset of their core or their core is a subset of the set of bankruptcy allocations. Also, we show how these results can be applied for finding new bankruptcy solutions.

1. Introduction

There are several situations when an estate is disputed by a set of agents, each one of them claiming a part of the estate. The problem arises when the total claim is greater than the estate, giving place to a *bankruptcy problem*. Typically, in a bankruptcy allocation, an agent must receive at least zero and at most his claim. This is the kind of problems we study in this paper. The bankruptcy problem was introduced by O'Neill [1] and later studied by several authors. See [2–5] for some seminal papers and analysis on this topic.

Under the normative approach to bankruptcy problems, the aim is to find *fair* bankruptcy solutions, and two methods for this purpose are identified. The direct method, introduced by Young [6], comes from an axiomatic study of bankruptcy solutions based on the original problems. In the indirect method, one considers transferable utility cooperative games (TU-games) that summarize appropriately the conflict of the bankruptcy problem. Such games are now called bankruptcy games and were first proposed by O'Neill [1]. The main idea of finding a solution in both settings is to give a rule (a bankruptcy solution) for distributing the estate among the agents satisfying uniquely a set of desirable properties: the interpretation of a bankruptcy solution is the recommendation of an external agent or arbitrator. A vast literature follows both approaches, partially gathered by Thomson [4]. The present work is based on the indirect approach. Particularly, given a bankruptcy problem, we construct a TU-game for every increasing-continuous function defined in a certain real interval.

For a fixed bankruptcy problem, we study TU-games where the characteristic function value for each coalition depends on a function of the agents' claims in that coalition, following the idea of the bankruptcy game proposed by O'Neill [1]. In [2], it is shown that, given a bankruptcy problem, the core of O'Neill's bankruptcy game is exactly the set of bankruptcy allocations. So, it seems tempting to ask if there exists another bankruptcy game, based on a convex function, with this property. Another TU-game with this property could help us to have a better understanding of the possible allocations for a bankruptcy problem. We prove that O'Neill's bankruptcy game is the only one with this property. To give a complement to our treatment of the problem, we provide two sets of TU-games: in the first one, each game has a core of allocations to the bankruptcy problem but not the converse; for every game in the second set, every allocation of the bankruptcy problem is in its core. Finally, we show how these last results can be applied to find new bankruptcy solutions.

If we use TU-game theoretic results to find allocations of a bankruptcy problem, it is desirable that the proposed allocation belongs to the core of the corresponding TU-game. If an allocation is not in the core of the associated TU-game, it means that there exists a subgroup of agents such that the amount assigned to them is lower than their

amount according to the TU-game (the allocation does not satisfy group rationality). Thus, is it more reasonable that the bankruptcy allocation proposed by a TU-game solution belongs to the core of the game? That is the main reason we are interested in using TU-games with a nonempty core for solving bankruptcy problems. Contrary to our expectation, there is just one TU-game based on convex functions with the additional property that its core and the set of bankruptcy allocations coincide. We show a way for finding TU-games such that all the elements of their cores are allocations for the original bankruptcy problem, and we show how this result can be applied for proposing new bankruptcy solutions.

The paper is organized as follows. In Section 2 we introduce the notation and some basic definitions related to the standard bankruptcy problem as well as transferable utility cooperative game theory concepts. In Section 3 we present our main results and illustrate them with some examples. Finally, in Section 4 we show an application of our results for finding bankruptcy solutions.

2. Notation and Definitions

Given a set of agents $N = \{1, \ldots, n\}$, a *bankruptcy problem* $(E; c)$ consists of a number $E \in \mathbb{R}_+$, the *estate*, and a *claims vector* $c \in \mathbb{R}^n$ where $c_i > 0$ for all $i \in N$ and the total claim, C, satisfies $C := c_1 + \cdots + c_n > E$. We denote the set of all possible bankruptcy problems with a set of agents N by B_N. This kind of problem is interpreted as a dispute between claimers, where the ith claimer demands a quantity c_i, from a fixed amount, E, which is not enough to satisfy all demands.

Given a bankruptcy problem $B = (E; c)$ a *bankruptcy allocation* for B is a vector $x \in \mathbb{R}^n$ such that

$$\sum_{i \in N} x_i = Ex \quad 0 \le x_i \le c_i, \ \forall i \in N. \tag{1}$$

A *solution* for bankruptcy problems is a function $\varphi : B_N \to \mathbb{R}^n$ such that $\varphi(B)$ is a bankruptcy allocation. In a bankruptcy allocation, the estate is totally allocated and each agent obtains a positive amount not exceeding his claim.

For $x \in \mathbb{R}^n$ and $S \subseteq N$, let $x(S) := \sum_{i \in S} x_i$. Notice that for a bankruptcy problem $(E; c)$, $C = x(N)$. Next, we present some standard notions of cooperative game theory which will be used in what follows. A *transferable utility cooperative game*, a TU-game, is defined as a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$, and is the *characteristic function* evaluated at the 2^N possible subsets of N with $v(\emptyset) = 0$. Given a coalition $S \subseteq N$, $v(S)$ is interpreted as the worth the agents in coalition S can obtain for their cooperation.

Let G_N be the set of TU-games with set of players N. Given a TU-game (N, v) its *dual game*, (N, v^*), is defined by $v^*(S) = v(N) - v(N \setminus S)$ for every $S \subseteq N$. A *solution* for a TU-game is a multivalued map $\phi : G_N \to \mathbb{R}^n$, where $\phi_i(N, v)$ is the amount that each agent obtains in the game $(N, v) \in G_N$. There are several solutions for TU-games; see, for example, [7, 8]. In particular, one can consider *the core* of (N, v) which is given by the set

$$\mathscr{C}(N, v) = \left\{ x \in \mathbb{R}^n \mid x(S) \ge v(S) \ \forall S \subsetneq N, x(N) = v(N) \right\}. \tag{2}$$

Considering nonemptiness, the core contains all allocations in which each coalition $S \in 2^N$ obtains at least its own worth and the amount $v(N)$ is distributed amongst all agents. An allocation x satisfying the last condition, $x(N) = v(N)$, is called *efficient*.

A TU-game $(N, v) \in G_N$ is called *convex* if for every player $i \in N$ and each pair of coalitions S, T such that $S \subseteq T \subseteq N \setminus \{i\}$ the next condition holds:

$$v(T \cup \{i\}) - v(T) \ge v(S \cup \{i\}) - v(S). \tag{3}$$

It is known that if (N, v) is convex then it has a nonempty core [9].

There is a vast literature where bankruptcy problems and TU-games are related in some way. O'Neill [1] proposed a TU-game for studying bankruptcy problems; with his model he summarized the conflict in an appropriated form. In general, a TU-game with this characteristic is called a *bankruptcy game*. O'Neill's bankruptcy game is as follows, for every problem $B = (E; c)$:

$$v^B(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} c_i \right\}, \quad \forall s \subseteq N. \tag{4}$$

The idea of this game is to assign as the worth of a coalition $S \subseteq N$ the remainder estate after satisfying the claims of the agents on $N \setminus S$. Curiel et al. [2] showed that the bankruptcy game proposed by O'Neill is convex and, additionally, that the core of O'Neill's bankruptcy game is exactly the set of all bankruptcy allocations.

3. Results

Let $B = (E; c) \in B_N$ be a bankruptcy problem and consider the next set of functions:

$$\mathscr{F}^B = \{ f \mid f : [0, C] \longrightarrow [0, E] \ \text{ increasing-continuous with } f(0) = 0, f(C) = E \}. \tag{5}$$

In all what follows we consider that the problem $B = (E; c)$ and the set $\mathscr{F}^B$ are given and fixed.

For $f \in \mathscr{F}^B$, the *dual function* is defined by

$$f^*(\tau) := f(C) - f(C - \tau), \quad \tau \in [0, C]. \tag{6}$$

An easy reference to f^* is provided by its graph: it is the graph of f rotated with respect to the line $\ell(\tau) = E\tau/C$ (the *positive* diagonal of the square $[0, C] \times [0, E]$ in $\mathbb{R}^2$).

For every $f \in \mathscr{F}^B$, we define the TU-game (N, v_f) associated with f as

$$v_f(S) := f\left(\sum_{i \in S} c_i \right), \quad \forall S \subseteq N. \tag{7}$$

We are interested in the core of certain TU-games (N, v_f). For this purpose, we focus on a kind of TU-games with a nonempty core.

Lemma 1. *For every bankruptcy problem $B = (E; c)$, if $f \in \mathcal{F}^B$ is a convex function, then (N, v_f) is a convex TU-game.*

We establish another useful result related to the set $\mathcal{F}^B$.

Lemma 2. *For every bankruptcy problem $B = (E; c)$ and every function $f \in \mathcal{F}^B$,*

(1) *$f^* \in \mathcal{F}^B$ and $(f^*)^* = f$;*

(2) *$v_{f^*} = v_f^*$;*

(3) *if f is a convex function, then f^* is concave; if f is concave, then f^* is convex;*

(4) *if $g \in \mathcal{F}^B$ then $f(\tau) \leq g(\tau)$ if and only if $f^*(\tau) \geq g^*(\tau)$, and $f(\tau) \geq g(\tau)$ if and only if $f^*(\tau) \leq g^*(\tau)$, for every $\tau \in [0, C]$.*

For the proofs of the previous lemmas, please refer to Appendix 4.

The next proposition is referred to Curiel et al. [2].

Proposition 3. *$x \in \mathbb{R}^n$ is a bankruptcy allocation for $(E; c)$ if and only if $x \in \mathcal{C}(N, w)$, where the TU- game (N, w) is defined as follows:*

$$w(S) = \max\left\{0, E - \sum_{i \in N \setminus S} c_i\right\}, \quad \forall S \subseteq N. \tag{8}$$

Example 4. Let us consider a bankruptcy problem with $n = 3$, $c = (4, 6, 20)$, and $E = 26$. For this situation, the TU-game (N, w) is as follows:

$$\begin{aligned} &w(\{1\}) = 0; \quad w(\{2\}) = 2; \quad w(\{3\}) = 16; \\ &w(\{1,2\}) = 6; \quad w(\{1,3\}) = 20; \quad w(\{2,3\}) = 22; \\ &w(N) = 26. \end{aligned} \tag{9}$$

The core of this game is the set

$$\mathcal{C}(N, w) = \{x \in \mathbb{R}^n \mid x_1 \in [0, 4], x_2 \in [2, 6], x_3 \in [16, 20], x_1 + x_2 + x_3 = 26\}. \tag{10}$$

As Proposition 3 establishes, this set contains the whole set of bankruptcy allocations for its corresponding bankruptcy problem. It is depicted in Figure 1 into the set of efficient solutions of (N, w).

Now, given a bankruptcy problem $B = (E; c)$, we consider the next set of functions:

$$\mathcal{F}_*^B = \left\{f \in \mathcal{F}^B : f \text{ is convex}\right\}, \tag{11}$$

and the related games

$$\mathcal{V}^B = \left\{v_f : f \in \mathcal{F}_*^B\right\}. \tag{12}$$

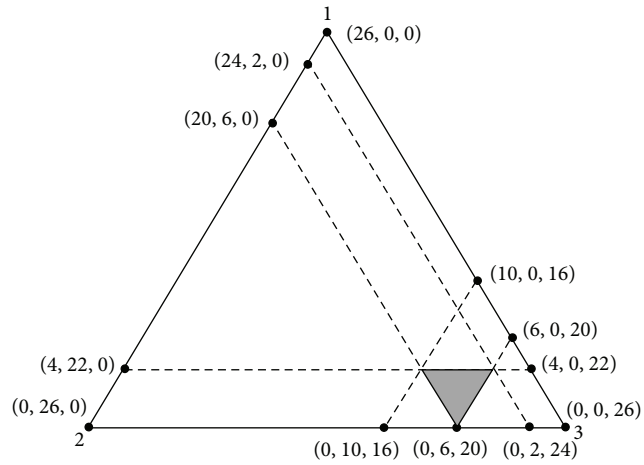


Figure 1: Barycentric representation of efficient solutions for the TU-game (N, w) associated with the bankruptcy problem [26; (4, 6, 20)]. The upper vertex corresponds to Agent 1, the lower-left to Agent 2, and the lower-right to Agent 3. The grey area is $\mathcal{C}(N, w)$.

We will show that a TU-game (N, v), with $v \in \mathcal{V}^B$, satisfies Proposition 3 if and only if it is O'Neill's game (N, w). The notation (N, w), or simply w, is maintained in the whole text to make reference to this special game.

Theorem 5. *Given a bankruptcy problem $B = (E; c)$, the core of a TU-game (N, v_f), with $v_f \in \mathcal{V}^B$, is exactly the set of allocations for the bankruptcy problem if and only if $v_f = w$.*

Proof. We will prove it by contradiction. If $v_f \in \mathcal{V}^B$ for certain $f \in \mathcal{F}_*^B$, with $v_f \neq w$ also satisfying Proposition 3, then there exists a coalition $S \subset N$ such that $v_f(S) \neq w(S)$.

Let us define the function $g \in \mathcal{F}^B$:

$$g(\tau) = \max\{0, E - C + \tau\}. \tag{13}$$

Notice that $w(S) = g(c(S))$ for all $S \subseteq N$. On one hand, if $v_f(S) < w(S)$, then $f(c(S)) < g(c(S))$ and, due to the monotonicity of f, we have

$$C - E < c(S) < C. \tag{14}$$

This inequality implies that there exists at least one $k \in N \setminus S$ such that $c_k < E$. For this agent k, the next condition holds:

$$C - E < c(S) \leq C - c_k < C, \tag{15}$$

and, because of the monotonicity and convexity of f and g, we have $f(C - c_k) < g(C - c_k)$ because $f(c(S)) < g(c(S))$.

According to Lemma 1, v_f is a convex game and then, $\mathcal{C}(N, v_f) \neq \varnothing$. A well-known result establishes that if a TU-game (N, v) is convex, then for every permutation σ of N the allocation $x^\sigma \in \mathbb{R}^n$, whose coordinates are

$$x_i^\sigma(v) = v(\sigma(1), \ldots, \sigma(i)) - v(\sigma(1), \ldots, \sigma(i-1)), \tag{16}$$

is in the core of the game. In particular, for the agent k defined previously and the TU-game (N, v_f), the kth coordinate of some of these vectors satisfies

$$\begin{aligned} x_k &= v_f(N) - v_f(N \setminus k) = E - f(C - c_k) \\ &> E - g(C - c_k) = c_k. \end{aligned} \tag{17}$$

It follows that x is not a solution to the bankruptcy problem $(E; c)$ and, then, we have reached a contradiction.

On the other hand, if $v_f(S) > w(S)$ there exists a subset $S \subset N$ such that

$$f(c(S)) > g(c(S)). \tag{18}$$

The continuity of f and g ensures that there exists an interval (α, β) where $f(\tau) > g(\tau)$ for every $\tau \in (\alpha, \beta)$. This fact, in addition with the convexity of f implies that $C - E \in (\alpha, \beta)$.

If

$$\alpha < c(S) \leq C - E < \beta, \tag{19}$$

then $E \leq c(N \setminus S)$. So, an allocation with $x_i = 0$ for each $i \in S$ and $x(N \setminus S) = E$ is not in the core of (N, v_f) because

$$v_f(S) = f(c(S)) > g(c(S)) = x(S) = 0. \tag{20}$$

Finally, if

$$\alpha < C - E < c(S) < \beta, \tag{21}$$

then $E > c(N \setminus S)$. So, an allocation such that $x_i = c_i$ for every $i \in N \setminus S$ and $x(S) = E - c(N \setminus S)$ is not in the core of (N, v_f) because

$$v_f(S) = f(c(S)) > g(c(S)) = E - c(N \setminus S) = x(S), \tag{22}$$

obtaining, again, a contradiction. □

In our last result we find two sets of convex TU-games; in the first one, the elements of their cores are allocations to a bankruptcy problem $(E; c)$; in the second set, those allocations are elements of their cores.

We recall the concept of *dual-core* of a TU-game (N, v):

$$\begin{aligned} \widehat{\mathscr{C}}(N, v) = \{ & x \in \mathbb{R}^n : x(N) = v(N), \\ & x(S) \leq v(S) \text{ for every } S \subset N \}. \end{aligned} \tag{23}$$

A related result about the relation between the core and the dual-core establishes that $x \in \mathscr{C}(N, v)$ if and only if $x \in \widehat{\mathscr{C}}(N, v^*)$ for every TU-game (N, v).

Theorem 6. *For every bankruptcy problem $B = (E; c)$, let us consider*

$$\begin{aligned} \mathscr{V}_1^B &= \left\{ v_f \in \mathscr{V}^B : \frac{E\tau}{C} \geq f(\tau) \geq g(\tau), \tau \in [0, C] \right\}, \\ \mathscr{V}_2^B &= \left\{ v_f \in \mathscr{V}^B : g(\tau) \geq f(\tau), \tau \in [0, C] \right\}, \end{aligned} \tag{24}$$

with g defined as in (13). *If $v_f \in \mathscr{V}_1^B$, then each element in the core of (N, v_f) is an allocation for the problem $(E; c)$. If $v_f \in \mathscr{V}_2^B$ then every bankruptcy allocation of $(E; c)$ belongs to the core of (N, v_f).*

Proof. Every TU-game (N, v_f), with $v_f \in \mathscr{V}_j^B$ $(j = 1, 2)$, is convex since f is a convex function. Then, it has a nonempty core. If $x \in \mathscr{C}(N, v_f)$ with $v_f \in \mathscr{V}_1^B$, then $x(N) = v_f(N) = E$ and

$$0 \leq x_i \leq E - f(c(N \setminus \{i\})) \leq E - \max\{0, E - c_i\} \leq c_i \tag{25}$$

for every $i \in N$. So, x is a bankruptcy allocation.

If $v_f \in \mathscr{V}_2^B$ and x is a bankruptcy allocation, then $x(N) = E = v_f(N)$ and $x(S) \leq c(S)$ for every $S \subset N$. Thus,

$$x(S) \leq \min\{E, c(S)\} = g^*(c(S)) \leq f^*(c(S)). \tag{26}$$

That is, $x \in \widehat{\mathscr{C}}(N, v_{f^*})$ and, recalling Lemma 2, $x \in \widehat{\mathscr{C}}(N, v_f^*)$. Then, $x \in \mathscr{C}(N, v_f)$. □

Example 7. Let $(E; c)$ be the bankruptcy problem of Example 4. In this case, the TU-game (N, w) referred to in Proposition 3 is $w(S) = \max\{0, c(S) - 1\}$ for every $S \subseteq N$, arising from the convex function $g_0(\tau) = \max\{0, \tau - 1\}$. For g_1 defined as

$$g_1(\tau) = \begin{cases} \dfrac{2}{5}\tau, & \text{if } \tau < 5, \\ \dfrac{24}{25}(\tau - 5) + 2, & \text{if } \tau \geq 5, \end{cases} \tag{27}$$

we have $v_{g_1} \in \mathscr{V}_1^B$ as follows:

$$\begin{aligned} & v_{g_1}(\{1\}) = 1.6; \quad v_{g_1}(\{2\}) = 2.96; \quad v_{g_1}(\{3\}) = 16.4; \\ & v_{g_1}(\{1, 2\}) = 6.8; \quad v_{g_1}(\{1, 3\}) = 20.24; \\ & v_{g_1}(\{2, 3\}) = 22.16; \quad v_{g_1}(N) = 26. \end{aligned} \tag{28}$$

The core of (N, v_{g_1}) contains all the efficient vectors (x_1, x_2, x_3) such that

$$\begin{aligned} & x_1 \in [1.6, 3.84]; \quad x_2 \in [2.96, 5.76], \\ & x_3 \in [16.4, 19.2]. \end{aligned} \tag{29}$$

Clearly, $\mathscr{C}(N, v_{g_1}) \subset \mathscr{C}(N, w)$. So, every element in $\mathscr{C}(N, v_{g_1})$ is a bankruptcy allocation for $(E; c)$, but there exists more.

On the other hand, for the next function g_2

$$g_2(\tau) = \begin{cases} 0, & \text{if } 0 \leq \tau < 4, \\ \dfrac{1}{2}(\tau - 4), & \text{if } 4 \leq \tau < 20, \\ \dfrac{9}{5}(\tau - 20) + 8, & \text{if } \tau \geq 20, \end{cases} \tag{30}$$

we have $v_{g_2} \in \mathscr{V}_2^B$ as follows:

$$\begin{aligned} & v_{g_2}(\{1\}) = 0; \quad v_{g_2}(\{2\}) = 1; \quad v_{g_2}(\{3\}) = 8; \\ & v_{g_2}(\{1, 2\}) = 3; \quad v_{g_2}(\{1, 3\}) = 15.20; \\ & v_{g_2}(\{2, 3\}) = 18.80; \quad v_{g_2}(N) = 26. \end{aligned} \tag{31}$$

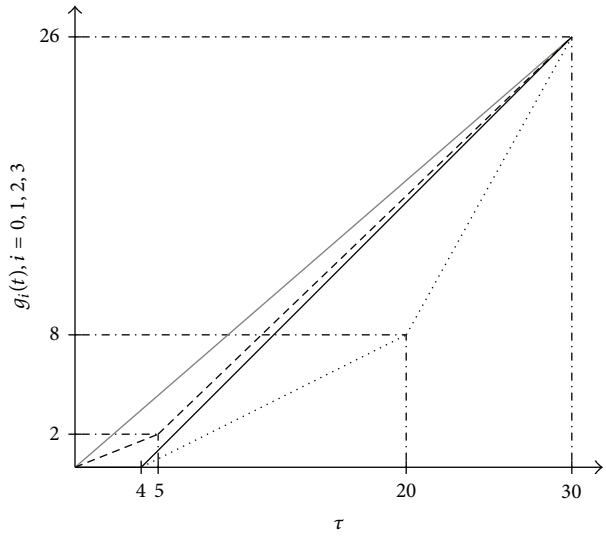


Figure 2: Graphics of the functions in Example 7. The solid black line corresponds to g_0, the dashed to g_1, and the dotted to g_2. The gray graphic corresponds to the function $g_3(\tau) = 26\tau/30$.

In this case, $\mathscr{C}(N, v_{g_2})$ contains all the efficient vectors (x_1, x_2, x_3) such that

$$x_1 \in [0, 7.2]; \qquad x_2 \in [1, 10.8], \qquad x_3 \in [8, 23]. \quad (32)$$

Now, we can easily check that $\mathscr{C}(N, w) \subset \mathscr{C}(N, v_{g_2})$, verifying Theorem 6 for the bankruptcy problem in Example 4 and functions g_1 and g_2 defined previously. In Figure 2, the functions g_0, g_1, and g_2 are depicted.

4. An Application

In this section we apply the results of Theorem 6 to find bankruptcy allocations belonging to the core of certain TU-games.

Given a bankruptcy problem $B = (E; c)$, let

$$\mathscr{W}^B = \left\{ v_{f_\alpha} : f_\alpha(\tau) = \max\{0, E - \alpha(C - \tau)\}, \alpha \in \left[\frac{E}{C}, 1\right] \right\}. \quad (33)$$

For every α, each TU-game (N, v_{f_α}) in $\mathscr{W}^B$ can be considered as a special case of the bankruptcy game introduced by O'Neill [1]. When $\alpha = 1$, (N, v_{f_1}) is precisely that TU-game is the function $f_1 = g$, with g defined as in (13). Moreover, from Theorem 6, the minimum value of α that ensures (N, v_{f_α}) does not have an empty core is $\alpha = E/C$. So, $\mathscr{W}^B \subseteq \mathscr{V}_1^B$ contains all possible generalizations of O'Neill's bankruptcy games with a nonempty core.

Let φ be a solution for TU-games such that $\varphi(N, v) \in \mathscr{C}(N, v)$ for convex games (e.g., the nucleolus or the Shapley value). So, we have that $\varphi(N, v_{f_\alpha}) \in \mathscr{C}(N, v_{f_\alpha})$ for every $\alpha \in [E/C, 1]$.

It is important that a bankruptcy allocation resulting from O'Neill's bankruptcy game generalization proposed before belongs to the core of the game. If this situation does not occur, there exist at least a pair of agents such that their assigned amount is less than the remaining estate after satisfying the claims of the other agents according to α, and we think that this will be an undesirable allocation. That is the reason we are considering the set of bankruptcy games $\mathscr{W}^B$ and a solution for TU-games such that it belongs to the core of the game.

By Theorem 6, $\varphi(N, v_{f_\alpha})$ is a solution to the problem $(E; c)$ for every $\alpha \in [E/C, 1]$. Now, we establish the next expression for every bankruptcy problem $B = (E; c)$ and every agent $i \in N$:

$$\rho_i^\varphi(B) = \frac{1}{1 - E/C} \int_{E/C}^1 \varphi_i\left(v_{f_\alpha}\right) d\alpha. \quad (34)$$

Proposition 8. *$\rho^\varphi(B) = (\rho_1^\varphi(B), \ldots, \rho_n^\varphi(B))$, with $\rho_i^\varphi(B)$ as in (34) being a solution to the bankruptcy problem $B = (E; c)$.*

Proof. $\varphi_i(N, v_{f_\alpha})$ is a solution to $(E; c)$. Then,

$$0 \le \varphi_i\left(N, v_{f_\alpha}\right) \le c_i, \quad \alpha \in \left[\frac{E}{C}, 1\right], \quad (35)$$

and, because of integral monotonicity,

$$\rho_i^\varphi(B) \le c_i. \quad (36)$$

Even more, $\varphi(N, v_{f_\alpha}) \in \mathscr{C}(N, v_{f_\alpha})$ implies that the sum of its entries is equal to E and, then,

$$\sum_{i=1}^n \rho_i^\varphi(B) = \frac{1}{1 - E/C} \int_{E/C}^1 \sum_{i=1}^n \varphi_i\left(N, v_{f_\alpha}\right) d\alpha = E. \quad (37)$$

□

The bankruptcy solution given by (34) is an expected value of the solutions φ taking a uniform distribution over the set of generalized O'Neill's bankruptcy games with a nonempty core. It is not our intention to provide a characterization or a full study of the properties of solution (34): we are only showing a way to use the results for providing new bankruptcy solutions.

Appendix

Proof of Lemmas

Proof of Lemma 1. First, we prove that, for every $\tau_1, \tau_2, \tau_3 \in [0, C]$ such that $\tau_1 + \tau_2 + \tau_3 \le C$, we have

$$f(\tau_2 + \tau_3) - f(\tau_1) \le f(\tau_1 + \tau_2 + \tau_3) - f(\tau_1 + \tau_2). \quad (A.1)$$

With $\tau_2 = 0$ we have an equality. Assume $\tau_2 > 0$ and recall that if f is a convex function on $[a, b]$ and $a < u_1 < u_2 < u_3 < b$, then

$$\frac{f(u_2) - f(u_1)}{u_2 - u_1} \le \frac{f(u_3) - f(u_1)}{u_3 - u_1} \le \frac{f(u_3) - f(u_2)}{u_3 - u_2}. \quad (A.2)$$

If $\tau_3 = \tau_2$ then $\tau_1 < \tau_1 + \tau_2 < \tau_1 + \tau_2 + \tau_3$, and using (A.2) we have

$$\frac{f(\tau_1+\tau_3)-f(\tau_1)}{\tau_3} \leq \frac{f(\tau_1+\tau_2+\tau_3)-f(\tau_1+\tau_2)}{\tau_3}. \quad (A.3)$$

In the same way, if $\tau_3 > \tau_2$ then $\tau_1 < \tau_1 + \tau_2 < \tau_1 + \tau_3 < \tau_1 + \tau_2 + \tau_3$. So,

$$\begin{aligned}\frac{f(\tau_1+\tau_3)-f(\tau_1)}{\tau_3} &\leq \frac{f(\tau_1+\tau_3)-f(\tau_1+\tau_2)}{\tau_3-\tau_2} \\ &\leq \frac{f(\tau_1+\tau_2+\tau_3)-f(\tau_1+\tau_2)}{\tau_3}\end{aligned} \quad (A.4)$$

and, if $0 < \tau_3 < \tau_2$ then $\tau_1 < \tau_1 + \tau_3 < \tau_1 + \tau_2 < \tau_1 + \tau_2 + \tau_3$, and consecutively

$$\begin{aligned}\frac{f(\tau_1+\tau_3)-f(\tau_1)}{\tau_3} &\leq \frac{f(\tau_1+\tau_2)-f(\tau_1+\tau_3)}{\tau_2-\tau_3} \\ &\leq \frac{f(\tau_1+\tau_2+\tau_3)-f(\tau_1+\tau_2)}{\tau_3}.\end{aligned} \quad (A.5)$$

These facts imply that (A.1) is true, specifically for $\tau_1 = c(S)$, $\tau_2 = c(T \setminus S)$, and $\tau_3 = c_i$, where $S \subseteq T \subseteq N \setminus \{i\}$. The lemma follows assigning the previous values to τ_1, τ_2 and τ_3; rewriting (A.1) we have

$$v_f(S \cup \{i\}) - v_f(S) \leq v_f(T \cup \{i\}) - v_f(T). \quad (A.6)$$

□

Proof of Lemma 2. From its definition, f^* is continuous and satisfies $f^*(0) = 0$, $f^*(C) = E$. For $\tau_1, \tau_2 \in [0, C]$ such that $0 \leq \tau_1 \leq \tau_2 \leq C$ we have

$$f^*(\tau_1) - f^*(\tau_2) = f(C-\tau_2) - f(C-\tau_1) \leq 0. \quad (A.7)$$

So, f^* is an increasing function and thus $f^* \in \mathcal{F}^B$. In addition, for every $\tau \in [0, C]$,

$$\begin{aligned}(f^*)^*(\tau) &= f(C) - f(C-C) - f(C) + f(C-C+\tau) \\ &= f(\tau).\end{aligned} \quad (A.8)$$

The second statement of the lemma follows from the equation

$$\begin{aligned}v_f^*(S) &= v_f(N) - v_f(N \setminus S) = f(C) - f\left(C - \sum_{i \in S} c_i\right) \\ &= f^*\left(\sum_{i \in S} c_i\right) = v_{f^*}(S).\end{aligned} \quad (A.9)$$

Now, supposing f is convex, we have, for every $t \in [0, 1]$ and $\tau_1, \tau_2 \in [0, C]$,

$$\begin{aligned}&tf^*(\tau_1) + (1-t) f^*(\tau_2) \\ &\quad = t(f(C) - f(C-\tau_1)) + (1-t)(f(C) - f(C-\tau_2)) \\ &\quad = f(C) - (tf(C-\tau_1) + (1-t) f(C-\tau_2)) \\ &\quad \leq f(C) - f(t(C-\tau_1) + (1-t)(C-\tau_2)) \\ &\quad = f(C) - f(C - (t\tau_1 + (1-t)\tau_2)) \\ &\quad = f^*(t\tau_1 + (1-t)\tau_2).\end{aligned} \quad (A.10)$$

It follows that f^* is concave. If f is concave, the previous inequality is reversed and we obtain the convexity of f^*. These facts prove the third statement of the lemma.

Finally, if $f(\tau) \leq g(\tau)$ for every $\tau \in [0, C]$, then

$$f^*(\tau) = f(C) - f(C-\tau) \geq g(C) - g(C-\tau) = g^*(\tau), \quad (A.11)$$

and, if $f^*(\tau) \geq g^*(\tau)$ then $f^*(C-\tau) \geq g^*(C-\tau)$, so $f(C) - f(\tau) \geq g(C) - g(\tau)$. Thus, $g(\tau) \geq f(\tau)$. Changing the corresponding inequality signs, the proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Breves semblanzas de Paloma Zapata y Sergio Hernández
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Algunas soluciones para problemas de división justa

Regresar

W. Olvera-López y F. Sánchez-Sánchez*

Resumen

En este trabajo mostraremos un algoritmo basado en gráficas bipartitas para realizar la repartición de un conjunto finito de bienes divisibles entre un conjunto de agentes, el cual puede verse como una generalización del algoritmo "*adjusted winner*". De igual manera, mostramos cómo adaptar el conocido procedimiento de Knaster para reparto de herencias al resultado obtenido mediante el algoritmo basado en gráficas para la distribución de bienes indivisibles.

Introducción

El problema conocido como "división justa" está relacionado con la manera de cómo dividir un conjunto de m objetos entre un conjunto de n agentes de una forma *justa*. Una problemática surge cuando quiere definirse con precisión lo que es *ser justo*. Aquí entenderemos por *división justa* una forma de repartir que satisfaga un conjunto de propiedades establecidas. Además, se considera que cada agente tiene una valuación positiva de cada objeto, por lo que de ahora en adelante nos referiremos a estos objetos como *bienes*. Con ello, el principal objetivo de la teoría de la división justa es construir procedimientos para llevar a cabo la distribución de los bienes (de acuerdo con alguna definición de justicia) y estudiar las propiedades de tales procedimientos.

Es posible reconocer dos tipos de problemas de división justa tomando en cuenta el tipo de bienes que tienen que dividirse: divisibles (continuos) e indivisibles (discretos). En este trabajo estudiaremos el problema de distribuir un conjunto de bienes divisibles. Este problema ha sido estudiado previamente por varios autores: Austin (1982), Brams y Taylor (1996), Neyman (1946), Steinhaus (1948) y Stromquist (1980). Iniciaremos este trabajo presentando el algoritmo de *adjusted winner*, propuesto por Brams y Taylor, que provee una manera libre de envidias, equitativa y eficiente para dividir un conjunto de bienes entre dos agentes. Y ahí radica su desventaja, ya que sólo está planteado para problemas bipersonales. De igual manera, mostraremos el procedimiento conocido como "algoritmo de Knaster", utilizado en problemas de reparto de herencias, y que se aplica, de manera general, en situaciones donde los bienes son indivisibles. Finalmente mostramos los fundamentos del algoritmo que proponemos, que puede considerarse una generalización del algoitmo de *adjusted winner* y cómo puede adaptarse via el algoritmo de Knaster para repartir bienes indivisibles de manera equitativa.

Para el caso general de bienes divisibles, estamos proponiendo una repartición que maximiza el valor de los bienes del agente que recibe la menor valuación. Para ello, resolvemos un problema de programación lineal usando su problema dual asociado y una gráfica bipartita para modelar las relaciones intrínsecas dadas por las condiciones Kuhn-Tucker del problema. Asociamos una gráfica bipartita con cada conjunto de variables duales y mostramos que la solución óptima tiene asociada un árbol. Por ello, resolvemos el problema enfocándonos en encontrar dicho árbol. Para hacer esto, iniciamos construyendo un árbol que represente una solución factible del problema dual: si esta solución factible no es óptima, realizamos cambios en el árbol de tal forma que represente un nuevo conjunto de variables duales factibles.

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Repetimos esta idea básica hasta tener el árbol óptimo. Además, la estructura de árbol permite encontrar los valores de la variables primales asociadas en forma secuencial.

En nuestra modelación, cada agente puede obtener cualquier fracción de cualquier bien. Además, suponemos que el valor que obtiene cada agente de cada bien es proporcional a la fracción que se le asignó, y el valor total asignado a cada agente es igual a la suma de sus fracciones. Con ello, es posible que cada agente obtenga bienes que él valora por más de $1/n$ de su monto total. Un problema surge cuando los bienes son indivisibles, ya que los resultados del algoritmo no son coherentes con las especificaciones del problema. Por ello, realizamos una adaptación del algoritmo de Knaster considerando que cada agente cuenta con alguna fracción antes de aplicar la idea de Knaster, pagándoles a cada agente la parte en la que valoran su fracción del bien para luego dividir el sobrante en partes iguales entre los agentes.

El trabajo está organizado de la siguiente manera: en la Sección 2 presentamos la terminología y definiciones que usaremos. En las secciones 3 y 4 mostramos el algoritmo de *adjusted winner* y el procedimiento de Knaster, respectivamente. Finalmente, en la sección 5 mostraremos el problema de programación lineal que resuelve la problemática que estamos modelando y mostramos las ideas detrás del algoritmo, incluyendo la formulación dual y el concepto de gráfica asociada con el problema. Terminamos con una sección donde se muestra la adaptación del procedimiento de Knaster a este algoritmo para la repartición de bienes indivisibles.

1. Definiciones y notación

Sean M un conjunto de m bienes divisibles (por simplicidad, *bienes*) y N el conjunto finito de n agentes. Sea $\mathbf{M}_{mn}$ el conjunto de matrices reales con entradas estrictamente positivas tal que $\sum_{i\in M} a_{ij} = c$ para cada $j \in N$, para cada $\mathbf{A} = [a_{ij}]_{i\in M, j\in N} \in \mathbf{M}_{mn}$ y algún $c \in \mathbf{R}$.

Cada matriz $\mathbf{A} \in \mathbf{M}_{mn}$ con entradas positivas representa un *problema de distribución*. La entrada a_{ij} indica la valoración que el agente j le da al bien i. En este problema supondremos que todos los agentes tienen una valoración positiva de cada bien.

Una solución para un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ es una matriz real $\mathbf{X} = [x_{ij}]_{i\in M, j\in N}$ tal que $\sum_{j\in N} x_{ij} = 1$ para cada $i \in M$. La entrada x_{ij} indica la fracción del bien i que es asignado al agente j.

Ejemplo 1 Consideremos M={1,2}, N={1,2,3} y el siguiente problema de distribución:

$$\mathbf{A} = \begin{bmatrix} 15 & 16 & 4 \\ 3 & 2 & 14 \end{bmatrix}$$

Una solución para este problema podría ser

$$\mathbf{X} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

De acuerdo con la solución anterior, la mitad del bien 1 se asigna al agente 1 y la otra mitad al agente 2. El agente 3 no está obteniendo ninguna fracción de este bien. También, nótese que los bienes se están distribuyendo en su totalidad. El agente 1 obtiene bienes que él valora en 9; el agente 2 está obteniendo un monto total de \$26/3, etcétera.

Definición 1. Diremos que una solución **X** para un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ es equitativa si el valor total obtenido de acuerdo con **X** para cada agente es el mismo. Esto es, **X** es equitativa si

$$\sum_{i\in M} a_{ij}x_{ij} = y\,;\, \forall j \in N \text{ y algún } y \in \mathbf{R}.$$

La solución mostrada en el Ejemplo 1 no es equitativa.

Definición 2 Una solución **X** para un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ es libre de envidias si cada agente $j \in N$ obtiene una valoración mayor o igual con los bienes que le asignaron que con cualquier otra asignación de los demás agentes. Esto es, **X** es libre de envidias si

$$\sum_{i\in M} a_{ij}x_{ij} \geq \sum_{i\in M} a_{ij}x_{ik} \;\; \forall k \in N$$

2. El método de *adjusted winner* (AW)

En esta sección mostraremos un algoritmo para la distribución de un conjunto M de bienes divisibles entre dos agentes (n=2). Fue creado por Steven J. Brams y Alan D. Taylor. En este algoritmo, un agente externo (un mediador o juez) determina qué bienes se le asignan a cada agente, así como determina cuál bien va a ser dividido entre ellos. Es bastante ilustrativo realizar la explicación del algoritmo mediante un ejemplo:

Ejemplo 2 Suponga que dos agentes quieren repartirse tres bienes M={A,B,C} y que tienen un total de 100 puntos para asignar a los bienes. Supóngase que ellos decidieron (cada uno en su columna correspondiente) distribuir los puntos como se indica en la siguiente matriz:

$$\mathbf{D} = \begin{bmatrix} 6 & 5 \\ 67 & 34 \\ 27 & 61 \end{bmatrix}$$

El algoritmo AW inicia asignando cada bien al agente que lo valore más. Así, durante esta primera etapa la asignación realizada es: los bienes A y B para el agente 1 y el bien C para el agente 2. Con esta asignación, el agente 1 está obteniendo bienes que él valora por 73 y el agente 2 bienes que él valora por 61. Denotemos por S_1 a la valoración que el agente 1 obtiene por los bienes que se le han asignado hasta el momento y por S_2 la valoración que está obteniendo el agente 2. Esta valoraciones varían en el método conforme van cambiando las asignaciones. El método AW busca una solución equitativa y claramente la asignación ofrecida en esta primera etapa no necesariamente lo es ($S_1 > S_2$). En este ejemplo, y con el fin de que los agentes obtengan en el mismo porcentaje, es necesario que el agente 1 transfiera bienes (o parte de ellos) al agente 2 para lograr la equitatividad ($S_1 = S_2$). Para ello, el algoritmo AW indica que la transferencia debe hacerse tomando en cuenta la razón entre los puntos del agente que va a transferir bienes y el agente al que los bienes van a transferirse, y que las transferencias deben iniciarse tomando en cuenta el orden ascendente de dichas razones. Para el ejemplo 2, estas razones para los bienes A y B (que tentativamente le fueron otorgados al agente 1) son las siguientes:

$$\frac{d_{A1}}{d_{A2}} = \frac{6}{5} = 1.2$$

$$\frac{d_{B1}}{d_{B2}} = \frac{67}{34} = 1.97$$

Con ello, el orden de los bienes a transferir del agente 1 al agente 2 será: A,B.

Con ello, el orden de los bienes a transferir del agente 1 al agente 2 será: A,B.

El algoritmo AW indica que se deben transferir bienes o la fracción de uno de ellos de acuerdo a este orden hasta que $S_1 = S_2$. Para el ejemplo anterior, se tienen las siguientes acciones:

- Se transfiere todo el bien A del agente 1 al agente 2, ya que si esto se hace se sigue cumpliendo que $S_1 > S_2$ (en este punto se tiene que S_1=67 y S_2=66). La nueva asignación es: el bien B para el agente 1 y los bienes A y C para el agente 2.
- Si se transfiere todo el bien B del agente 1 al agente 2 ya no se cumpliría que $S_1 > S_2$ (se tendría S_1=0). Esto indica que el bien B debe dividirse de alguna manera entre los agentes 1 y 2 para lograr la equitatividad. Sea p la fracción del bien B que conservará el agente 1 y $(1-p)$ la fracción que se le transferirá al agente 2. Con ello, el agente 1 obtendría una valoración $67p$ y el agente 2, $5+34(1-p)+61=100-34p$. Por ello, habrá que elegir p tal que

  $$67p = 100 - 34p.$$

  Esto conlleva a p=100/101.
- Por lo anterior, ya se tiene una repartición equitativa. La solución es

  $$\begin{bmatrix} 0 & 1 \\ \frac{100}{101} & \frac{1}{101} \\ 0 & 1 \end{bmatrix}$$

  Cada agente está obteniendo bienes que valora en 6700/101.

La descripción general del algoritmo es la siguiente:

Algoritmo 1 El método de *adjusted winner* (AW)
Entrada: Un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ para dos agentes
Salida: Una solución equitativa y libre de envidias.

1. Cada agente, por separado, distribuye 100 puntos entre los bienes.
2. Asígnese cada bien al agente que más lo valore.
3. Si la solución no es equitativa, sea i el agente con valoración total mayor y M_i el conjunto de bienes que se le asignaron. Sea j el otro agente.
4. Ordene los bienes en M_i ascendentemente respecto a a_{ki}/a_{kj}, $k \in M_i$.
5. Transfiera bienes de acuerdo al orden anterior del agente i al agente j mientras la nueva valoración total de i sea mayor a la de j.

Las pruebas de que el algoritmo AW es equitativo y libre de envidias pueden hallarse en Brams y Taylor (1996).

3. El procedimiento de Knaster (KP)

Este procedimiento permite la distribución de un conjunto de bienes indivisibles entre un conjunto finito de agentes. A menudo es referido como un procedimiento para reparto de herencias e involucra dinero externo para llevar a cabo la repartición. Asegura que cada agente obtiene al menos $1/n$ de su valoración total de los bienes.

Inicialmente se requiere que cada agente indique el valor monetario que para él tiene cada bien. Es necesario que esta información sea mediante un número real no negativo, y que también fungirá como un precio de compra/venta que el agente está comprometido a pagar en caso de que el bien se le asigne a él; o en caso contrario, a vender su fracción del bien. Con ello, se tiene un problema de distribución A.

El esquema general de este procedimiento se explica en el algoritmo 2:

Algoritmo 2 El procedimiento de Knaster (KP)
Entrada: Un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$
Salida: Una solución en la que cada agente obtiene al menos $1/n$ de su valoración total.
Para cada bien:

1. Véndase al agente que más lo valore. Debe pagar la valoración que propuso
2. A cada agente se le da, en dinero, $1/n$ de lo que él valora el bien.
3. El dinero sobrante se reparte en partes iguales.

Mostraremos el funcionamiento del procedimiento de Knaster anterior mediante un ejemplo.

Ejemplo 3 Resolvamos la situación del Ejemplo 2, suponiendo que las entradas de la matriz son los precios en los que los agentes valuaron los bienes. Con ello, el primer bien se le vende al agente 1. Él paga 6 unidades por ese bien. De ese dinero, al primer agente se le da 6/2=3 y al segundo, 5/2. Sobra 1/2 unidad, y se guarda. Para el caso del segundo bien, se le vende nuevamente al agente 1. Él paga 67 unidades, de las cuales 67/2 se le dan a él y 34/2=17 se le dan al agente 2; sobran 33/2, guardándose junto con 1/2 sobrante del primer bien. Finalmente, el tercer bien se le vende al agente 3, pagando un precio de 61 unidades. 27/2 se le dan al agente 1 y 61/2 al agente 2, sobrando 17 unidades. En total sobraron 34 unidades. Se dividen en partes iguales, asignándoles 17 de ellas a cada agente. Tanto los bienes como el dinero sobrante han sido repartidos, por lo que los resultados del algoritmo para este ejemplo son:

- El agente 1 obtuvo los bienes 1 y 2 y paga 6 unidades monetarias.
- El agente 2 obtuvo el tercer bien y recibe 6 unidades monetarias.

Ambos agentes terminaron con 67% de la valoración total del conjunto de bienes. Este hecho fue una mera coincidencia del ejemplo, ya que en general no tiene porqué ocurrir que todos obtengan la misma valoración total.

Una observación importante es que el procedimiento de Knaster sólo garantiza que cada agente reciba al menos $1/n$ de su valoración total. Para mayor información y detalles de este procedimiento, remítase a Steinhaus (1948) y Knaster (1946). Nótese que la entrada para el algoritmo AW son puntos, mientras que para el procedimiento de Knaster son precios.

4. Un algoritmo basado en gráficas

El esquema general que suponemos es que, para obtener el problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$, un juez pide a los agentes que distribuyan, de acuerdo a sus preferencias, un número fijo de puntos $c \in \mathbf{R}$ entre los bienes. Cada agente realiza esta tarea independientemente y sin saber la distribución que realizan los otros agentes. Entonces, el juez realizará la distribución de acuerdo a la respuesta que se obtenga.

En general, un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ puede tener varias soluciones equitativas. Un ejemplo natural de solución puede ser la matriz $\mathbf{X}$ donde para cada $i \in M$, $j \in N$. De acuerdo con esta solución, cada bien se divide en n partes iguales y una de ellas se le da a cada agente. Con ello, cada agente está obteniendo bienes que él valora por $1/n$ de su valoración total, dado que estamos asumiendo que el conjunto de bienes es el mismo para cada agente. Entonces, cada uno obtiene el mismo monto. El algoritmo que mostramos indica cómo encontrar una solución equitativa tal que el monto del agente que obtiene menor valuación sea tan grande como sea posible.

Para ello, es necesario resolver el siguiente problema de programación lineal:

$$Max\ z \qquad (1)$$

sujeto a

$$\sum_{i \in M} \boldsymbol{a_{ij} x_{ij}} \geq \boldsymbol{z} \qquad \forall \boldsymbol{j} \in \boldsymbol{N} \qquad (2)$$

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall i \in M \qquad (3)$$

$$x_{ij} \geq 0 \qquad \forall i \in M; \forall j \in N \qquad (4)$$

De acuerdo con las restricciones (4), la fracción que cada agente obtiene de cada bien debe ser un número real no negativo. Las restricciones (3) indican que cada bien se distribuye en su totalidad. Las desigualdades (2) establecen que el agente que está obteniendo el menor monto obtiene el máximo valor posible. Si todas estas restricciones se satisfacen como igualdades, entonces la solución dada por este problema de programación lineal es una solución equitativa, dado que estamos asumiendo que la valoración de cada agente al conjunto total de bienes es la misma. Nótese que este supuesto no se establece explícitamente en el problema de programación lineal.

Proposición 1 Sea $\mathbf{A} \in \mathbf{M}_{mn}$ un problema de distribución, $\mathbf{X}^*$ una solución óptima del problema de programación lineal asociado (1-4) y z^* el valor de la función objetivo correspondiente. Entonces,

$$\sum_{i \in M} a_{ij} x_{ij}^* = z^* \quad \forall j \in N$$

Gracias al resultado anterior, una solución al problema (1-4) es siempre una solución igualitaria. Aún más, dado que (1-4) es un problema de maximización, esta solución es tan grande como sea posible.

El problema dual correspondiente de (1-4)

$$Min\ w = \sum_{i \in M} v_i \qquad (5)$$

sujeto a

$$v_i \geq a_{ij} u_j \qquad \forall i \in M; \forall j \in N \qquad (6)$$

$$\sum_{j \in N} u_j = 1 \qquad (7)$$

$$u_j \geq 0 \qquad \forall j \in N \qquad (8)$$

donde estamos considerando los valores negativos de las variables u_j, $j \in N$ que corresponden con (2) para obtener un problema más conveniente.

Definiremos *vectores duales* a $v \in \mathbf{R}^M$ y $u \in \mathbf{R}^N$, donde sus coordenadas están indexadas de acuerdo con los conjuntos M y N, respectivamente. Dado un problema de distribución

, definimos una gráfica bipartita $g=(M,N,E)$, donde el conjunto de aristas está dado por $E=\{(i,j)|x_{ij}$ *es una variable básica de* (1-4)$\}$. Nótese además que para cada arista $(i,j)\in E$ se debe cumplir que $v_i=a_{ij}u_j$, donde v y u son las correspondientes variables duales de (5-8), y viceversa. Nos referiremos a esta gráfica como la a-gráfica de u y v. De aquí en adelante, consideraremos que el problema $\mathbf{A} \in \mathbf{M}_{mn}$ está dado y es fijo.

Diremos que los vectores u y v duales son *factibles* si satisfacen (6-8). También diremos que su a-gráfica asociada es factible. De manera análoga, definimos vectores duales *óptimos* y a-gráficas *óptimas*.

Estamos interesados en encontrar vectores duales óptimos **u** y **v** para un problema de distribución dado, ya que si obtenemos estos vectores, podremos encontrar una solución óptima **X** para (1-4). Lo haremos trabajando con a-gráficas básicas. Por todo lo mencionado anteriormente, el problema de encontrar soluciones óptimas para (1-4) y (5-8) es equivalente a encontrar una a-gráfica óptima. Es posible probar que existe al menos una a-gráfica óptima que es óptima. Estamos interesados en estas a-gráficas.

Ahora presentaremos el esquema general de nuestro algoritmo.

Algoritmo 3 Encontrar la mayor solución equitativa de un problema de distribución
Entrada: Un problema de distribución **A**
Salida: La mayor solución equitativa **X**

1. Hallar una a-gráfica factible de **A** con vectores duales u,v.
2. Asúmase que
3. Encontrar el valor de las variables primales **X**. Si **X** es una solución factible-primal, el algoritmo termina.
4. En otro caso, encontrar una nueva a-gráfica factible y regresar al paso 2.

Cuando un par de vectores duales **u** y **v** satisfacen (6) y (8), es posible construir un nuevo par de vectores $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$ que adicionalmente satisfagan (7):

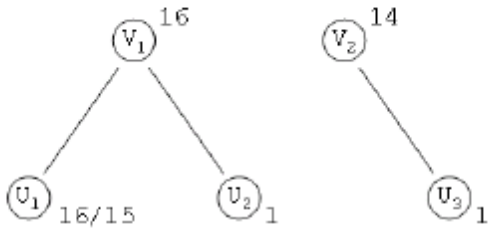
— ; — ; con (9)

Iniciaremos una explicación más detallada de cada paso del algoritmo anterior. Primeramente, mostraremos la propuesta para encontrar una a-gráfica básica factible. Iniciaremos con una a-gráfica sin aristas y conectaremos dos componentes hasta que tengamos una a-gráfica básica factible. Mostramos cómo hacer esto en el Procedimiento 3, cuya descripción se muestra en el Apéndice.

Proposición 2 La salida del Procedimiento 3 es una pareja de vectores duales u y v factibles y su a-gráfica asociada es básica.

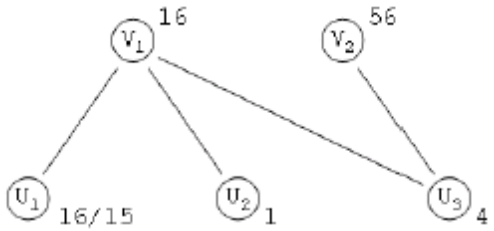
Ejemplo 4 Trabajaremos con el problema de distribución planteado en el Ejemplo 1. Aplicando el Procedimiento 3 a este problema, obtenemos inicialmente los vectores duales **u**=(1,1,1) y **v**=(16,14) y una a-gráfica con aristas (1,2) y (2,3). Necesitamos añadir aristas que unan las tres componentes que se formaron tal que las restricciones (6) continúen factibles. La arista (1,1) proporciona el menor valor de α, α=16/15 (si existiesen varias aristas con el mismo valor α, romper empates arbitrariamente). Actualizamos el valor de las variables duales asociadas con nodos en C_1, $1 \in N$, multiplicándolas por α. Luego, añadimos la arista (1,1). Obtenemos entonces la a-gráfica de la Figura 1.

Figura 1. Primera iteración del Procedimiento 3 aplicado al problema dado en el Ejemplo 1



Aún no se tiene una a-gráfica con $m+n-1$ aristas. Iterando nuevamente, obtenemos $(h,k)=(1,3)$ con α=4. Actualizamos el valor de las variables duales asociadas con nodos en C_3, $3 \in N$, y añadimos esta arista. Así, obtenemos la a-gráfica de la Figura 2.

Figura 2. Segunda iteración del Procedimiento 3 aplicado a la a-gráfica de la Figura 1



Ya se tiene una a-gráfica con $m+n-1$ aristas. Así, el Procedimiento 3 ha finalizado. La Figura 2 muestra la a-gráfica básica final. Aplicando las ecuaciones (9), obtenemos los vectores duales $\hat{\mathbf{v}} = \frac{1}{91}(240,280)$ y $\hat{\mathbf{u}} = \frac{1}{91}(16,15,90)$. Calculando el valor de la función objetivo para estas variables duales, obtenemos w=1080/91.

Ahora veremos cómo encontrar los valores de las variables primales **X** dados **u** y **v**, vectores duales factibles de un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$ siendo g su correspondiente a-gráfica básica. De acuerdo con el Algoritmo 3, asumiremos que se ha encontrado el óptimo y que por lo tanto coinciden las soluciones dual y primal. En el Procedimiento 4, que se muestra en el Apéndice, y bajo este supuesto, proponemos un método para hallar la solución del problema primal asociado **u** y **v** tal que se satisfagan las restricciones (2) y (3), posiblemente fallando en las restricciones (4).

Proposición 3 Las variables primales **X** dadas como salida del Procedimiento 4 satisfacen las restricciones (2) y (3).

Ejemplo 5 Consideremos los vectores cuales $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$ asociados con el problema del Ejemplo 1 dados en la parte final del Ejemplo 4 y su a-gráfica dada en la Figura 2. Aplicando los pasos 2 y 3 del Procedimiento 4, obtenemos

$$x_{23} = 1, \quad x_{11} = \frac{72}{91}, \text{ y } \; x_{12} = \frac{135}{182}.$$

Con esta información, podemos encontrar los valores de las demás variables básicas. Sabemos que $x_{11} + x_{12} + x_{13} = 1$. Entonces, obtenemos

$$x_{13} = -\frac{97}{182},$$

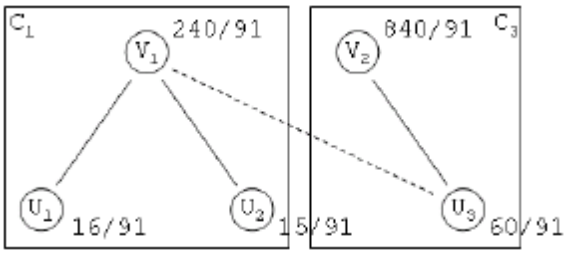
x_{ij}=0 en otro caso. De igual forma, este valor de x_{13} verifica $14x_{23} + 4x_{13} = \frac{1080}{91}$. Claramente, esta no es una solución factible del problema primal (1-4) ya que $x_{13} < 0$ (no se satisface (4)). Así, los vectores duales $\bar{u}, \bar{v}$ no son óptimos.

Es posible que los vectores duales $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$ de un problema de distribución **A** arrojados por el Procedimiento 3 no sean óptimos. Esto es, si **u** y **v** y su a-gráfica básica son la entrada del Procedimiento 4, puede ser que algunas entradas de la salida **X** de este procedimiento sean números negativos. Este hecho indica que **X** no es una solución factible de (1-4). Entonces, **u** y **v** no son óptimos. En el Procedimiento 5 (ver Apéndice), mostramos cómo modificar una a-gráfica básica g tal que tenga asociados un nuevo conjunto de vectores duales factibles.

Proposición 4 Los vectores duales $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$ y su a-gráfica, resultantes del Procedimiento 5 son factibles y son diferentes de los vectores de entrada.

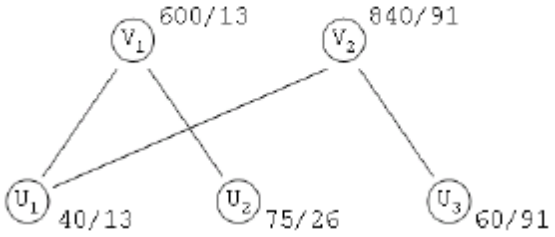
Ejemplo 6 Consideremos los vectores duales **u** y **v**, su a-gráfica g resultante del Ejemplo 4 y la matriz de distribución **X** resultante del Ejemplo 4 como entrada al Procedimiento 5. En este caso, existe una única variable primal negativa, x_{13}. Entonces, removemos la arista (1,3) (Pasos 1 y 2). Esta remoción produce la desconexión de g en dos componentes, C_1, $1 \in N$, y C_3, como se muestra en la Figura 3.

Figura 3. Desconexión de g en dos componentes, C_1 y C_3, luego de remover la arista (1,3)



Necesitamos añadir una arista para obtener una a-gráfica básica. Cuando hacemos nuevamente el paso 3, notamos que debemos verificar únicamente las aristas (2,1) y (2,2). La arista con mínimo valor α es (2,1), α=35/2. Ahora, actualizamos el valor de las variables duales en C_1, $1 \in M$, multiplicándolas por α. Luego, añadimos la arista. Entonces, obtenemos la a-gráfica de la Figura 4.

Figura 4. *a*-gráfica resultante de añadir la arista (2,1) a la gráfica de la Figura 4



Entonces, normalizamos las variables duales para obtener $\hat{\mathbf{v}} = \frac{1}{241}(1690.336)$ y $\hat{\mathbf{u}} = \frac{1}{241}(112{,}105{,}24)$. Ahora, luego de explicar los pasos del algoritmo, citaremos el resultado principal.

Teorema 1 El Algoritmo 3 produce, como salida, una solución equitativa maximal.

Las pruebas de las proposiciones y teoremas, así como un resultado sobre la convergencia y aplicabilidad de este algoritmo, puede encontrarse en Olvera-López y Sánchez-Sánchez (2013).

Ejemplo 7 Continuando la aplicación del Algoritmo 3 para los vectores duales $\bar{u}$ y $\bar{v}$ del problema de distribución dado en el Ejemplo 1 y su *a*-gráfica obtenidos en el Ejemplo 6. Así, necesitamos aplicar el Procedimiento 4 para encontrar las correspondientes variables primales **X**. Luego de hacer esto, notamos que **X** es una solución factible primal. Entonces, el algoritmo termina. La Figura 4 muestra una a-gráfica óptima y sus vectores duales asociados son variables óptimas de (5-8), con w=2016/41≈8.3651. Entonces, una solución equitativa para el Ejemplo 1 es de la siguiente manera:

$$\mathbf{X} = \frac{1}{241}\begin{bmatrix} 115 & 126 & 0 \\ 97 & 0 & 144 \end{bmatrix} \qquad (10)$$

Hay que hacer notar que si asignamos 1/3 de cada bien a cada agente cada uno de ellos obtiene bienes que valoran únicamente en 6 unidades.

5. Algoritmo basado en gráficas + KP

En esta sección mostraremos cómo adaptar el procedimiento anterior para dividir un conjunto de bienes indivisibles. Para ello, usaremos el procedimiento de Knaster y al igual que éste, necesita dinero externo para realizar la asignación.

Supóngase que se aplica el algoritmo basado en gráficas para encontrar la máxima solución equitativa de un conjunto de bienes, cuyas valoraciones están dadas de acuerdo a un problema de distribución **A**. Esta matriz se interpretará como el precio que cada agente está dispuesto a comprar o vender cada bien o parte de él. Al igual que en KP, cada bien es vendido al agente con el precio más alto. Luego, por cada bien, se le paga a cada agente que es dueño de una fracción del bien lo que para él significa esa fracción (las fracciones provienen del resultado del algoritmo basado en gráficas). El dinero sobrante por cada bien se reparte de manera igualitaria entre todos los agentes.

Supongamos que los bienes a repartirse en el Ejemplo 1 son indivisibles. Aplicando el algoritmo basado en gráficas explicado en la sección anterior se obtiene la matriz de asignaciones dada en (10). De acuerdo al procedimiento explicado en el párrafo anterior, el primer bien es vendido al agente 2 a un precio de 16 unidades monetarias, mientras que el segundo bien se vende al agente 3 por 14 unidades. Luego, de acuerdo a (10) el agente 1 posee 115/241 del bien 1, que para él representan 1725/241 unidades monetarias. Se le da esta cantidad al agente 1. El agente 2 tiene los restantes 126/141 bien, por lo que a él se la dan 2016/241 unidades monetarias. Esto implica que sobran 115/241 de las originales 16 unidades que se pagaron por el bien 1, las cuales se dividirán en partes iguales para los tres agentes, por lo que cada uno de ellos recibe 115/273 unidades monetarias adicionales. Se realiza el mismo procedimiento con el segundo bien. La siguiente matriz denota las cantidades que cada agente recibe de la venta de cada bien

Con ello, puede resumirse el resultado de la repartición de la siguiente manera:

- El agente 1 recibe 10 unidades monetarias.
- El agente 2 recibe el primer bien y paga 6 unidades.
- El agente 3 recibe el segundo bien y paga 4 unidades.

Así, cada agente está recibiendo (ya sea en unidades monetarias o en bienes) un 55.55% de su valoración total.

Proposición 5 La solución resultante de aplicar el procedimiento de Knaster a la salida del algoritmo basado en gráficas como se indica al inicio de esta sección es una solución equitativa.

Demostración. Como se indica en el Teorema 1, la salida del algoritmobasado en gráficas es una solución equitativa. Entonces, de acuerdo con el procedimiento explicado en esta sección, primeramente cada agenta *cambia* por dinero la fracción del bien que le fue asignado. Luego, como el agente que se queda con el bien fue el que más lo valora y pagó por él dicho valor, el valor sobrante entre lo que pagó y el cambio que se les hizo por dinero a los agentes que tenían fracciones del bien se reparte en partes iguales entre todos los agentes. Además, esto se repite para cada bien. Con esto la solución es claramente equitativa.

Apéndice

En esta sección mostraremos el esquema de los procedimientos referidos en la Sección 5.

Procedimiento 1 actualización de una variable dual u_k y sus relaciones

Entrada: Vectores duales $\mathbf{v} \in \mathbf{R^M}, \mathbf{u} \in \mathbf{R^N}$; un número real α; un natural k y $N\}$.

Salida: Vectores duales actualizados

1. Establezca $u_k := \alpha \cdot u_k$
2. Para cada
   Ejecutar el procedimiento 2 con entrada).

Procedimiento 2 Actualización de una variable dual v_h y sus relaciones

Entrada: Vectores duales $\mathbf{v} \in \mathbf{R^M}, \mathbf{u} \in \mathbf{R^N}$; un número real α; un natural $k \in N$ y $S \subseteq \{(i,j) | i \in M, j \in N\}$.

Salida: Vectores duales actualizados $\mathbf{v} \in \mathbf{R^M}, \mathbf{u} \in \mathbf{R^N}$.

1. Establezca $v_h := \alpha \cdot v_k$
2. Para cada $(l,k) \in S$
3. Ejecutar el procedimiento 1 con entrada $(\mathbf{v}, \mathbf{u}, \alpha, l, S \setminus \{(l,k)\})$.

En estos procedimientos existe una relación, indicada por *S*, entre el par de vectores duales **u** y **v**. Así, en el Procedimiento 1, el valor de u_k se actualiza multiplicándola por α y luego, todas las variables en **v** relacionadas con u_k, de acuerdo con *S*, se actualizan también. En el Procedimiento 2 el proceso es similar, pero actualizando primeramente una variables dual v_k y luego, todas las variables en **u** relacionadas con v_k.

Dada una gráfica bipartita $g=(M,N,E)$, denotamos por $C(g)$ al conjunto de componentes de g. Para una componente $\in C(g)$, denotamos por $N(C)$ al conjunto de nodos en C. Para una nodo k, C_k denotará la componente $C \in C(g)$ que contiene al dicho nodo $k \in N(C)$.

Procedimiento 3 Encontrando una *a*-gráfica factible

Entrada: Un problema de distribución $\mathbf{A} \in \mathbf{M}_{mn}$

Salida: Vectores duales $\mathbf{v} \in \mathbf{R^M}, \mathbf{u} \in \mathbf{R^N}$ para **A** y su *a*-gráfica *g=(M,N,E)*

1. Iniciar con $\varnothing$ *E:=*
2. Incluir a *E* la arista con la entrada máxima por cada renglón.
3. Incluir a *n*-1 aristas adicionales manteniendo factibilidad dual. Se debe incluir la arista

$$(h,k) := argmin\left\{\alpha := \frac{v_i}{a_{ij}u_j} \middle| (i,j) \notin E, i \notin N(C_j)\right\}$$

4. Aplicar (9) a las variables duales obtenidas previamente.

Dada una gráfica bipartita *g= (M,N,E)* y un nodo *i,* deg(*i*,*g*) denota el número de aristas conectando a *i* con cualquier otro nodo en *g*.

Procedimiento 4 encontrando valores de las variables primales dada una *a*-gráfica.

Entrada: Vectores duales factibles $\mathbf{v} \in \mathbf{R^M}, \mathbf{u} \in \mathbf{R^N}$ para un problema $\mathbf{A} \in \mathbf{M}_{mn}$ y su *a*-gráfica asociada *g=(M,N,E)*

Salida: Variables primales **X** que satisfagan (2) y (3).

1. Calcular el valor de la función objetivo *w* del problema dual.
2. Para cada $(i,j) \in E$ tal que deg(*i*,*g*)=1, establecer $x_{ij} := 1$
3. Para cada $(i,j) \in E$ tal que deg(*i*,*g*)=1, establecer $x_{ij} := w/a_{ij}$
4. Encontrar el valor de las demás variables usando la ecuación (3) y

$$\sum_{i \in M} a_{ij} x_{ij} = w \quad \forall j \in N.$$

En un adecuado orden secuencial.

Procedimiento 5 Calculando nuevos vectores duales factibles.

Entrada: Vectres duales no óptimos **u** y **v** para un problema **A**, su a-gráfica básica y las variables primales **X** resultantes del procedimiento 4.

Salida: Vectores duales $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$ para **A** y su a-gráfica básica $g´=(M.N,E´)$

1. Sea $(h,k) \in E$ tal que $x_{hk}<0$. Si existen varias variables primales negativas, escójase la que tenga el menor valor.
2. Establezca $E´:=E \setminus (h,k)$.
   Calcular

$$(i,j) := argmin\left\{\alpha := \frac{v_i}{a_{ij}u_j}\middle|(i,j) \notin E', i \in N(C_k), j \in N(C_h)\right\}$$

3. Llamar el procedimiento 1 con entrada (**v**,**u**, α,j,$E´$) y luego $E´:=E´U(i,j)$. Aplicar la normalización dada por (9) a **u** y **v** para obtener $\hat{\mathbf{u}}$ y $\hat{\mathbf{v}}$.

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Cooperative games with homogeneous groups of participants

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Cooperative games with homogeneous groups of participants

L. Hernández-Lamoneda · Francisco Sánchez-Sánchez

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Abstract We consider a class of games, which we call "groups' games", in which players are partitioned into classes within which players may be considered indistinguishable. We compute explicit formulas for some of the principal values restricted to these games. This is particularly useful for dealing with games with a large number of players where the usual formulas would be impractical (or impossible) to compute. We give several examples that illustrate how this idea may be applied to concrete situations.

Keywords Shapley value · Cooperative games · Values for TU games · Homogeneous groups games

1 Introduction

One of the principal practical problems in cooperative game theory arises when the number of agents is large. As the number of players grows, the number of all possible coalitions grows exponentially, making it impractical (or even impossible) to compute typical solutions, such as Shapley's value. Nevertheless, in some instances, large populations of agents may be grouped in somewhat homogeneous classes.

In this article, we propose a method to deal with this situation. In particular, we give easily computable versions of the principal values in cooperative game theory adapted to, what we call, groups' games. In general, though we consider players individually, for groups' games the number of evaluations needed to compute Shapley's (or any other solution) is a polynomial function in the number of players.

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There have been different approaches to deal with large numbers of agents. Perhaps, most notable, is the idea of considering a partition of the set of players and then look at each partition class as a (weighted) player in a new game. This idea may be the only sensible thing to do in some applications, but we believe that in certain situations (that we exemplify ahead) it might be preferable to still consider players individually, while acknowledging that they can be grouped into homogenous classes. As the examples show, solutions under these two different approaches will vary significantly, and we argue that the axiomatics of Shapley's value (for example) support our choice.

The article contains the following. The first two sections introduce groups' games and compute explicit formulas for linear, symmetric and efficient solutions on them (in particular, the one for Shapley's value) first and also for semivalues (an explicit formula for Banzhaff's is given here). Precise statements are given, though the proofs are delayed to an appendix. Whereas computation of these solutions for arbitrary games requires 2^n evaluations, for game groups', partitioned into m classes, the number of evaluations is of the order of n^m.

Section 4 consists of several classes of examples to which the notion of groups' games may be applied. It is important to clarify that some of these problems (and their solutions) are well known in the literature, yet they illustrate some advantage of our setup. Examples 1 and 3 show how to obtain the "classical" solution in a much faster and efficient way. Example 2 shows how one obtains a "fairer" solution than with the usual partition methods. Other examples point to possible applications.

2 Warmup: games with 2 homogeneous groups

Let $N = N_1 \cup N_2$ be a partition of $N = \{1, 2, ..., n\}$ and $v : 2^N \to \mathbb{R}$ a cooperative game, in TU form. We assume that, for the game v, the players in each N_j are indistinguishable (or symmetric, or substitutes); that is, we assume that

$$v(\theta S) = v(S),$$

for every permutation $\theta : N \to N$ leaving invariant the given partition (i.e. $\theta(N_j) = N_j$, $j = 1, 2$).

We will say that the players are "of two types" and that (N, v) is a "groups' game".

It is clear that the value of such a game in a given coalition depends only on the number of players, of each type, that the coalition gathers. Thus, every groups' game v can be written as

$$v(S) = q(s_1, s_2)$$

for some function $q : \{0, 1, ..., n_1\} \times \{0, 1, ..., n_2\} \to \mathbb{R}$, where $s_j = |S \cap N_j|$ and $n_j = |N_j|$, $j : 1, 2$.

Before stating the formula for a linear, symmetric and efficient solution, on groups' games, recall that such solutions are uniquely determined by $n-1$ coefficients $\{\beta_s\}_1^{n-1}$ (see Ruiz et al. 1998 or Hernández-Lamoneda et al. 2007).

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Theorem 1 *Let $v(S) = q(s_1, s_2)$ a groups' game, as before, and ϕ a linear, symmetric and efficient solution. Then*

$$\phi_i(v) = \frac{q(n_1, n_2)}{n} + \sum_{(s_1,s_2)\in\Lambda_0} \left[\frac{s_1 n_2 - s_2 n_1}{n_1 s(n-s)}\right] \binom{n_1}{s_1}\binom{n_2}{s_2} \beta_s q(s_1, s_2),$$

where $\Lambda_0 = \{0, 1, ..., n_1\} \times \{0, 1, ..., n_2\} \setminus \{(0, 0), (n_1, n_2)\}$, $s = |S| = s_1 + s_2$, $n = |N| = n_1 + n_2$ and $i \in N_1$ (a corresponding formula may be stated for the players in N_2).

For Shapley's value: $\beta_s = \binom{n}{s}^{-1}$, therefore

Corollary 1 *Shapley's value, on groups' games (for a partition into two groups), is given by*

$$Sh_i(v) = \frac{q(n_1, n_2)}{n} + \sum_{(s_1,s_2)\in\Lambda_0} \left[\frac{s_1 n_2 - s_2 n_1}{n_1 s(n-s)}\right] \frac{\binom{n_1}{s_1}\binom{n_2}{s_2}}{\binom{n}{s}} q(s_1, s_2).$$

3 Homogeneous groups

The more general setup is as follows: we consider a partition $\{N_j\}_{j\in M}$ of N. We consider games, in TU form, such that for every $j \in M$, the players inside N_j are indistinguishable among themselves. That is, we look for games v such that

$$v(\theta S) = v(S), \quad \text{for all permutations } \theta : N \to N, \text{ such that } \theta(N_j) = N_j, \quad j \in M.$$

We will say, then, that v is a groups' games with $m = |M|$ types. Again as before, such a game can be uniquely written as

$$v(S) = q(s_1, s_2, ..., s_m),$$

where $s_j = |S_j| = |S \cap N_j|$, for $j \in M = \{1, ..., m\}$.

Set

$$\Lambda = \{0, 1, ..., n_1\} \times \{0, 1, ..., n_2\} \times \cdots \times \{0, 1, ..., n_m\},$$

and $\Lambda_0 = \Lambda \setminus \{(0, ..., 0), (n_1, ..., n_m)\}$. Then we can state

Theorem 2 *Let (N, v) be a groups' game with m types. Let $q : \Lambda \to \mathbb{R}$ be such that $v(S) = q(s_1, ..., s_m)$. Then if ϕ is the linear, symmetric and efficient solution correpsonding to $\{\beta_s\}_1^{n-1}$, we have*

$$\phi_i(v) = \frac{q(n_1, ..., n_m)}{n} + \sum_{(s_1,...,s_m)\in\Lambda_0} \left[\frac{s_k n - n_k s}{n_k s(n-s)}\right] \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} \beta_s q(s_1, ..., s_m)$$

for $i \in N_k$.

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Corollary 2 *Shapley's value on the groups' game* $v(S) = q(s_1, ..., s_m)$ *is given by*

$$Sh_i(v) = \frac{q(n_1, ..., n_m)}{n} + \sum_{(s_1,...,s_m)\in\Lambda_0} \left[\frac{s_k n - n_k s}{n_k s(n-s)}\right] \frac{\binom{n_1}{s_1}\cdots\binom{n_m}{s_m}}{\binom{n}{s}} q(s_1, ..., s_m)$$

for $i \in N_k$.

3.1 Semivalues for groups' games

It is shown in Dubey et al. (1981) that all linear, symmetric and null solutions (semivalues) are of the form

$$\psi_i(v) = \sum_{S\not\ni i} p_s \left[v(S\cup\{i\}) - v(S)\right],$$

where the p_s, $s : 0, 1, ..., n-1$, are arbitrary real numbers.

Theorem 3 *Let* (N, v) *be a groups' game with m types. Let* $q : \Lambda \to \mathbb{R}$ *be such that* $v(S) = q(s_1, ..., s_m)$. *Then if* ψ *is a semivalue, then*

$$\psi_i(v) = \sum_{(s_1,...,s_m)\in\Lambda\setminus\{0\}} \frac{s_k p_{s-1} - (n_k - s_k)p_s}{n_k} \binom{n_1}{s_1}\cdots\binom{n_m}{s_m} q(s_1, ..., s_m)$$

for $i \in N_k$.

Corollary 3 *Banzhaf's value for groups' games may be computed by*

$$B_i(v) = \sum_{(s_1,...,s_m)\in\Lambda\setminus\{0\}} \frac{2s_k - n_k}{2^{n-1}n_k} \binom{n_1}{s_1}\cdots\binom{n_m}{s_m} q(s_1, ..., s_m)$$

for $i \in N_k$.

4 Applications

4.1 The United Nations Security Council

The United Nations Security Council is formed by 15 nations, of which 5 are permanent members and 10 are non-permanent ones. The five permanent ones are China, France, Russia, United Kingdom and United States. The ten non-permanent members are regional representatives chosen according to some fair geographical distribution. Each member in the Council has a vote. Security Council decisions require the affirmative votes of nine members. Nevertheless, the five permanent members have veto power.

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Let N_1, N_2 be the sets of permanent and non-permanent (respectively) members, and let $N = N_1 \cup N_2$. Thus, a voting scenario may be represented by the following game:

$$v(S) = \begin{cases} 1 & \text{if } N_1 \subset S \quad \text{and} \quad |S \cap N_2| \geq 4; \\ 0 & \text{otherwise.} \end{cases}$$

A solution for this cooperative game gives us a measure of the voting power of the countries in the Security Council. Notice that the above game is a groups' game. Thus, we can easily compute its Shapley value. For example, for a permanent member we obtain:

$$Sh_1(v) = \frac{1}{15} + \sum_{s_2=4}^{9} \left[\frac{50 - 5s_2}{5(5+s_2)(10-s_2)} \right] \frac{\binom{10}{s_2}}{\binom{15}{5+s_2}} = \frac{421}{2145}.$$

Whereas for the non-permanent ones, Shapley predicts a voting power of

$$Sh_6(v) = \frac{1}{15} + \sum_{s_2=4}^{9} \left[\frac{5s_2 - 50}{10(5+s_2)(10-s_2)} \right] \frac{\binom{10}{s_2}}{\binom{15}{5+s_2}} = \frac{4}{2145}.$$

Remark 1 One may instead compute the voting power of each UN member using Banzhaf's value namely, the formula from Corollary 3 above; again, this homogeneous groups' version of the usual semivalue may be easily computed and yields:

$$B_1(v) = \frac{212}{4096}, \quad \text{and} \quad B_6(v) = \frac{21}{4096}.$$

Which gives, roughly, a tenfold difference, rather than the hundredfold difference that gives Shapley's.

4.2 Managing retention groups

We take an example similar to one in Lemaire (1991).

Insurance companies usually charge their insurance costs in such a way that their probability of insolvency is very low. On the other hand, it is known that the probability that a random variable hits a value below $\mu+3\sigma$, here μ is the mean and σ^2 the variance, is very high. For example, if $X \sim N(\mu, \sigma^2)$ then $P(X \leq \mu+3\sigma) = 0.99865$. Thus, if X stands for the amount to be paid (by the insurance company) for all losses occurred in a fixed period of time, the total insurance cost for the period could be

$$E(X) + 3\sigma_X.$$

The problem that remains to be solved is how to distribute this cost among the policy-holders.

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To show how we suggest to attack this problem, we assume as a first scenario, that we have two groups of insured people: N_1 and N_2. The first one, with n_1 elements, has each individual insured for an amount c_1 and with probability $p_1 = Pr(X_i = 1)$ that policy-holder $i \in N_1$ will suffer an accident; the second group, N_2, has n_2 members, each insured for an amount of c_2, and with $p_2 = Pr(Y_j = 1)$ the probability of loss for $j \in N_2$. Assume that the individuals' risks are independent and that the distributions have finite variance.

Let $N = N_1 \cup N_2,\ S \subset N$ and

$$Z^S = \sum_{i \in S_1} c_1 X_i + \sum_{j \in S_2} c_2 Y_j,$$

where $S_k = S \cap N_k,\ k = 1, 2$.

Thus, the total insurance cost $v(S)$ that must be payed by the insured in S should equal

$$v(S) = E[Z^S] + 3\sigma_{Z^S},$$

where

$$E[Z^S] = \sum_{i \in S_1} c_1 E[X_i] + \sum_{j \in S_2} c_2 E[Y_j] = s_1 p_1 c_1 + s_2 p_2 c_2,$$

and

$$Var(Z^S) = Var\left(\sum_{i \in S_1} c_1 X_i\right) + Var\left(\sum_{j \in S_2} c_2 Y_j\right),$$

where $s_k = |S_k|$.

From these, it follows that

$$v(S) = E[Z^S] + 3\sigma_{Z^S} = s_1 p_1 c_1 + s_2 p_2 c_2 + 3\sqrt{s_1 p_1 (1 - p_1) c_1 + s_2 p_2 (1 - p_2) c_2}$$

from which we get a cooperative game and we can apply to it Shapley's value (or some other solution) to distribute the cost. Since v is actually a groups' game, we can compute the solution using the formulas presented in this article.

In the general case, we would have m different groups of policy-holders $N_1, ..., N_m$, $N = N_1 \cup \cdots \cup N_m$ and a cooperative game

$$v(S) = \sum_{k=1}^{m} s_k p_k c_k + 3\sqrt{\sum_{k=1}^{m} s_k p_k (1 - p_k) c_k},$$

where $s_k = |S_k| = |S \cap N_k|,\ p_k$ is the accident probability and c_k the amount covered by the insurance for an individual in group N_k.

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For example, assume we have 3 groups: the first one with 100 insured people, the second one with 50 and the third with 20. Each person insures one unit (i.e. $c_k = 1,\ k = 1, 2, 3$), and the probabilities of loss are $p_1 = 0.01,\ p_2 = 0.02,\ p_3 = 0.03$.

We can define, as above, a groups' game v and apply to it Shapley's value (from Theorem 2) to compute how much each individual must pay. The answer is:

Each insured $i \in N_1$ pays:	Each insured $i \in N_2$ pays:	Each insured $i \in N_3$ pays:
0.02909644285623	0.05644633496039	0.08302665489064

Instead, one could try to solve this problem considering each group as a single player ($\bar{k} := N_k$); one would define the following (three player) game w:

T	$w(T)$
$\{\bar{1}\}$	3.984962311320
$\{\bar{2}\}$	3.969848480984
$\{\bar{3}\}$	2.888667734731
$\{\bar{1}, \bar{2}\}$	6.210700654285
$\{\bar{1}, \bar{3}\}$	5.361382724478
$\{\bar{2}, \bar{3}\}$	5.349399951992
$\{\bar{1}, \bar{2}, \bar{3}\}$	7.392494131452.

Shapley's value for this game is $Sh(w) = (2.794946690768, 2.781398389357, 1.816149051327)$; therefore, it predicts a per capita cost of:

$$0.02794946691, 0.05562796778 \quad \text{and} \quad 0.09080745255,$$

respectively, for individuals in groups $N_1,\ N_2$ and N_3.

Remark 2 Notice that there is a significant discrepancy between the two solutions. We argue that the solution for the groups' game v is the "fairer" of the two, since it is taking into account the insurance cost reduction obtained when the individual players group into homogeneous classes.

4.3 A transportation problem

Consider the problem of transporting a single type of goods from a certain number of origins to m destinations. At each origin i, there are b_i units of the product in question and at each destination j, s_j of them are required. Let the players for this game be the individual units of the goods that are to be shipped, and consider indistinguishable those that are to be transported to the same destination. Set N_j the total amount of units sent to destination j and $f : N_1 \times \cdots \times N_m \to \mathbb{R}$, such that $f(s_1, s_2, ..., s_m)$ is the minimum cost at which the demand $(s_1, s_2, ..., s_m)$ at the m destinations is satisfied. We further assume that different destinations correspond to different owners.

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For example, in Samet et al. (1984), the authors give a transportation problem with two origins and two destinations, where a cooperative games' solution is proposed to avoid the two destinations fighting for the control of the nearest supplier.

Each destination requires 20 units of some good, each supplier has exactly 20 units of that same good, and the transportation cost is given by the function

$$f(s_1, s_2) = \begin{cases} 10s_1 + 15s_2 & \text{if } s_1 + s_2 \leq 20 \\ 1000s_1 + 1005s_2 - 19800 & \text{if } 20 \leq s_1 + s_2 \leq 40, \end{cases}$$

where s_k, $k : 1, 2$, is the amount of units delivered to destination k.

We set $N_k =$ units to be delivered to destination k, $N = N_1 \cup N_2$ all goods to be transported and define $v(S) = f(s_1, s_2)$, $s_k = |S \cap N_k|$. Then v is a groups' game and we may apply to it Shapley's value (Corollary 1) to obtain the cost that each destination must pay: $Sh(v) = (505, 510)$.

This solution (computed quite trivially) coincides with the solution given in the above mentioned article (Samet et al. 1984) where it is obtained using non-atomic games with a continuum of players.

4.4 Cost sharing

Consider a discrete costs distribution problem: There is a finite set $M = \{1, 2, ..., m\}$ of agents, each of them demanding a quantity $x_k \in N_k = \{1, 2, ..., n_k\}$ of goods. If $\Lambda = N_1 \times N_2 \times \cdots \times N_m$, let $f : \Lambda \to \mathbb{R}_+$, $f(0, ..., 0) = 0$ be the cost of producing the vector $x \in \Lambda$ of goods. One wishes to find a production prices' vector $p(f, x) \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{m} p_i(f, x)x_i = f(x). \tag{1}$$

We assume that for each $k : 1, ..., m$, the x_k goods are indistinguishable among themselves; therefore, we set N_k as the collection of all possible goods (up to n_k) of "type k". $N = N_1 \cup \cdots \cup N_m$ contains all goods of all types, and we define a groups' game v on N by

$$v(S) = f(x), \quad \text{where } x_k = |S \cap N_k| \text{ is the number of goods, of type } k, \text{ to be produced.}$$

Then any efficient solution on v will be a production prices' vector satisfying (1).

For example, for two different types of goods and a game v defined by

$$v(S) = f(x_1, x_2) = ax_1 + bx_2 + cx_1x_2$$

one computes

$$Sh(v) = \left(a + \frac{cn_2}{2}, b + \frac{cn_1}{2}\right)$$

as expected.

For a detailed survey on this problem we refer to Moulin (2002).

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4.5 Multi-choice games

Multi-choice games are a generalization of traditional cooperative games where each player may cooperate at different levels of participation (see, for example, Branzei et al. 2008). Consider a multi-choice game with m players and where the levels of participation are given in identical increments. This situation may be modeled by a groups' game where the participation levels of player j are the agents of type j.

More precisely, a multi-choice game is a triple $(\bar{N}, \bar{m}, \bar{v})$, where $\bar{N} = \{1, 2, ..., \bar{n}\}$ is a set of players, $\bar{m} = (m_1, m_2, ..., m_{\bar{n}})$ are the participation levels available for each of the players, i.e. player i may participate in the game at any level in the set $M_i = \{0, 1, ..., m_i\}$. Finally, $\bar{v} : M_1 \times \cdots \times M_{\bar{n}} \to \mathbb{R}$, such that $\bar{v}(0, 0, ..., 0) = 0$, is the characteristic function for the game.

If the levels of participation are given by identical, indistinguishable, increments (for each particular agent), we form a groups' game v, on the set $\coprod N_k$ (where each homogeneous group N_k has m_k indistinguishable players), by

$$v(S) = \bar{v}(s_1, s_2, ..., s_{\bar{n}}),$$

where $s_k = S \cap N_k$.

Let us consider a concrete example. We assume there is a project that has to be completed by a certain determined date. The project consists of several stages, where each of them is the responsibility of a different work team, in agreement with a certain PERT and/or CPM stipulated in advance. We also assume that the contract includes a penalty in case the signed deadline is not met. The problem we wish to address consists in distributing this penalty cost between the participating teams.

So, let

- $\bar{N}$ the set of work teams involved in the project;
- m_i the delay, in days, incurred by team i, we are thinking of each delay as the "participation level" of player i;
- $M_i = \{0, 1, ..., m_i\}$, $M = M_1 \times \cdots \times M_{\bar{n}}$;
- $q : M \to \mathbb{R}$, where $q(t_1, ..., t_{\bar{n}})$ is the penalty cost when each team i has a delay of t_i days.

This setups a multi-choice game $(\bar{N}, \bar{m}, q)$ which induces a groups' game v as above. Applying Shapley's value (for example) to v will give us each crew's fair share of the penalty.

5 The proofs

We show the proof of Theorem 2.

In Ruiz et al. (1998) (also see Hernández-Lamoneda et al. 2007), it is proved that a solution φ is linear, symmetric and efficient if and only if there is a (unique, but arbitrary) collection of real numbers $\{\beta_s\}_{s=1}^{n-1}$ such that

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} \beta_s \frac{v(S)}{s} - \sum_{S \not\ni i} \beta_s \frac{v(S)}{n - s}$$

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for every cooperative game v and player $i \in N$.

Assume, now, that $N = N_1 \cup \cdots \cup N_m$, $v(S) = q(s_1, ..., s_m)$ is a groups' game, and take $i \in N_k$.

For each cardinality s, fix an m-tuple $(s_1, s_2, ..., s_m)$ of non-negative integers, such that $s_1 + \cdots + s_m = s$. How many coalitions $S \subset N$ are there such that $|S \cap N_j| = s_j$? For each index $j \neq k$, one may choose s_j elements from a set with cardinality n_j, i.e. $\binom{n_j}{s_j}$ possible sets S_j; for $k \in S_k$, there are $\binom{n_k-1}{s_k-1}$ possible $S'_k s$; likewise, for $k \notin S_k$, there are only $\binom{n_k-1}{s_k}$ possible $S'_k s$. Thus, for each fixed cardinality s, there are exactly

$$\binom{n_1}{s_1} \cdots \binom{n_k - 1}{s_k - 1} \cdots \binom{n_m}{s_m}$$

sets S with $|S \cap N_j| = s_j$ and $S \ni i$ and exactly

$$\binom{n_1}{s_1} \cdots \binom{n_k - 1}{s_k} \cdots \binom{n_m}{s_m}$$

sets S with $|S \cap N_j| = s_j$ and $S \not\ni i$. Thus,

$$\begin{aligned}\varphi_i(v) = {} & \tfrac{q(n_1,\ldots,n_m)}{n} + \textstyle\sum_{s_1+\cdots+s_m=s, s_k\neq 0} \tfrac{\beta_s}{s} \binom{n_1}{s_1} \cdots \binom{n_k-1}{s_k-1} \cdots \binom{n_m}{s_m} q(s_1, ..., s_m) \\ & - \textstyle\sum_{s_1+\cdots+s_m=s, s_k\neq n_k} \tfrac{\beta_s}{n-s} \binom{n_1}{s_1} \cdots \binom{n_k-1}{s_k} \cdots \binom{n_m}{s_m} q(s_1, ..., s_m).\end{aligned}$$

Since $\binom{n_k-1}{s_k-1} = \frac{s_k}{n_k}\binom{n_k}{s_k}$ and $\binom{n_k-1}{s_k} = \frac{n_k-s_k}{n_k}\binom{n_k}{s_k}$, we can rewrite the above as

$$\begin{aligned}\varphi_i(v) = {} & \tfrac{q(n_1,\ldots,n_m)}{n} + \sum_{\substack{s_1+\cdots+s_m=s \\ s_k\neq 0}} \tfrac{s_k}{sn_k} \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} \beta_s q(s_1, ..., s_m) \\ & - \sum_{\substack{s_1+\cdots+s_m=s \\ s_k\neq n_k}} \tfrac{n_k-s_k}{(n-s)n_k} \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} \beta_s q(s_1, ..., s_m) \\ = {} & \tfrac{q(n_1,\ldots,n_m)}{n} + \textstyle\sum_{s_1+\cdots+s_m=s} \left[\tfrac{s_k}{sn_k} - \tfrac{n_k-s_k}{(n-s)n_k}\right] \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} \beta_s q(s_1, ..., s_m) \\ = {} & \tfrac{q(n_1,\ldots,n_m)}{n} + \textstyle\sum_{s_1+\cdots+s_m=s} \left[\tfrac{s_k n - n_k s}{sn_k(n-s)}\right] \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} \beta_s q(s_1, ..., s_m). \qquad \square\end{aligned}$$

Corollary 2 follows immediately after recalling that Shapley's value corresponds to $\beta_s = \binom{n}{s}^{-1}$.

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Proof of Theorem 3 we will assume that $i \in N_1$. Then

$$\varphi_i(v) = \sum_{S \not\ni i} p_s\,[v(S \cup \{i\}) - v(S)] = \sum_{S \ni i} p_{s-1} v(S) - \sum_{S \not\ni i} p_s v(S)$$

$$= \sum_{(s_1,\ldots,s_m)\in\Lambda; s_1 \neq 0} p_{s-1} \binom{n_1-1}{s_1-1}\binom{n_2}{s_2}\cdots\binom{n_m}{s_m} q(s_1, \ldots, s_m)$$

$$- \sum_{(s_1,\ldots,s_m)\in\Lambda; s_1 \neq n_1} p_s \binom{n_1-1}{s_1}\binom{n_2}{s_2}\cdots\binom{n_m}{s_m} q(s_1, \ldots, s_m);$$

so, simplifying as in the proof of Theorem 2,

$$\varphi_i(v) = \sum_{(s_1,\ldots,s_m)\in\Lambda} p_{s-1} \frac{s_1}{n_1}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m} q(s_1, \ldots, s_m)$$

$$- \sum_{(s_1,\ldots,s_m)\in\Lambda} p_s \frac{n_1-s_1}{n_1}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m} q(s_1, \ldots, s_m)$$

$$= \sum_{(s_1,\ldots,s_m)\in\Lambda} \frac{s_1 p_{s-1} - (n_1-s_1)p_s}{n_1}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m} q(s_1, \ldots, s_m) \qquad \square$$

Corollary 3 follows by substituting the values $p_s = \frac{1}{2^{n-1}}$, for all s, corresponding to Banzhaf's semivalue.

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An elementary characterization of the Gini index

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HIGHLIGHTS

- We offer an original and novel characterization for the Gini index.
- The treatment of distributions that maintain the same income ordering is essential.
- The characterization works very well for the discrete case and is simple.
- The axioms are imposed directly on the index and not on welfare orderings.
- A simpler way to compute the Gini index is derived.

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ABSTRACT

The Gini index is one of the most used indicators of social and economic inequality. In this paper we characterize the Gini index as the unique function that satisfies the properties of scale independence, symmetry, standardization and comonotone separability. Furthermore, we propose a simpler way to compute it.
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1. Introduction

In this article we offer a novel characterization of the Gini index,[1] exploiting the geometry of the contour lines of this index restricted to the unit simplex. In the case of three agents, these contour lines are hexagons whose edges are line segments on distribution sets that maintain a similar ordering (see Foster and Sen, 1997). There are six permutations of the agents, which lead to the six edges of any contour line of the Gini index. The basic idea is to find appropriate axioms that take advantage of this fact. The treatment of extreme points of subsets of distributions that maintain the same ordering of incomes is essential. If we impose the condition that the index has a certain value at these extreme points, we can use it as a stepping stone to extend the index to the rest of the simplex by making use of symmetry and linearity arguments.

Our characterization is substantially different from others proposed in the literature and throughout this introduction we will discuss the main differences. The main idea of the characterization is the following. An inequality index is a real function whose domain is the set of vectors with nonnegative real entries. We impose four properties or axioms directly on functions that constitute potential inequality indices. Our first two axioms are well known. The first one, *scale independence*, allows us to restrict the domain of the index to the unit simplex in $\mathbb{R}^n$.[2]

A second property, *symmetry* or anonymity, is used to restrict our domain even more; it will be sufficient to define the index on sets of income distribution whose elements are non-decreasingly arranged. We say that two income distributions are comonotone if their elements are ordered in the same way—we give the precise definition in the next section. In other words, symmetry implies that the inequality index of any distribution is independent of the ordering of its entries.

The third axiom, *standardization*, allows us to define the index at the extreme points of these special sets. The coordinates of these extreme points may only take two possible values: zero or identical

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[1] The coefficient attributed to Gini (1912) has been discussed and analyzed by Dalton (1920), Atkinson (1970) and Sen (1973), among others. The discrete formula of the Gini coefficient as a standardized average of income differences is presented in several equivalent versions, a good explanation is available in the classic paper of Sen (1973) or in the survey of Dutta (2002). Atkinson (1970) provides the basis to discuss the relation between inequality and social welfare orders.

[2] We restrict our attention to a fixed population of n agents ($n \geq 2$).

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values, such that the coordinates add up to one. For the case of three individuals and distributions with $y_1 \leq y_2 \leq y_3$, the extreme points are $(1/3, 1/3, 1/3)$, $(0, 1/2, 1/2)$ and $(0, 0, 1)$. From the standardization axiom we define the inequality index at extreme points, as the ratio of the number of individuals with no income divided by the total number of individuals.

Finally, the fourth axiom, *comonotone separability*,[3] extends linearly the inequality index, from the extreme points to the rest of the corresponding special set.

One of the main differences between the characterization offered in this work and alternative axiomatizations in the literature lies in the fact that we want to characterize a cardinal inequality index, whereas many authors look instead for a representation of an inequality ordering: the axiom of standardization is crucial in this respect. Among the advantages of our characterization we note the following: it works very well for the discrete case and it is simple; the axioms are imposed directly on indices and not on welfare orderings; there is no need to appeal to standard axioms of rationality (completeness and transitivity) and continuity to obtain numerical representations of welfare orderings.

A related work, closer to the approach followed in the paper, is the one by Thon (1982). This author proposes an axiomatization of the Gini coefficient in a general case. He works in a framework with income distributions, whose aggregate incomes and populations are not necessarily the same. Its axioms are imposed directly over the numerical function representing the inequality index, as in our case. But the axioms used by Thon are different from the ones used in this paper; although some of those axioms refer to the pure distribution case and have a clear interpretation from the welfare point of view, he has also taken into account considerations which transcend the pure distribution case, when comparing distributions with different populations and comparability of income distributions with the same population but different total incomes. Basically, those axioms show us other properties of the Gini index, more concerned with asymptotic properties, when the size of the population increases. In our framework the population size is fixed and we explore algebraic distribution properties instead of asymptotic characteristics of the Gini coefficient.

The rest of the paper is organized as follows. In Section 2 we introduce our notation and recall some definitions, particularly that of comonotonicity. The four axioms that we employ are presented in Section 3. We state in Section 4 our main result and proofs, plus some remarks and examples.

2. Framework and notation

Let $N = \{1, 2, \ldots, n\}$ be a set of individuals. When the income distribution function is discrete, such incomes take n values that can be denoted by a vector $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$,[4] where x_i denotes the income of the agent i.

For a given $x \in \mathbb{R}^n_+$, let x^* denote the vector obtained from x by rearranging its elements in a non-decreasing way; i.e., $x^*_1 \leq x^*_2 \leq \cdots \leq x^*_n$. For example, if $x = (9, 7, 0, 11, 3, 5, 9, 0)$, then $x^* = (0, 0, 3, 5, 7, 9, 9, 11)$.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$ acts on the space of income vectors $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ in a natural way; i.e., for $\theta \in S_n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, we define

$x^\theta := (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$.[5]

Notice that for an arbitrary $x \in \mathbb{R}^n_+$, there exists at least one permutation θ such that $x^\theta = x^*$.

Let $x, y \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ be two income distributions. We say that x and y are *comonotone* if $(x_i - x_j)(y_i - y_j) \geq 0$, $\forall i, j \in N$. In other words, two income distributions are comonotone if the income position for every agent is the same in both distributions x^* and y^*. For example, the income distributions $x = (6, 0, 3, 18, 1, 24)$ and $y = (11, 5, 10, 19, 7, 33)$ are comonotone; since for both distributions, agent 2 is the poorest one, agent 5 is the second poorest, agent 6 is the richest one, and so on.

An *inequality index* is a function $I : \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \to \mathbb{R}$ that assigns to each income vector a real number, which represents the society's inequality level.

The attractiveness of the Gini index to many economists is that it has an intuitive geometric interpretation. That is, the Gini index can be defined geometrically as the ratio of the area that lies between the line of perfect equality and the Lorenz curve, over the total area under the line of equality.

Sen (1973) defined the Gini index as a function $G : \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \to \mathbb{R}$ such that

$$G(x) = \frac{1}{n}\left[n + 1 - 2\frac{\sum_{i=1}^{n}(n+1-i)x^*_i}{\sum_{i=1}^{n}x^*_i}\right]. \tag{1}$$

This formulation illustrates the fact that the income-rank-based weights are inversely related to the incomes sizes. That is, the highest incomes get lower weights while the incomes of the poor get higher weights. Notice that for this discrete version, $0 \leq G(x) \leq \frac{n-1}{n}$ for every $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.

Example 1. Let the income distribution be $x = (30, 20, 60, 10)$, then $x^* = (10, 20, 30, 60)$ and

$$G(x) = \frac{1}{4}\left[5 - 2\frac{4(10) + 3(20) + 2(30) + 1(60)}{120}\right] = \frac{1}{3}.$$

This paper will show that the Gini index as it is defined above is uniquely determined by a certain set of properties, described in the next section.

3. The axioms

We introduce below a set of axioms which an inequality index is required to satisfy in this work.

The first axiom deals with scales; that is, if every person's income x_i in an economy is multiplied by any positive constant, then inequality should not change (it is independent of the aggregate level of income). So, if the income is measured in other currencies, then the index remains the same.

Axiom 1 (*Scale Independence*)**.** The inequality index I is said to be scale-independent if

$$I(\lambda x) = I(x)$$

for all $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ and $\lambda > 0$.

The next axiom requires that the inequality index be independent of any characteristic, other than their incomes, of the individuals. More precisely,

Axiom 2 (*Symmetry*)**.** The inequality index I is symmetric if and only if

$$I(x^\theta) = I(x)$$

for every $\theta \in S_n$ and $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.

The previous two axioms are common and standard in the literature, they are satisfied by most of the well-known inequality

[3] The precise definitions will be provided in Section 3.

[4] Here, we discard the null distribution $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$ in the space of admissible income vectors.

[5] This permutation implies a change of position of the individuals within the income distribution.

indices. However, the next two axioms are new and crucial for our characterization of the Gini index.

Consider a society where there are only two types of individuals: rich people who share equally 1 unit of income and the poor with no income. The next property requires that in this case, the inequality index is equal to the proportion of individuals with no income. Let $v^{k+1} = (\underbrace{0, \ldots, 0}_{k}, \underbrace{\frac{1}{n-k}, \ldots, \frac{1}{n-k}}_{n-k})$, $k = 0, \ldots, n-1$, be the income vector of one such society.

Axiom 3 (*Standardization*)**.** The inequality index I is said to be standardized if

$$I(v^{k+1}) = \frac{k}{n}.$$

The reader may wonder why we used the term *standardization*. The intuition behind the previous axiom is the following. Suppose we start with an income vector where one unit of income is equally divided among all individuals. Next, one agent is deprived of income and the unit is equally divided among the remaining agents. We are interested in the difference of inequality:

$$I\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) - I\left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right).$$

Again in a second step, one more individual is deprived of income and we focus in the difference:

$$I\left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right) - I\left(0, 0, \frac{1}{n-2}, \ldots, \frac{1}{n-2}\right).$$

We repeat the process until we reach the distribution where exactly one agent has the entire income:

$$\vdots$$

$$I\left(0, \ldots, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - I\left(0, \ldots, 0, \frac{1}{2}, \frac{1}{2}\right)$$

$$I\left(0, \ldots, 0, \frac{1}{2}, \frac{1}{2}\right) - I(0, \ldots, 0, 1).$$

What we required is that all such differences (of inequality) are equal to a common constant, such that $I\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = 0$ and $I(0, \ldots, 0, 1) = \frac{n-1}{n}$.[6] Those conditions capture the idea of inequality in a very particular way by restricting attention to particular distributions. To some extent, it is reminiscent of the idea of deprivation in cases where the extent of deprivation (or poverty) is assimilated with the percentage of the poor.

Finally, the next property represents an independence or separability condition[7] that requires an specific treatment for comonotone income distributions. It enables an overall index value to be computed from subaggregates. These subaggregates share a common characteristic: they are comonotone income vectors with same total income. In particular, the index applied to a mixture of two comonotone income vectors can be expressed as a weighted sum of the index value calculated for their components.

Axiom 4 (*Comonotone Separability*)**.** The inequality index I is comonotone separable if

$$I[\beta x + (1-\beta)z] = \beta I(x) + (1-\beta)I(z)$$

for every $\beta \in [0, 1]$, every $x, z \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ that are comonotone and $\sum_{i=1}^n x_i = \sum_{i=1}^n z_i$.

[6] Thus, one can think that such an axiom has a component of normalization and a component of constant inequality changes.

[7] This condition is weaker than the Strong Separability Condition used in the Expected Utility model (Yaari, 1987, 1988), since it applies to a restricted set of distributions: precisely, those that are comonotone.

It is important to notice that an index satisfying the last axiom, has a separability property only on sets of comonotone distributions with the same total income.[8] For example, suppose that I is an index satisfying the above axioms and that we have a situation where there are four individuals with income distributions (with same total income) $x = (2, 2, 0, 0)$ and $z = (0, 0, 2, 2)$. Then we have that $I(x) = I(z) = \frac{1}{2}$ and $I\left(\frac{1}{2}x + \frac{1}{2}z\right) = I(1, 1, 1, 1) = 0$. Thus $I\left(\frac{1}{2}x + \frac{1}{2}z\right) \neq \frac{1}{2}I(x) + \frac{1}{2}I(z)$, which is compatible with Axiom 4 since x and z are not comonotone.

On the other hand, take two comonotone income vectors (for two individuals) $x = (0, 10)$ and $z = (20, 20)$. In this case we get $I(x) = \frac{1}{2}$, $I(z) = 0$ and $I\left(\frac{4}{5}x + \frac{1}{5}z\right) = I(4, 12) = I\left[\frac{1}{2}(8, 8) + \frac{1}{2}(0, 16)\right] = \frac{1}{4}$. Then $I\left(\frac{4}{5}x + \frac{1}{5}z\right) \neq \frac{4}{5}I(x) + \frac{1}{5}I(z)$, which is again compatible with Axiom 4 since x and z have different total income.

4. The main result

This section is devoted to the characterization of the Gini index by means of the previous axioms, which constitutes the main result of this work. First of all, we shall present some preliminary results that are used in the proof of our main result.

The key idea is to show that Axioms 3 and 4 characterize the Gini index on the convex subset

$$K = \left\{ y \in \mathbb{R}^n_+ \mid \sum_{i=1}^n y_i = 1 \text{ and } y_1 \leq y_2 \leq \cdots \leq y_n \right\}.$$[9]

Then Axioms 1 and 2, allow us to extend the Gini index on the whole space of income vectors $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. To wit, let $I : \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \to \mathbb{R}$ be a symmetric and scale-independent index. For $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, there exists $\theta \in S_n$ such that $x_{\theta(1)} \leq x_{\theta(2)} \leq \cdots \leq x_{\theta(n)}$ and hence $\frac{1}{\sum_{i=1}^n x_i} x^\theta \in K$. Therefore,

$$I(x) = I(x^\theta) = I\left[\frac{1}{\sum_{i=1}^n x_i} x^\theta\right].$$

For subsequent developments, we need to express each element of K as a convex combination of the vectors $\{v^j\}_{j=1}^n$, which are the extreme points of K and are given by

$$v^j_i = \begin{cases} \dfrac{1}{n-j+1} & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We know that such decomposition always exists by Carathéodory's Theorem, in convex analysis. The precise values of the components in the decomposition are indicated in the following

Lemma 1. *For every $y \in K$, there exist unique non-negative scalars $\{\alpha_j\}_{j=1}^n$ with $\sum_{j=1}^n \alpha_j = 1$, such that:*

$$y = \sum_{j=1}^n \alpha_j v^j. \tag{3}$$

Moreover, the scalars are given by

$$\alpha_j = (n-j+1)(y_j - y_{j-1})$$

[8] However, we note that this axiom has some implications for comonotone distributions with different total incomes. For example, if x and y are comonotone distributions with total income a and b, respectively; then from scale independence and comonotone separability, we have $I(x+y) = I\left(\frac{a}{a+b}\frac{x}{a} + \frac{b}{a+b}\frac{y}{b}\right) = \frac{a}{a+b}I(x) + \frac{b}{a+b}I(y)$, which could have an intuitive interpretation.

[9] Note that if $y \in K$, then $y^* = y$.

for $j = 1, 2, \ldots, n$, where $y_0 = 0$. The α_j's are called the barycentric coordinates of y with respect to $\{v^j\}$.

Proof. Let V be a $n \times n$ matrix with entries v_i^j, and let α be the $1 \times n$ matrix $\alpha = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]$. Notice that $y = \sum_{j=1}^n \alpha_j v^j$ can be represented by the system of equations $y = V\alpha$. Since V is non-singular, then the scalars $\{\alpha_j\}_{j=1}^n$ are unique.

Now, for arbitrary $y \in K$:

$$\begin{aligned} y_i - y_{i-1} &= \sum_{j=1}^n \alpha_j v_i^j - \sum_{j=1}^n \alpha_j v_{i-1}^j \\ &= \sum_{j=1}^i \frac{\alpha_j}{n-j+1} - \sum_{j=1}^{i-1} \frac{\alpha_j}{n-j+1} \\ &= \frac{\alpha_i}{n-i+1}. \end{aligned}$$

Hence,

$$\alpha_i = (n-i+1)\,(y_i - y_{i-1})$$

and they are not negative since $y \in K$.

For the other direction:

$$\begin{aligned} \sum_{j=1}^n \alpha_j v_i^j &= \sum_{j=1}^i (n-j+1)(y_j - y_{j-1}) \frac{1}{n-j+1} \\ &= \sum_{j=1}^i y_j - \sum_{j=1}^i y_{j-1} = y_i. \end{aligned}$$

Finally, those scalars add up to 1:

$$\begin{aligned} \sum_{j=1}^n \alpha_j &= \sum_{j=1}^n (n-j+1)(y_j - y_{j-1}) \\ &= \sum_{j=1}^n (n-j+1) y_j - \sum_{j=0}^{n-1} (n-j) y_j \\ &= \sum_{j=1}^n y_j = 1. \quad \blacksquare \end{aligned}$$

Lemma 2. *Let $w^j \in K$, $j = 1, \ldots, m$ and $\beta_j \geq 0$ be such that $\sum_{j=1}^m \beta_j = 1$. If $I : \mathbb{R}_+^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ verifies comonotone separability, then*

$$I\left(\sum_{j=1}^m \beta_j w^j\right) = \sum_{j=1}^m \beta_j I(w^j)$$

Proof. First, it is clear that every pair of elements in K is comonotone, since by definition they all are ordered increasingly.

The proof is done by induction on m. Assume that the statement holds for $k < m$:

$$I\left(\sum_{j=1}^k \beta_j w^j\right) = \sum_{j=1}^k \beta_j I(w^j)$$

for $w^j \in K$, $j = 1, \ldots, k$ and $\beta_j \geq 0$ are such that $\sum_{j=1}^k \beta_j = 1$.

Now, let $w^j \in K$, $j = 1, \ldots, k+1$ and let $\gamma_j \geq 0$ be such that $\sum_{j=1}^{k+1} \gamma_j = 1$ (assume $\gamma_{k+1} < 1$). Since I is separable:

$$\begin{aligned} I\left(\sum_{j=1}^{k+1} \gamma_j w^j\right) &= I\left(\sum_{j=1}^k \gamma_j w^j + \gamma_{k+1} w^{k+1}\right) \\ &= (1-\gamma_{k+1}) I\left(\sum_{j=1}^k \frac{\gamma_j}{1-\gamma_{k+1}} w^j\right) + \gamma_{k+1} I(w^{k+1}). \end{aligned}$$

Notice that $\frac{\gamma_j}{1-\gamma_{k+1}} \geq 0$ and $\sum_{j=1}^k \frac{\gamma_j}{1-\gamma_{k+1}} = 1$. Thus, by the induction hypothesis:

$$\begin{aligned} I\left(\sum_{j=1}^{k+1} \gamma_j w^j\right) &= (1-\gamma_{k+1}) \sum_{j=1}^k \frac{\gamma_j}{1-\gamma_{k+1}} I(w^j) + \gamma_{k+1} I(w^{k+1}) \\ &= \sum_{j=1}^{k+1} \gamma_j I(w^j). \quad \blacksquare \end{aligned}$$

Note that the scale independence and standardization axioms determine the inequality index on the extreme points of K, whereas comonotone separability, extends the index on the rest of K. Finally, symmetry extends I on the entire simplex.

We are now ready to state our main result.

Theorem 1. *Let $I : \mathbb{R}_+^n \setminus \{0\} \to \mathbb{R}$. Then, I equals the Gini index, given by (1), if and only if it satisfies the axioms of scale independence, symmetry, standardization and comonotone separability.*

Proof. It is straightforward to prove that the Gini index given by (1) satisfies the four properties.

For the converse, let $I : \mathbb{R}_+^n \setminus \{0\} \to \mathbb{R}$ be any index satisfying the four axioms and let $z \in \mathbb{R}_+^n \setminus \{0\}$ be an income vector. Define $y_i = \frac{z_i^*}{\sum_{j=1}^n z_j}$. It is clear that $y \in K$ and so, it can be decomposed as $y = \sum_{j=1}^n \alpha_j v^j$ for unique real numbers $\{\alpha_j \geq 0\}_{j=1}^n$ with $\sum_{j=1}^n \alpha_j = 1$.

Then, using the symmetry and scale independence axioms, and appealing to Carathéodory's Theorem, we have:

$$I(z) = I(z^*) = I(y) = I\left(\sum_{i=1}^n \alpha_i v^i\right).$$

Next, invoking Lemmas 2 and 1, we obtain

$$I(z) = \sum_{i=1}^n \alpha_i I(v^i) = \sum_{i=1}^n (n-i+1)(y_i - y_{i-1}) I(v^i).$$

By the standardization axiom, we get:

$$\begin{aligned} I(z) &= \sum_{i=1}^n (n-i+1)(y_i - y_{i-1}) \frac{i-1}{n} \\ &= \sum_{i=1}^n \frac{(n-i+1)\,(i-1)}{n} y_i - \sum_{i=2}^n \frac{(n-i+1)\,(i-1)}{n} y_{i-1} \\ &= \sum_{i=1}^n \frac{(n-i+1)\,(i-1)}{n} y_i - \sum_{i=1}^{n-1} \frac{(n-i)i}{n} y_i \\ &= \frac{n-1}{n} y_n + \sum_{i=1}^{n-1} \left[\frac{(n-i+1)\,(i-1)}{n} - \frac{(n-i)i}{n}\right] y_i \\ &= \frac{n-1}{n} y_n + \sum_{i=1}^{n-1} \frac{2i-n-1}{n} y_i = \sum_{i=1}^n \frac{2i-n-1}{n} y_i \\ &= \frac{1}{n}\left[-n-1+2\sum_{i=1}^n i y_i\right] \\ &= \frac{1}{n}\left[-n-1+2\sum_{i=1}^n i \left(\frac{z_i^*}{\sum_{j=1}^n z_j}\right)\right] \\ &= \frac{1}{n}\left[n+1-2\frac{\sum_{i=1}^n (n+1-i) z_i^*}{\sum_{j=1}^n z_j^*}\right] \\ &= G(z) \end{aligned}$$

which is precisely (1). $\blacksquare$

Note that, an index that verifies Axioms 1, 2 and 4, can be written as $I(z) = \sum_{i=1}^{n} \alpha_i I(v^i)$. Thus, different ways of assigning the value of $I(v^i)$ can lead to different indices. We can redefine the Standardization Axiom and obtain other possible indices. For example, we could give a measure of "poverty" by setting $I(v^{k+1}) = k - (n-k)$ (that is, the number of poor people less the number of rich people). Such an index is given by $I(z) = \sum_{i=1}^{n}(-3n-2+4i)y_i$.[10]

Example 2. As an illustration of the above derivation, we compute the Gini index of Example 1.

For the income distribution $z = (30, 20, 60, 10)$, we have $y = \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}\right)$. Furthermore,

$$\begin{aligned} y &= \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{3}v^1 + \frac{1}{4}v^2 + \frac{1}{6}v^3 + \frac{1}{4}v^4 \\ &= \frac{1}{3}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4}\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &\quad + \frac{1}{6}\left(0, 0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{4}(0, 0, 0, 1) \end{aligned}$$

and hence,

$$\begin{aligned} G(z) &= G(y) = \frac{1}{3}G\left(v^1\right) + \frac{1}{4}G\left(v^2\right) + \frac{1}{6}G\left(v^3\right) + \frac{1}{4}G\left(v^4\right) \\ &= \frac{1}{3}(0) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{6}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{3}{4}\right) \\ &= \frac{1}{3}. \end{aligned}$$

The proof of Theorem 1 suggests a simple way to compute the Gini index. That is, for an arbitrary income vector $z \in \mathbb{R}^n_+ \setminus \{0\}$, it is shown that the Gini index is obtained from

$$G(z) = \sum_{i=1}^{n} \gamma_i y_i \tag{4}$$

where $\gamma_i = \frac{2i-n-1}{n}$ and $y_i = \frac{z_i^*}{\sum_{j=1}^{n} z_j^*}$.[11]

In order to compute the coefficients $\{\gamma_i\}_{i=1}^{n}$, it is sufficient to calculate half of them, since the relation $\gamma_i + \gamma_{n+1-i} = 0$ holds for every $i = 1, 2, \ldots, n$. When n is an odd number, it turns out that the central coefficient $\gamma_{\frac{n+1}{2}}$ vanishes.

Example 3. Let us give the precise values of coefficients γ_i for the case $n = 7$:

i	1	2	3	4	5	6	7
γ_i	$-\frac{6}{7}$	$-\frac{4}{7}$	$-\frac{2}{7}$	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{6}{7}$

Example 4. As a final example, we present the computation of the Gini index from expression (4) for the income distribution $z = (75, 5, 10, 20, 6, 58, 6)$.

According to the previous discussion, $y = \left(\frac{1}{36}, \frac{1}{30}, \frac{1}{30}, \frac{1}{18}, \frac{1}{9}, \frac{29}{90}, \frac{5}{12}\right)$ and we then get:

$$\begin{aligned} G(z) &= -\frac{6}{7}\left(\frac{1}{36}\right) - \frac{4}{7}\left(\frac{1}{30}\right) - \frac{2}{7}\left(\frac{1}{30}\right) + \frac{2}{7}\left(\frac{1}{9}\right) \\ &\quad + \frac{4}{7}\left(\frac{29}{90}\right) + \frac{6}{7}\left(\frac{5}{12}\right) \\ &= \frac{164}{315} \approx 0.5206. \end{aligned}$$

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[10] This index reflects in some way an impact of extreme poverty and takes values in $[-n, n-2]$. When there are an even number of individuals, half with no income and half with the total income equally distributed, the index is equal to 0. We are in a situation of latent conflict because both groups are large and antagonistic, a kind of maximum polarization. The greatest value of the index, occurs when all individuals have zero income, except for one individual. It is the maximum impact of poverty; however, we do not necessarily associate this case with conflict as it would require the organization and coordination of the dispossessed to generate conflict. The lowest value of the index is $-n$, where the distribution of income is egalitarian.

[11] Silber (1989) proposed an equivalent expression to compute the Gini coefficient:

$$G(z) = \sum_{j=1}^{n} \frac{z_j}{\sum_{j=1}^{n} z_j}\left(\frac{n-j}{n} - \frac{j-1}{n}\right)$$

where z is arranged in a non-increasing order. Xu (2004) noted that by simply reversing the order and making use of $i = n - j + 1$, the resulting expression is exactly the one given in (4).

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A characterization of the symmetric Banzhaf value for Games with a Coalition Structure

Research Article

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Abstract: We provide a new characterization of the symmetric Banzhaf value via the coalitional neutral player, intracoalitional balanced contributions and coalitional transfer invariance axioms. The latter axiom says that a union's total payoff is invariant to a transfer of worth between two coalitions he or she belongs to, provided that such coalitions do not contain subsets of any union.

MSC: 91A12 • 91A40

Keywords: Games • Value • Coalition Structure

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1. Introduction

The Shapley value [1] and the Banzhaf value [2] are the most popular values in games with transferable utilities (or TU-games). Both values assign to each player the sum of weighted marginal contributions. In the Banzhaf value each player has the same probability to enter any coalition and on the Shapley value this probability depends on the number of elements of a coalition. These two values have been extended to TU-games with a coalition structure [3, 4] or a coalition configuration [5, 6], TU-games on antimatroids [7, 8], and TU-games on convex geometries [9, 10], etc.

Alonso-Meijide and Fiestras-Janeiro [11] proposed another extension of the Banzhaf value to TU-games with a coalition structure, called the symmetric Banzhaf value. This value is the composition of the Banzhaf value and the Shapley value in the following sense: suppose that in a TU-game with a union system there are two levels of negotiation; first, between unions and then between players of the same union. The distribution between unions is done by assigning to each union their Banzhaf value in the quotient game. The amount assigned to each union in this first share is divided between the players of each union using the Shapley value of each union's TU-game.

We provide a new and concise characterization of the extension given by Alonso-Meijide and Fiestras-Janeiro in [11] of the Banzhaf value for TU-games with a coalition structure by means of three axioms. The first one is the well-known intracoalitional balanced contribution axiom. The second axiom, called coalitional neutral player, indicates that if a union is a neutral player in the quotient game, then the payoff to that union is what its members obtain on their own in the TU-game. The third axiom, called coalitional transfer invariance, says that a union's total payoff is invariant to a transfer of worth between two coalitions he or she belongs to, provided that such coalitions do not contain subsets of any union. This means that if the worth of a coalition varies a certain amount and, at the same time, the worth of another coalition varies by the opposite amount, then for unions whose members belong to both coalitions the total payoff allocation does not change, provided that such coalitions do not contain subsets of any union. The result so achieved is therefore placeable into the crossing zone between Operations research and management science through a suitable application of decision theory. Due to this feature, the treatment pursued in this work concerns applied mathematics and, for the arguments and disciplines herein involved, it has been kindly accepted by the new IJAMM which has already published many papers drawn up upon the same cross-disciplinary area in which

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Operations research and management science fruitfully intertwine amongst them. See for instance from [12], [13], [14] and [15].

The rest of this paper is organized as follows. section 2 contains the notation and basic definitions and introduces the axiom of coalitional transfer invariance and coalitional neutral player. Equation 3 is devoted to the proof of the main result of the article.

2. Preliminaries

2.1. TU-game

The cardinality of a finite set T will be denoted by a lowercase letter t and the power set of T by 2^T. For simplicity of notation, we write i instead of $\{i\}$. A *cooperative game with transferable utilities (or simply, TU-game)* is a pair (N,v), where N is the set of players and $v:2^N\to\mathbb{R}$ with $v(\emptyset)=0$. We denote by G^N the set of TU-games on N. For any pair of TU-games $v,w\in G^N$ and any $\alpha\in\mathbb{R}$, the TU-game $\alpha v+w$ is defined as: $(\alpha v+w)(S)=\alpha v(S)+w(S)$, for all $S\subseteq N$. For each non-empty coalition $T\subseteq N$, the Dirac TU-game $\delta_T\in G^N$ is defined as: $\delta_T(S)=1$ if $S=T$ and $\delta_T(S)=0$ for any other $S\subseteq N$. A player $i\in N$ is a *neutral player* in (N,v) if $\forall S\ni i$, $v(S)-v(S\setminus i)=v(i)$.

A *value* ϕ in G^N is a mapping $\phi:G^N\to\mathbb{R}^n$ which uniquely determines, for all $(N,v)\in G^N$ and all $i\in N$, a payoff $\phi_i(N,v)\in\mathbb{R}$ for participating in this TU-game $(N,v)\in G^N$.

Dummy player axiom (DP) *A value ϕ satisfies DP if, for each $i\in N$ such that i is a neutral player in (N,v), it holds that: $\phi_i(N,v)=v(i)$.*

Two values in G^N that satisfy DP are the *Shapley value* [1] and the *Banzhaf value* [2]. The Shapley value is the mapping Sh that assigns to each TU-game $(N,v)\in G^N$ and each $i\in N$ the real number $Sh_i(N,v)\in\mathbb{R}$ given by

$$Sh_i(N,v)=\sum_{S\ni i}\frac{(s-1)!(n-s)!}{n!}[v(S)-v(S\setminus i)].$$

The Banzhaf value is the mapping Bz that assigns to each TU-game $(N,v)\in G^N$ and each $i\in N$ the real number $Bz_i(N,v)\in\mathbb{R}$ given by

$$Bz_i(N,v)=\sum_{S\ni i}\frac{1}{2^{n-1}}[v(S)-v(S\setminus i)].$$

For further information about the meaning and properties of the Shapley value and the Banzhaf value, refer to [1] and [16], respectively.

2.2. TU-games with a coalition structure

A *coalition structure* over the set of players, N, is a partition of N, that is, $\mathscr{B}=\{B_1,\dots,B_m\}$ is a coalition structure if it satisfies that $\bigcup_{1\le l\le m}B_l=N$ and $B_k\cap B_l=\emptyset$ if $k=l$. We also assume $B_l\neq\emptyset$ for all l. The sets $B_l\in\mathscr{B}$ are called "unions" or "blocks". We denote by $\mathscr{B}(N)$ the set of all coalition structures over N. A TU-game with a coalition structure is a triplet $(N,\mathscr{B},v)$, where $(N,v)\in G^N$ and $\mathscr{B}\in\mathscr{B}(N)$. The coalition structure $\mathscr{B}$ gives an a priori structure of cooperation among players. We denote by Γ^N the set of TU-games with a coalition structure.

If $(N,\mathscr{B},v)\in\Gamma^N$ with $\mathscr{B}=\{B_l:l\in M:=\{1,\dots,m\}\}\in\mathscr{B}(N)$, the *quotient game* $(M,v^{\mathscr{B}})$ is a TU-game played by the unions, i.e,

$$(M,v^{\mathscr{B}})\in G^M\qquad and\qquad v^{\mathscr{B}}(K)=v\left(\bigcup_{k\in K}B_k\right)\ \ \forall K\subseteq M.$$

A *value* f in Γ^N is a mapping $f:\Gamma^N\to\mathbb{R}^n$ which uniquely determines, for all $(N,\mathscr{B},v)\in\Gamma^N$ and all $i\in N$, a payoff $f_i(N,\mathscr{B},v)\in\mathbb{R}$ for participating in this TU-game with a coalition structure $(N,\mathscr{B},v)\in\Gamma^N$.

For a value f in Γ^N and $S\subseteq N$, let $f(N,\mathscr{B},v)[S]=\sum_{i\in S}f_i(N,\mathscr{B},v)$. For all $l\in M$ with $b_l>1$ and $i\in B_l$, define $\mathscr{B}_{-i}=\{B_1,\dots,B_l\setminus i,\dots B_m\}$, that is, $\mathscr{B}_{-i}$ is the new coalition structure when the player i leaves the TU-game.

Our interest is focused on the value proposed in [11], for TU-games with a coalition structure, called the symmetric Banzhaf value. Given a TU-game with a coalition structure $(N,\mathscr{B},v)\in\Gamma^N$, this value assigns to each player $i\in N$ the real number

$$\Pi_i(N,\mathscr{B},v)=\sum_{K\ni k}\sum_{\substack{S\ni i\\ S\subseteq B_k}}\frac{1}{2^{m-1}}\frac{(s-1)!(b_k-s)!}{b_k!}[v(Q_K\cup S)-v(Q_K\cup S\setminus i)]$$

where $B_k\in\mathscr{B}$ is the union containing i ($i\in B_k$) and where we set $Q_R=\bigcup_{r\in K}B_r$.

Since $\Pi(N,\mathscr{B},v)[B_l]=Bz_l(M,v^{\mathscr{B}})$ for all $l\in M$ and the Banzhaf value satisfies DP, then the symmetric Banzhaf value, Π, satisfies the following axiom:

Coalitional neutral player (CNP) *A value f in Γ^N satisfies CNP if, for all $(N,\mathscr{B},v)\in\Gamma^N$ and all $l\in M$ such that l is a neutral player in $(M,v^{\mathscr{B}})$, it holds that: $f(N,\mathscr{B},v)[B_l]=v(B_l)$.*

We note that given $\mathscr{B}\in\mathscr{B}(N)$ and $(N,v),(N,w)\in G^N$ the quotient game of $(N,\mathscr{B},(v+w))$ is $(M,\mathscr{B},v^{\mathscr{B}}+w^{\mathscr{B}})$. From this linearity, the equation $\Pi(N,\mathscr{B},v)[B_l]=Bz_l(M,v^{\mathscr{B}})$, $\forall l\in M$, the equality $\delta^{\mathscr{B}}_{Q_K}(R)=\delta_K(R)$ for all $R\in 2^M$ and all $K\in 2^M\setminus\emptyset$, plus the fact that $Bz_l(M,v^{\mathscr{B}})$ only depends on the sum of the marginal contributions of coalitions of unions that contain l, we obtain that the symmetric Banzhaf value, Π, satisfies the following axiom:

Coalitional transfer invariance (CTI) *A value f in Γ^N satisfies CTI if, for all $(N,\mathscr{B},v)\in\Gamma^N$, all $\alpha\in\mathbb{R}$ and all $K,R\in 2^M\setminus\emptyset$, it holds that: $f(N,\mathscr{B},v)[B_l]=f(N,\mathscr{B},v+\alpha(\delta_{Q_K}-\delta_{Q_R}))[B_l],\forall l\in K\cap R$.*

3. Results

It turns out that the combination of CNP, CTI and the following axiom characterizes Π on Γ^N.

Intracoalitional balanced contributions (IBC) *A value f in Γ^N satisfies IBC if, for all $(N,\mathscr{B},v)$, for all $l\in M$ and all $\{i,j\}\subseteq B_l$, it holds that:*

$$f_i(N,\mathscr{B},v)-f_i(N\setminus j,\mathscr{B}_{-j},v)=f_j(N,\mathscr{B},v)-f_j(N\setminus i,\mathscr{B}_{-i},v).$$

The IBC axiom indicates that if two players $i,j\in N$ belong to the same union, then the contribution of player j to the payoff of i is equal to the contribution of player i to the payoff of j.

The characterization theorem is:

Theorem 3.1.
The only value defined in Γ^N that satisfies CNP, CTI and IBC is the symmetric Banzhaf value.

Proof. **Existence.** In section 2 we saw that the value Π satisfies CNP and CTI. From the results of Myerson [17] and since the value Π is written as

$$\Pi_i(N,\mathscr{B},v)=\frac{1}{2^{m-1}}\sum_{K\ni k}Sh_i(B_k,v_{Q_K})\quad\forall i\in N,$$

where $v_{Q_K}(S)=v(Q_K\cup S)-v(Q_K)$ for all $S\subseteq B_k$, we deduce that the value Π satisfies IBC.

Uniqueness. Let $(N,\mathscr{B},v)\in\Gamma^N$ be a TU-game with a coalition structure, f a value in Γ^N satisfying the axioms listed in the hypotheses, and $q=(q_S)_{S\in 2^N}$ such that $q_S\in\mathbb{R}$ and $q_\emptyset=0$. Let us first prove that $f(N,\mathscr{B},v)[B_l]=\Pi(N,\mathscr{B},v)[B_l]$ for all $l\in M$. To do this, we define a value φ^q in G^M such that for all $(M,w)\in G^M$ and all $l\in M$, $\varphi^q_l(M,w)=f(N,\mathscr{B},\overline{w})[B_l]$, where for all $S\subseteq N$, $\overline{w}(S)=w(T)$ if $S=Q_T$ and $\overline{w}(S)=q_S$ otherwise. As f is a value in Γ^N, then φ^q is a value in G^M.

We next prove that φ^q satisfies DP and *strong transfer invariance (STI)* in G^M.

DP. Let $l\in M$ such that l is a neutral player in the TU-game $(M,w)\in G^M$. Given that the quotient game of the TU-game with a coalition structure $(N,\mathscr{B},\overline{w})\in\Gamma^N$ is precisely $(M,w)\in G^M$ and f satisfies CDP, then $\varphi^q_l(M,w)=f(N,\mathscr{B},\overline{w})[B_l]=\overline{w}(B_l)=w(l)$. This implies that φ^q satisfies DP in G^M.

STI. Let $l\in M$ and (K^+,K^-,α) be a transfer with $l\in K^+\cap K^-$. Since the TU-game $\overline{w+\alpha(\delta_{K^+}-\delta_{K^-})}\in G^N$ coincides with the TU-game $\overline{w}+\alpha(\delta_{Q_{K^+}}-\delta_{Q_{K^-}})\in G^N$ we obtain that $\varphi^q_l(M,w+\alpha(\delta_{K^+}-\delta_{K^-}))=f(N,\mathscr{B},\overline{w}+\alpha(\delta_{Q_{K^+}}-\delta_{Q_{K^-}})[B_l]$.

Given that $l\in K^+\cap K^-$ and f satisfies CTI, then $\varphi^q_l(M,w+\alpha(\delta_{K^+}-\delta_{K^-}))=f(N,\mathscr{B},\overline{w}+\alpha(\delta_{Q_{K^+}}-\delta_{Q_{K^-}})[B_l]=f(N,\mathscr{B},\overline{w})[B_l]=\varphi^q_l(M,w)$, which implies that φ^q satisfies STI in G^M.

Béal et al. [18] proved that the Banzhaf value is the only value in G^M that satisfies DP and STI. Hence, $\varphi^q=Bz$ in G^M.

Consider q such that $q_S=v(S)$ for all $S\in 2^N$. Now, it is trivial to deduce that

$$Bz_l(M,v^{\mathscr{B}})=f(N,\mathscr{B},v)[B_l]\quad\forall l\in M. \tag{1}$$

From (1) and the fact that $\Pi(N,\mathscr{B},v)[B_l]=Bz_l(M,v^{\mathscr{B}})$, for all $l\in M$, we conclude that

$$f(N,\mathscr{B},v)[B_l]=\Pi(N,\mathscr{B},v)[B_l]. \tag{2}$$

Let $l\in M$. We next prove that $f_i(N,\mathscr{B},v)=\Pi_i(N,\mathscr{B},v)$ for all $i\in B_l$, by induction on b_l. If $B_l=\{j\}$, then the equality (2) implies that $f_j(N,\mathscr{B},v)=\Pi_j(N,\mathscr{B},v)$. Let us assume that $b_l\geq 2$. By IBC for each $\{i,j\}\subseteq B_l$ it holds that

$$f_i(N,\mathscr{B},v)-f_i(N\setminus j,\mathscr{B}_{-j},v)=f_j(N,\mathscr{B},v)-f_j(N\setminus i,\mathscr{B}_{-i},v). \tag{3}$$

Using the induction hypothesis, we have $f_i(N\setminus j,\mathscr{B}_{-j},v)=\Pi_i(N\setminus j,\mathscr{B}_{-j},v)$ and $f_j(N\setminus i,\mathscr{B}_{-i},v)=\Pi_j(N\setminus j,\mathscr{B}_{-j},v)$. By (3) we obtain that $f_i(N,\mathscr{B},v)-f_j(N,\mathscr{B},v)=\Pi_i(N\setminus j,\mathscr{B}_{-j},v)-\Pi_j(N\setminus i,\mathscr{B}_{-i},v)$. This implies that $f_i(N,\mathscr{B},v)-f_j(N,\mathscr{B},v)=\Pi_i(N,\mathscr{B},v)-\Pi_j(N,\mathscr{B},v)$. Adding over all $j\in B_l$ in the above equality and using (2) we obtain that $b_lf_i(N,\mathscr{B},v)=b_l\Pi_i(N,\mathscr{B},v)$. From this equality and since $b_l\geq 2$ we conclude that $f_i(N,\mathscr{B},v)=\Pi_i(N,\mathscr{B},v)$ for all $i\in B_l$.

□

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Envy-Free Solution and Allocation Rule of Indivisible Goods

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Abstract

Given n indivisible goods and n agents we want to assign the goods among the agents, whose valuations on the goods determine the vector of prices. We prove that there always exists a solution that is envy-free and individually rational with non negative prices, under the assumption that agents have quasi - linear preferences on goods. Also, we provide an allocation rule that is strategy-proof and envy-free.

Mathematics Subject Classification: 90C90, 91A12, 91B08, 91B26

Keywords: Linear Programming, Envy-free, Strategy-proof, Allocation rule, Indivisible goods, Cooperative game

1 Introduction

In this work we treat the problem of fair division where a finite number of indivisible goods must be distributed among a finite number of agents. Supposing that the valuations of the agents over the good are known, we must determine

a price for each good. We assume that each agent is interested in acquiring only one good and has associated a utility function that is quasi - linear on prices. Under this setting, there are studies that have been done by several authors such as Brams et al. [1], Shao Chin Sung et al. [4] and Haake et al. [5].

Assuming that the valuations of every agent over the goods are not negative, we prove that there always exists a way to assign these goods among agents and determine a price for each good. This price is set such that no agent wishes the good of another. Moreover, the price of each good is at most the valuation of the agent who receives it and is not negative. Also, we provide an allocation rule that is strategy-proof and the obtained solution by the rule is envy-free and individually rational.

A situation where our rule can be applied is as follows: there are n houses that have to be distributed among n families. Each family has its valuations on the houses, what is interpreted as what the family is willing to pay for each house. The problem is how to assign the houses among the families and determine a price for each of them such that no family wishes the house of another. In this paper we provide a way to assign the houses and calculate the prices of these.

This paper is divided in four parts. In part 2 we give the notations and definitions that will be used throughout the text. In part 3 we give the proof of the existence of envy-free solutions. In part 4 we provide the allocation rule.

2 Notations and definitions

Let $M := \{1, \ldots, n\}$ be a finite set of goods that has to be distributed among the agents in $N := \{1, \ldots, n\}$. We suppose that each agent is interested in acquiring only one good and he can receive any of them. We represent the valuations of the agents over the goods by a matrix $A := (a_{ij}) \in \mathbb{R}_+^{M \times N}$, where a_{ij} is the valuation of the agents j on the ith good. We will denote by A^T and A_j the transpose and the jth column of the matrix A, respectively. M and N are fixed, a *problem* will be a matrix $A \in \mathbb{R}_+^{M \times N}$. Given a problem $A \in \mathbb{R}_+^{M \times N}$, $j \in N$ and $B_j \in \mathbb{R}_+^M$ we define a new problem $A(B_j)$ by making the substitution of the column A_j, in A, by B_j.

An *assignment* for the problem $A \in \mathbb{R}_+^{M \times N}$ is a permutation θ of the set $\{1, \ldots, n\}$, i.e, $\theta \in \mathcal{S}_n$ (the permutation group of n elements). Given an assignment θ, $\theta(j) = i$ means that good i is assigned to agent j. An assignment

θ such that

$$\sum_{j=1}^{n} a_{\theta(j)j} \geq \sum_{j=1}^{n} a_{\delta(j)j}, \quad \forall \delta \in \mathcal{S}_n$$

is called an *efficient assignment*. A *price* will be a vector $p \in \mathbb{R}_+^M$. A *solution* is a pair $(\theta, p) \in \mathcal{S}_n \times \mathbb{R}_+^M$ representing an assignation of goods to agents and a price for each one of them.

We assume that each agent $j \in N$ has a utility function $u_j : M \times \mathbb{R} \to \mathbb{R}$, given by $u_j(i, p) = a_{ij} - p_i, \quad \forall j \in N$. This quantity represents the utility of the agent j once he receives the *ith* good.

A solution $(\theta, p) \in \mathcal{S}_n \times \mathbb{R}_+^M$ for a problem A is called *individually rational* if $a_{\theta(j)j} \geq p_{\theta(j)}, \forall j \in N$, i.e the pay of each agent is at most what he is willing to pay for the good that he receives. It is called *envy-free* if $a_{\theta(j)j} - p_{\theta(j)} \geq a_{ij} - p_i$, $\forall j \in N$ y $\forall i \in M$, which means that no agent wishes the good of another.

An *allocation rule (or simply a rule)* is a (non-empty) correspondence $\varphi : \mathbb{R}_+^{M \times N} \to P(\mathcal{S}_n \times \mathbb{R}_+^M)$ where $P(Q)$ is the power set of Q. This correspondence is set such that $u_j(\theta(j, p)) = u_j(\sigma(j), \overline{p})$ for all $j \in N$ whenever $(\theta, p), (\sigma, \overline{p}) \in \varphi(A)$, i.e., all agents are indifferent between any two solutions selected by the rule. A rule φ is called *envy-free* if for every problem A, all elements in $\varphi(A)$ are envy-free. It is called *strategy-proof* if for every problem A and any two solutions $(\theta, p) \in \varphi(A)$, $(\delta, \overline{p}) \in \varphi(A(B_j))$ it holds that

$$a_{\theta(j)j} - p_{\theta(j)} \geq a_{\delta(j)} - p_{\delta(j)} \quad \forall j \in N.$$

Given a problem $A \in \mathbb{R}_+^{M \times N}$, $S \subseteq N$, $T \subseteq M$ with $|S|, |T| \geq 1$ we define a linear programming problem $P_A(S, T)$ and its dual $D_A(S, T)$:

$P_A(S,T):$	$D_A(S,T):$
$Max \sum_{i \in S} \sum_{j \in T} a_{ij} x_{ij}$	$Min \sum_{i \in S} v_i + \sum_{j \in T} u_j$
$s.a$	$s.a$
$\sum_{i \in S} x_{ij} \leq 1 \quad \forall j \in T$	$u_j + v_i \geq a_{ij} \quad \forall i \in S$ y $\forall j \in T$
$\sum_{j \in T} x_{ij} \leq 1 \quad \forall i \in S$	$u_j \geq 0 \quad \forall j \in T$
$x_{ij} \geq 0$	$v_i \geq 0 \quad \forall i \in S.$

Dantzig [3] proved that there always exist optimal solutions of $P_A(S, T)$ such that $x_{ij} \in \{0, 1\}$ for all $i \in M$ and $j \in N$. Then, this kind of solution which is optimal for the problem $P_A(N, M)$, gives an efficient assignment of the problem A as a result. So, the optimal value of $P_A(N, M)$ is the total value for all goods in M, which we will denote by $P_A^*(N, M)$.

In the next section the optimal solutions of the linear programming problem $D_A(N, M)$ will play an important role in the existence of envy-free solutions.

3 Envy-free solutions

In this section we will prove that every problem A has an envy-free and individually rational solution.

Proposition 3.1. *If $A \in \mathbb{R}_+^{M\times N}$ is a problem then there exist $(\theta, p) \in \mathcal{S}_n \times \mathbb{R}_+^M$ such that (θ, p) is envy-free and individually rational for the problem A.*

Proof. We know by Dantzig [3] that the linear programming problem $P_A(N, M)$ has optimal solutions with $x_{ij} \in \{0, 1\}$ for all $i \in M$ and $j \in N$. Suppose that x^* is an optimal solution of this kind and (u^*, v^*) is an optimal solution of $D_A(N, M)$. We consider the solution (θ, v^*) with $\theta(j) = i$ if $x^*_{ij} = 1$, for all $j \in N$. Then, by the complementary slackness conditions we have that $u^*_j = a_{ij} - v^*_i$. By the previous equality and that $u^*_j \geq a_{ij} - v^*_i$ for all $i \in M$ we have that (θ, v^*) is an envy-free solution. The inequality $a_{\theta(j)j} \geq v^*_{\theta(j)}$ follows from $u^*_j = a_{\theta(j)j} - v^*_{\theta(j)} \geq 0$ for all $j \in N$. □

Remark 3.2. *Note that the shadow price vector v of the goods in the linear programming problem $P_A(N, M)$ has the following property: if (θ, v) is a solution with θ an efficient assignment, then (θ, v) is envy-free and individually rational.*

In the next lemma we say that any assignment in an envy-free solution is an efficient assignment.

Lemma 3.3. *If $(\theta, p) \in \mathcal{S}_n \times \mathbb{R}_+^M$ is an envy-free solution, then $\theta \in \mathcal{S}_n$ is an efficient assignment.*

Proof. Let $(\theta, p) \in \mathcal{S}_n \times \mathbb{R}_+^M$ be an envy-free solution. Then,

$$a_{\theta(j)j} - p_{\theta(j)} \geq a_{\sigma(j)j} - p_{\sigma(j)}$$

for all $j \in N$. This implies that

$$\sum_{j\in N} a_{\theta(j)j} - \sum_{j\in N} p_{\theta(j)} \geq \sum_{j\in N} a_{\sigma(j)j} - \sum_{j\in N} p_{\sigma(j)}$$

for all $\sigma \in \mathcal{S}_n$. Hence, $\sum_{j\in N} a_{\theta(j)j} \geq \sum_{j\in N} a_{\sigma(j)j}$ for all $\sigma \in \mathcal{S}_n$. □

Now, we characterize the core of the associated game of problem $A \in \mathbb{R}_+^{M\times N}$ via envy-free and individually rational solutions. For this, we need some notation and a few definitions.

A *n-person game* in characteristic function form (or a TU game) is a pair (N, w), where N is a finite set of players and w is a function $w : 2^N \to \mathbb{R}$ with $w(\emptyset) = 0$. The function w specifies the value of $S \subseteq N$. The *core* of the TU game (N, w) is defined by

$$\mathcal{C}(N, w) = \{x \in \mathbb{R}^N \mid x(S) \geq w(S) \ \ \forall S \subsetneq N, x(N) = w(N)\}.$$

For every problem $A \in \mathbb{R}_+^{M \times N}$ we associate the TU game $(\overline{N}, w_A)$, where $\overline{N} = N \cup M$ and for all $R \subseteq \overline{N}$

$$w_A(R) = \begin{cases} \max\limits_{\{S \subseteq R \cap N, T \subseteq R \cap M \ \mid \ |S|, |T| \geq 1\}} P_A^*(S, T), & \text{if } R \cap N \neq \emptyset \ \text{ and } \ R \cap M \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

According to the TU game $(\overline{N}, w_A)$ the value for every coalition $R \subseteq \overline{N}$, with $R \cap N \neq \emptyset$ and $R \cap M \neq \emptyset$, is the maximum value that we obtain by the goods in R under any efficient assignment among the agents in R. The coalitions with only elements in N or M have zero value.

Proposition 3.4. *Let $A \in \mathbb{R}_+^{M \times N}$ be a problem and $(\overline{N}, w_A)$ its associated TU game. We have that $(u, v) \in \mathcal{C}(\overline{N}, w_A)$ iff there exist $\theta \in \mathcal{S}_n$ such that (θ, v) is an envy-free and individually rational solution for the problem A and u satisfies $u_j = a_{\theta(j)j} - v_{\theta(j)}$ for all $j \in N$.*

Proof. Let $A \in \mathbb{R}_+^{M \times N}$ be a problem, $(\overline{N}, w_A)$ its associated TU game and $\theta \in \mathcal{S}_n$ an efficient assignment. We suppose that $(u, v) \in \mathcal{C}(\overline{N}, w_A)$. Shapley et al. [6] proved that every element of the core of $(\overline{N}, w_A)$ is an optimal solution of the linear programming problem $D_A(N, M)$. From Remark 3.2 (θ, v) is an envy-free and individually rational solution for problem A.

Conversely, let $(u, v) \in \mathbb{R}_+^{n \times n}$ and $\theta \in \mathcal{S}_n$. We assume that (θ, v) is an envy-free and individually rational solution and u satisfies $u_j = a_{\theta(j)j} - v_{\theta(j)}$ for all $j \in N$. Then $u_j \geq 0$ for all $j \in N$. Given that $a_{\theta(j)j} - p_{\theta(j)} \geq a_{ij} - p_i$, $\forall j \in N$ and $\forall i \in M$, then $u_j \geq \max\{a_{ij} - v_i : i \in M\}$ $\forall j \in N$. Hence (u, v) is a feasible solution of the linear programming problem $D_A(N, M)$. We note that

$$\begin{aligned} \sum_{j=1}^{n} u_j + \sum_{i=1}^{n} v_i &= \sum_{j=1}^{n} (a_{\theta(j)j} - v_{\theta(j)}) + \sum_{i=1}^{n} v_i, \\ &= \sum_{j=1}^{n} a_{\theta(j)j}, \\ &= w_A(\overline{N}), \end{aligned}$$

since $\theta \in \mathcal{S}_n$ is an efficient assignment, which follows from Lemma 3.3. Hence (u, v) is an optimal solution of the linear programming problem $D_A(N, M)$. The result follows from the fact that every optimal solution of the linear programming problem $D_A(N, M)$ is an element of $\mathcal{C}(\overline{N}, w_A)$ (see Shapley et al. [6]). □

4 An allocation rule

In the previous section we prove that for every $A \in \mathbb{R}_+^{M\times N}$ there exists an envy-free and individually rational solution. In this section we will provide a rule that is envy-free and strategy-proof.

Let θ be an assignment. We define $p^\theta \in \mathbb{R}^n$ in the following way

$$p_i^\theta = P_A^*(N \setminus \{\theta^{-1}(i)\}, M) - P_A^*(N \setminus \{\theta^{-1}(i)\}, M \setminus \{i\}), \quad \forall i \in M.$$

Now, we define the allocation rule ϕ by

$$\phi(A) = \{(\theta, p^\theta) \mid \theta \text{ is an efficient assignment}\} \tag{1}$$

for all $A \in \mathbb{R}_+^{M\times N}$.

Proposition 4.1. *The allocation rule (**??**rul) is envy-free and strategy-proof.*

Proof. Let A be a problem and $(\theta, p^\theta) \in \phi(A)$. Demange [2] proved that every vector $(u, v) \in \mathbb{R}^{n\times n}$ with $u_j = P_A^*(N, M) - P_A^*(N \setminus \{j\}, M)$, for all $j \in N$ and $v_i = p_i^\theta$, for all $i \in M$, is an element of $\mathcal{C}(\overline{N}, w_A)$. Thus, by Proposition 3.4, ϕ is envy-free. Now, we suppose that there exist $j \in N$, $B_j \in \mathbb{R}_+^M$ and two solutions $(\theta, p^\theta) \in \phi(A)$, $(\delta, p^\delta) \in \phi(A(B_j))$ such that

$$a_{\theta(j)j} - p^\theta_{\theta(j)} < a_{\delta(j)} - p^\delta_{\delta(j)}.$$

The previous inequality and equality $P_A^*(N \setminus \{j\}, M) = P_{A(B_j)}^*(N \setminus \{j\}, M)$ imply that

$$a_{\theta(j)j} + P_A^*(N \setminus \{j\}, M \setminus \{\theta(j)\}) < a_{\delta(j)j} + P_{A(B_j)}^*(N \setminus \{j\}, M \setminus \{\delta(j)\}),$$

which is a contradiction since θ is an efficient assignment. To see this, note that $P_{A(B_j)}^*(N \setminus \{j\}, M \setminus \{\delta(j)\})$ is the maximum value that is obtained for the goods in $M \setminus \{\delta(j)\}$ when those are assigned efficiently among the agents in $N \setminus \{j\}$; in that case

$$a_{\theta(j)j} + P_A^*(N \setminus \{j\}, M \setminus \{\theta(j)\})) = \sum_{j=1}^{n} a_{\theta(j)j}.$$

□

Example 4.2. *We suppose that there are three houses to be distributed among three families, whose valuations on the houses are given by the following matrix:*

$$A = \begin{bmatrix} 18 & 24 & 24 \\ 10 & 9 & 15 \\ 15 & 18 & 18 \end{bmatrix}.$$

We note that a unique efficient assignment is $\theta(1) = 3$, $\theta(2) = 1$ *and* $\theta(3) = 2$. *With this assignment the total value of the houses is* 54 *monetary units. Using the explicit formula of* p_i^θ *and with a few calculations we obtain that* $p^\theta = (9, 0, 3)$. *If from each column of* A *we subtract the vector* p^θ, *we obtain the following matrix*

$$\begin{bmatrix} 9 & 15 & 15 \\ 10 & 9 & 15 \\ 12 & 15 & 15 \end{bmatrix}.$$

Each entry $a_{\theta(j)j}$, *with* $j \in N$, *in this matrix is positive and greater or equal to any other entry in the same column. This implies that the solution* $(\theta, p^\theta) = ((2, 3, 1), (9, 0, 3))$ *is envy-free and individually rational.*

If family 1 *changes its valuations on the houses, then we obtain the following matrix*

$$A(B_1) = \begin{bmatrix} 18 & 24 & 24 \\ 5 & 9 & 15 \\ 10 & 18 & 18 \end{bmatrix}$$

where $B_1 = (18, 5, 10)^T$. *In this case the unique efficient assignment is* $\delta(1) = 1$, $\delta(2) = 3$ *and* $\delta(3) = 2$. *With the assignment* δ *the total value of the houses is* 51 *monetary units. Calculating the price vector we obtain* $p^\delta = (9, 0, 3)$. *We note that* $a_{\theta(1)1} - p^\theta_{\theta(1)} = 12 > 9 = a_{\delta(1)1} - p^\delta_{\delta(1)}$, *which implies that family* 1 *got less for changing its valuations.*

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Linear symmetric rankings for TU-games

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Abstract We define ranking as an equivalence relation on the set of power indices and study those that have a linear and symmetric representative. Moreover, we classify—or parametrize—those rankings that reward "positive" payoffs for "positive" participation. It is shown that these are in 1-1 correspondence with the points of the standard simplex. Moreover, this correspondence is convex. Finally, we contrast this classification with Saari–Sieberg's approach via "positive" semi-values.

Keywords Ranking · Linear power index

1 Introduction

This article draws original inspiration from Saari and Sieberg's, "Some surprising properties of power indices", (Saari and Sieberg 2001). We have tried to continue their work with a systematic study of (linear) rankings, analyzing several of their properties and giving a classification of them, which could prove a useful decisions tool in mathematical economics.

To do this, we start by giving a rigorous definition of "ranking" (Definition 3) as an equivalence class of power indices (or solutions) defined on TU cooperative games. Two such indices are "equivalent" if they assign the same ordering to all players, in all games. This definition is quite general and applies to arbitrary solutions. Nevertheless, soon enough we restrict to rankings given by linear and symmetric (see Definition 1)

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solutions. Still, this is slightly more general than the power indices consider in Saari and Sieberg (2001), as we will explain below.

Thus, we call LS-rankings those that contain a linear and symmetric representative (Definition 4). It is observed (Theorem 1) that every such LS-ranking contains a linear, symmetric and efficient (LSE) representative. Moreover, this LSE- representative is unique up to a positive scaling factor (Theorem 2). Since LSE indices are parametrized by vectors in $\mathbb{R}^{n-1}$ (see Ruiz et al. 1998 or Hernández-Lamoneda et al. 2007), where n is the number of players in the TU games considered, we obtain a 1-1 correspondence (Theorem 3) between all LS-rankings and the unit sphere $S^{n-2} \subset \mathbb{R}^{n-1}$. This should be understood in the sense that if we pick two power indices corresponding to two different points on S^{n-2}, then there will exist a game v and players i and j, whose rank in the game v will be interchanged by the chosen indices. This result also tells you that there are exactly $n-2$ dimensions of linear and symmetric rankings for n players.

Starting in Sect. 6 we specialize our study to (linear and symmetric) power indices that assign a positive contribution to players with a positive participation. The precise definition of positivity for LSE-indices is given in Definition 7 and discussed in the lines prior to it. Whereas Proposition 2 says that—except for the egalitarian ranking—every LS-ranking corresponds uniquely to a semi-ray on $\mathbb{R}^{n-1}$; positivity of the ranking corresponds, then, to semi-rays on the positive orthant.

One may choose a point—an LSE power index representative—from each semi-ray. For example, the point where the semi-ray intersects the unit sphere—as stated above. Nevertheless, for positive LS-rankings there is a more meaningful choice (from a game theoretic point of view): pick the point where the semi-ray intersects the standard simplex $\Sigma = \{b = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1} \mid b_1 + \cdots + b_{n-1} = 1, b_j \geq 0\}$. It is proved—Lemma 3—that this choice corresponds to asking that the power index satisfies the inessential game condition (Definition 8).

In other words, it is proved (Theorem 5) that there is a 1-1 correspondence between positive LS-rankings and the points of the simplex Σ, since both are in 1-1 correspondence with positive LSE-indices that further satisfy the inessential game condition (we call these, replicative indices). It is also shown that this correspondence is convex, Corollary 2. This result may be interpreted as a classification (or parametrization) of all positive LS-rankings.

Next, examples are given. In particular, we see that the ranking given by Shapley's value corresponds to the baricenter of Σ. Similarly, the point on Σ corresponding to Banzhaf's is computed. It is readily seen (as was first observed in Saari and Sieberg 2001) that these two points coincide for $n = 3$, but are different for every other $n > 3$. Hence, the rankings they induce behave accordingly (Fig. 1).

Figure 2, at the end of Sect. 7, depicts the position in Σ of a few rankings corresponding to well known indices. It is observed that some of them enjoy a certain geometric symmetry. This symmetric position inside Σ has a game theoretic meaning which is explained in Theorem 7.

The last part of the article deals with semi-values and their relationship with rankings. Thus, we look at solutions of the form

$$\psi_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S) \right]$$

for arbitrary numbers $r_0, r_1, \ldots, r_{n-1}$. It is well known that these are precisely the solutions that satisfy linearity, symmetry and the null player condition. We will say that ψ is a semi-value if, in addition, it satisfies the inessential game condition. In the literature (e.g. Dubey et al. 1981), it is also customary to further ask that each $r_i \geq 0$.

These are exactly the power indices considered in Saari and Sieberg (2001), i.e. semi-values with all $r_i \geq 0$. Let us call these, "positive semi-values".

We prove a few technical results relating semi-values with the work in previous sections, culminating in Theorem 8, where it is shown that every positive LS-ranking contains a semi-value. In other words, there is no loss—as far as ranking is concerned—in considering only semi-values. As expected from the difference in dimensions, there is a 1-parameter family of semi-values giving the same ranking. It is explicitly given just after Theorem 8.

Saari and Sieberg consider a simplex $\Delta \subset \mathbb{R}^n$ that parametrizes their power indices. At the end of the section we give the linear map that projects this simplex into Σ and which explains the redundancy in ranking inside Δ, found by these authors. Perhaps unexpectedly, it is readily seen that, when $n > 3$, this projection is not onto. This tells us that even though *every positive LS ranking contains a semi-value, it does not necessarily contain a "positive" semi-value.* In other words: every positive ranking (in our terminology) will always have a representative that is a semi-value, but this semi-value cannot always be chosen to have all coefficients r_i to be non-negative.

Thus, some potentially interesting rankings are not included in Saari–Sieberg's Δ.

Other authors have taken a different point of view, concentrating on classifying rankings for special subsets of games. For example, Tomiyama (1987) or Friedman et al. (2006), consider subclasses of voting games. More general families of TU-games—and thus, making that work closer to this article—were considered by Freixas (2010). In particular, Freixas studies subsets of TU-games whose players are ranked equally by all semi-values of a certain subclass. For example, he characterizes those monotone TU-games (which he calls "coherent") for which Shapley and Banzhaf values give the same ranking.

2 Notation

Let $G = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ be the space of games, in TU-form, for the players in $N = \{1, 2, \ldots, n\}$.

Definition 1 A solution $\phi : G \to \mathbb{R}^n$ is said to be symmetric if, for every permutation θ of N, and every game $v \in G$, we have that

$$\phi(\theta^* v) = \theta \cdot \phi(v);$$

recall that $\theta^* v$ is the game that maps $S \mapsto v(\theta^{-1} S)$ and that $\theta \cdot (\phi_1, \ldots, \phi_n) = (\phi_{\theta^{-1}(1)}, \ldots, \phi_{\theta^{-1}(n)})$, for every vector $(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$.

Recall, also, that linearity means that $\phi(ru + sv) = r\phi(u) + s\phi(v)$, for arbitrary games $u, v \in G$ and scalars $r, s \in \mathbb{R}$.

In this note we will consider solutions that are linear and symmetric in the above sense.

3 Rankings

Any solution $\phi : G \to \mathbb{R}^n$ provides a power index ranking for the players with respect to any game v, in which they are involved, by ordering increasingly (or decreasingly) the coordinates of $\phi(v)$. For this reason, in the rest of the article we will usually prefer to call ϕ an "index" instead of a solution.

For example, suppose that for a five player game v and a certain index ϕ we have $\phi(v) = (8, 0, -1, 4, 3)$. Then ϕ, for this game v, induces the following ranking:

$$3 \prec 2 \prec 5 \prec 4 \prec 1.$$

Definition 2 Let $\phi, \psi : G \to \mathbb{R}^n$ be two indices (or solutions). We say that $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$ give the same ranking if and only if for every game v, and players i, j,

$$\phi_i(v) > \phi_j(v) \quad \text{iff} \quad \psi_i(v) > \psi_j(v).$$

We will write $\phi \sim \psi$ to denote that both indices give the same ranking.

Observe that $\sim$ is an equivalence relation. In other words:

1. $\phi \sim \phi$ for every ϕ;
2. $\phi \sim \psi$ if and only if $\psi \sim \phi$; and,
3. $\phi \sim \psi$ and $\psi \sim \eta$ imply $\phi \sim \eta$.

Thus, the equivalence relation $\sim$ determines equivalence classes: given any index $\phi : G \to \mathbb{R}^n$, its equivalence class is the set

$$[\phi] = \{\psi : G \to \mathbb{R}^n \mid \phi \sim \psi\}.$$

Definition 3 We call "a ranking" such an equivalence class $[\phi]$ of indices.

Observe that the collection of rankings partitions the set of all indices. Also notice that, if $\psi \sim \phi$ and if $\psi_i(v) = \psi_j(v)$, for some pair of players i, j and some game v, then $\phi_i(v) = \phi_j(v)$.

Definition 4 We will say that a ranking is a "linear symmetric ranking" (an LS ranking) if there exists a linear symmetric representative for it: i.e., a ranking $[\phi]$ is an LS ranking if there exists a linear, symmetric ψ such that $\psi \sim \phi$.

It will be useful, in the sequel, to consider the following related, but stronger, notion.

Definition 5 Let $\phi, \psi : G \to \mathbb{R}^n$ be two indices. We say that ϕ and ψ agree up to a constant, if there exists a function $k : G \to \mathbb{R}$, such that, for every game $v \in G$ and every player $j \in N$

$$\phi_j(v) = \psi_j(v) + k(v).$$

Remark 1 – Clearly, two indices that agree up to a constant will induce the same ranking. We will see below (Corollary 1) that, perhaps surprisingly, a certain converse of this statement holds for LS indices.
– Agreeing up to a constant is also an equivalence relation. As noted in the previous line, it is a finer relation than $\sim$.
– If ϕ is an LS index and $\psi \sim \phi$, it is not necessarily true that ψ is linear. For example, let $k : G \to \mathbb{R}$ be any function, then $\psi(v) = \phi(v) + k(v)(1, 1, \ldots, 1)$ is equivalent to ϕ and not linear for almost any choice of k. It might be worthwhile to try to characterize those rankings that are LS, but we wont pursue this problem here.

4 Efficiency

Definition 6 An index $\phi : G \to \mathbb{R}^n$ is said to be efficient if, for every game $v \in G$

$$\sum_{j \in N} \phi_j(v) = v(N).$$

Now we will see that all LS rankings admit a linear, symmetric and efficient (LSE) representative. In other words, every LS ranking is an LSE ranking. In fact, let us show the following:

Theorem 1 *Given any linear, symmetric index $\phi : G \to \mathbb{R}^n$ there exists a unique LSE-index ψ, such that ϕ and ψ agree up to a constant.*

Proof Let $v \in G$ be any game. For each player $j \in N$, define

$$\psi_j(v) = \phi_j(v) + \frac{1}{n}\left[v(N) - \sum_{i \in N} \phi_i(v) \right].$$

This rule defines an index ψ. Let us check that ψ is LSE:

Linearity:

$$\begin{aligned}
&\psi_j(av + bw) \\
&= \phi_j(av + bw) + \frac{1}{n}\left[(av + bw)(N) - \sum_{i \in N} \phi_i(av + bw) \right] \\
&= a\phi_j(v) + b\phi_j(w) + \frac{1}{n}\left[av(N) + bw(N) - \sum_{i \in N} (a\phi_i(v) + b\phi_i(w)) \right] \\
&= a\phi_j(v) + a\frac{1}{n}\left[v(N) - \sum_{i \in N} \phi_i(v) \right] + b\phi_j(w) + b\frac{1}{n}\left[w(N) - \sum_{i \in N} \phi_i(w) \right] \\
&= a\psi_j(v) + b\psi_j(w).
\end{aligned}$$

Symmetry:

$$\begin{aligned}\psi_j(\theta^* v) &= \phi_j(\theta^* v) + \frac{1}{n}\left[(\theta^* v)(N) - \sum_{i\in N}\phi_i(\theta^* v)\right]\\ &= \phi_{\theta_j^{-1}}(v) + \frac{1}{n}\left[v(\theta^{-1}N) - \sum_{i\in N}\phi_{\theta_i^{-1}}(v)\right]\\ &= \phi_{\theta_j^{-1}}(v) + \frac{1}{n}\left[v(N) - \sum_{k\in N}\phi_k(v)\right]\\ &= \psi_{\theta_j^{-1}}(v).\end{aligned}$$

Efficiency:

$$\begin{aligned}\sum_{j\in N}\psi_j(v) &= \sum_{j\in N}\phi_j(v) + \sum_{j\in N}\left\{\frac{1}{n}\left[v(N) - \sum_{i\in N}\phi_i(v)\right]\right\}\\ &= \sum_{j\in N}\phi_j(v) + n\left\{\frac{1}{n}\left[v(N) - \sum_{i\in N}\phi_i(v)\right]\right\}\\ &= v(N).\end{aligned}$$

For every $v \in G$, $\psi_j(v) = \phi_j(v) + k(v)$, where $k(v) = \frac{1}{n}\left[v(N) - \sum_{i\in N}\phi_i(v)\right]$.

To prove uniqueness, assume ϕ and ψ are LSE and that $\phi_j(v) = \psi_j(v) + k(v)$ for some function $k : G \to \mathbb{R}$, for all $v \in G$ and for every $j \in N$. Adding over all players, and using efficiency, we get

$$v(N) = v(N) + nk(v),$$

for every game $v \in G$. Thus $k(v) = 0$. □

Thus any ranking induced by a linear symmetric index can be realized by a linear, symmetric, efficient one.

4.1 Ruiz–Valenciano–Zarzuelo's parametrization of LSE indices

There is a 1-1 correspondence between $(n-1)$-tuples $b = (b_1, b_2, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$ and LSE-indices for n players. This was probably first observed in Ruiz et al. (1998) and later, in an equivalent way, in Hernández-Lamoneda et al. (2007). We follow this last reference.

The correspondence is given by $b \mapsto \phi^b$ where

$$\phi_i^b(v) = \frac{v(N)}{n} + \sum_{\substack{S\ni i\\ S\neq N}}\frac{n-1}{n\binom{n-1}{s-1}}b_s v(S) - \sum_{T\not\ni i}\frac{n-1}{n\binom{n-1}{t}}b_t v(T). \qquad (1)$$

Here s, t represent the cardinalities of coalitions S, T.

Later on we will provide an interpretation of this formula. For the moment, let us recall a characterization (from Hernández-Lamoneda et al. 2007) of LSE-indices.

Consider a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. For every $j : 1, 2, \ldots, n$ one can define a game x^j as follows:

$$x^j(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = j \\ 0 & \text{if } |S| \neq j. \end{cases}.$$

Every vector x decomposes uniquely as a sum $x = c + z = \bar{x}(1, 1, \ldots, 1) + z$, where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and z is a vector whose coordinates add up to zero: $z_1 + \cdots + z_n = 0$. Accordingly, one obtains a decomposition for the game $x^j = c^j + z^j$. Notice that the game c^j is a symmetric game, i.e. its value on a given coalition depends only on that coalition's cardinality. Now, every symmetric and efficient index ϕ has already determined its values on every symmetric game v: by symmetry $\phi_k(v) = \phi_l(v)$ for every pair of players k, l; by efficiency, $n\phi_l(v) = \sum_{i \in N} \phi_i(v) = v(N)$. So we look at the values of ϕ on the games z^j.

First of all, notice that $z^n = 0$, therefore we consider only $j : 1, 2, \ldots, n-1$. Now we can state the characterization.

Proposition 1 *Let $\varphi : G \to \mathbb{R}^n$ be an LS index. For every $j : 1, \ldots, n-1$, there is a constant $\lambda_j \in \mathbb{R}$, such that, for every $z \in \mathbb{R}^n$ with $z_1 + \cdots + z_n = 0$,*

$$\varphi(z^j) = \lambda_j z.$$

Moreover, for the LSE-index ϕ^b, $b \in \mathbb{R}^{n-1}$, $\lambda_j = b_j$. In other words,

$$\phi^b(z^j) = b_j z.$$

Thus, if two LSE-indices coincide on all z^j, then they are equal.

Proof We give only a sketch of the proof. For more details, we refer the reader to Hernández-Lamoneda et al. (2007).

The proof is an exercise in the theory of representations for the group S_n, of permutations of n players. Fix $j : 1, \ldots, n-1$, and consider the subspace of games $V_j = \{z^j \in G \mid z_1 + \cdots + z_n = 0\}$. This subspace is invariant under S_n and in fact isomorphic to the standard representation space $\{z \in \mathbb{R}^{n-1} \mid z_1 + \cdots + z_n = 0\}$ (this is simply the equation $\theta * z^j = (\theta \cdot z)^j$). Since φ is symmetric (S_n-equivariant in the language of group representations) and the spaces in question irreducible, the result follows from Schur's Lemma.

To verify the assertion for LSE-indices, one needs only to compute $\phi^b(z^j)$ on any z, say on $z = (1, 0, 0, \ldots, 0, -1)$. □

Remark 2 In fact, this proposition tells you that every LSE-solution is determined by its values (for $j : 1, 2, \ldots, n-1$) on z^j for a single non-zero vector z whose coordinates add up to zero. For example $z = (1, 0, \ldots, 0, -1)$.

Back to rankings.

Take the LSE-index ϕ^b as defined before, multiply it by a positive constant, $r > 0$, and look at the unique LSE-index which agrees with $r\phi^b$ up to a constant.

Lemma 1 *The unique LSE-index that agrees with $r\phi^b$ up to a constant is the LSE-index ϕ^{rb}.*

Proof Let ψ be the LSE-index that agrees up to a constant with $r\phi^b$: $\psi(v) = r\phi^b(v) + k(v)$, for some function $k : G \to \mathbb{R}$.

Then, according to Proposition 1, it is enough to verify that $\psi(z^j) = \phi^{rb}(z^j)$ for every vector $z = (z_1, \ldots, z_n)$, $z_1 + \cdots + z_n = 0$ and for every $j : 1, \ldots, n-1$. Just as in the proof of the last theorem, adding over all players, $i : 1, \ldots, n$, the equations

$$\psi_i(z^j) = r\phi_i^b(z^j) + k(z^j),$$

and using efficiency, we get $z^j(N) = rz^j(N) + nk(z^j)$; but $z^j(N) = 0$, therefore $k(z^j) = 0$ and

$$\psi(z^j) = r\phi^b(z^j) = rb_j z = \phi^{rb}(z^j).$$

□

Now, clearly, if $r > 0$, then ϕ and $r\phi$ induce the same ranking ($\phi \sim r\phi$); therefore, $\phi^b \sim \phi^{rb}$ if $r > 0$. The next theorem states that this is the only occurrence of equivalence within LSE indices.

Theorem 2 *Let $b, b' \in \mathbb{R}^{n-1}$. Then $\phi^b \sim \phi^{b'}$ if and only if $b' = rb$, for some constant $r > 0$.*

Proof If $b = 0$, ϕ^0 is the egalitarian solution: $\phi^0(v) = \frac{v(N)}{n}(1, 1, \ldots, 1)$; any other linear index that ranks all players equally for every game, must be a multiple of ϕ^0. If in addition it is efficient, then it must be equal to ϕ^0. Therefore, assume $b \neq 0$.

Consider $a_1, a_2, \ldots, a_{n-1}$ arbitrary real numbers. Define vectors $z_s = (a_s, 0, 0, \ldots, 0, -a_s)$, $s : 1, \ldots, n-1$, and the game $u = \sum_{s=1}^{n-1} z_s^s$. Then

$$\phi_1^b(u) - \phi_2^b(u) = \sum_{s=1}^{n-1} \left(\phi_1^b(z_s^s) - \phi_2^b(z_s^s)\right) = \sum_{s=1}^{n-1} (b_s a_s - 0) = \sum_{s=1}^{n-1} b_s a_s;$$

Similarly, $\phi_1^{b'}(u) - \phi_2^{b'}(u) = \sum_{s=1}^{n-1} b'_s a_s$.

So, if $\phi^b \sim \phi^{b'}$, then

$$\sum_{s=1}^{n-1} b_s a_s = \phi_1^b(u) - \phi_2^b(u) > 0 \quad \text{iff} \quad \sum_{s=1}^{n-1} b'_s a_s = \phi_1^{b'}(u) - \phi_2^{b'}(u) > 0.$$

The a_s's are arbitrary real numbers, hence the fact that ϕ^b and $\phi^{b'}$ induce the same ranking implies that

$$b \cdot a > 0 \quad \text{iff} \quad b' \cdot a > 0$$

for every vector $a \in \mathbb{R}^{n-1}$, where $\cdot$ stands for the usual euclidean inner product. This clearly implies that b' is a positive multiple of b. □

5 Classifying LS rankings

We use the results in the previous section to give a classification of all linear symmetric rankings.

Proposition 2 *There is a 1-1 correspondence between the set of all possible linear symmetric index rankings and the set consisting of the origin in $\mathbb{R}^{n-1}$ union the set of positive semi-rays in $\mathbb{R}^{n-1}$:*

$$\{\textit{LS rankings}\} \overset{1-1}{\longleftrightarrow} \{0\} \cup \left\{\{rb\} \mid r > 0, 0 \neq b \in \mathbb{R}^{n-1}\right\}.$$

Proof Take an LS ranking $[\psi]$, then by Theorem 2 there exists $b \in \mathbb{R}^{n-1}$ such that $\phi^b \in [\psi]$. Moreover, if $\phi^a \in [\psi]$ then $\phi^a \sim \phi^b$ and therefore $a = rb$, for some $r > 0$. Thus, the map $[\psi] \mapsto \{rb\}$ is well defined and 1-1. It is obviously onto (with inverse $\{rb\} \mapsto [\phi^b]$). □

Equivalently,

Theorem 3 *There is a 1-1 correspondence between*

$$\{\textit{LS rankings}\} \overset{1-1}{\longleftrightarrow} \{0\} \cup S^{n-2},$$

where S^{n-2} stands for the unit sphere in $\mathbb{R}^{n-1}$.

Yet another way of rephrasing this: *There is a 1-1 correspondence between all linear symmetric index rankings (different from the egalitarian one) and LSE indices ϕ^b with $\|b\| = 1$.*

Finally, as a further consequence of Theorem 2, we have the following converse mentioned in Remark 1.

Corollary 1 *Let $\eta, \psi : G \to \mathbb{R}^n$ be linear and symmetric. Assume that η and ψ give the same ranking (i.e., $\eta \sim \psi$). Then there exists a constant $r > 0$ such that ψ and $r\eta$ agree up to a constant.*

Proof Let $\phi^b, \phi^{b'}$ be the LSE indices that agree up to a constant with η and ψ respectively. Then, $\eta \sim \psi$ implies $\phi^b \sim \phi^{b'}$, hence $b' = rb$, for some $r > 0$. But ϕ^{rb} agrees up to a constant with $r\phi^b$. Observe that "agrees up to a constant" is also an equivalence relation. Hence, ψ agrees up to a constant with $\phi^{b'}$ which agrees up to a constant with $r\phi^b$ which agrees up to a constant with $r\eta$. □

In other words, two linear and symmetric solutions η and ψ induce the same ordering of the players for every game v, if and only if there exists a constant $r > 0$ and a function $k : G \to \mathbb{R}$ such that, for every player j and every game $v \in G$

$$\eta_j(v) = r\psi_j(v) + k(v).$$

6 Replicative indices

We have seen that every possible ranking, induced by a linear symmetric index, can be achieved by a linear, symmetric and efficient one (i.e. any LS ranking contains a ϕ^b, for some $b \in \mathbb{R}^{n-1}$). We have also observed that such an LSE-index is not unique, but we get a whole semi-ray worth of equivalent LSE-indices (corresponding to rb, $r > 0$). To associate a unique LSE index to every LS-ranking, some kind of normalization must be performed.

The geometrically natural normalization given by setting $\|b\| = \sum_k b_k^2 = 1$ -as in Theorem 3 above- does not seem to imply any game theoretic properties (for example, Shapley's value does not satisfy it). In this section we will look at a different normalization, namely $\sum_k b_k = 1$, which is equivalent to the inessential game condition, as will be proved below.

6.1 Positivity of indices

Let us recall, first of all (see Hernández-Lamoneda et al. 2007, for example), how one may interpret $\phi_i^b(v)$, for each player i and for an arbitrary game v.

Firstly, the total payoff to be shared among the players is, by efficiency, $v(N)$. Thus, in a first step we split this amount equally among all players: thus each receives $v(N)/n$.

Next, one needs to compensate among players according to their different participation. For this, one interprets $v(S)$ as the worth of coalition S, or as the amount claimed by S. These compensations may be understood as some sort of transference among the players—in which the total payoff is preserved—following a simple rule: for every coalition $S \neq N$, players in S receive, from those outside of S, an equal share of

$$\frac{n-1}{\binom{n}{s}} b_s v(S).$$

Likewise, each player not in S (i.e., in $N \backslash S$), pays an equal share of the above amount. And so, the final payoff that every player receives is given by

$$\phi_i^b(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} \frac{1}{s} \left[\frac{n-1}{\binom{n}{s}} b_s v(S) \right] - \sum_{T \not\ni i} \frac{1}{n-t} \left[\frac{n-1}{\binom{n}{t}} b_t v(T) \right];$$

which is equivalent to

$$\phi_i^b(v) = \frac{v(N)}{n} + \frac{n-1}{n}\left[\sum_{\substack{S\ni i\\ S\neq N}} \frac{1}{\binom{n-1}{s-1}} b_s v(S) - \sum_{T\not\ni i} \frac{1}{\binom{n-1}{t}} b_t v(T)\right].$$

Each b_k can be thought of as the relative weight that coalitions of cardinality $k \leq n-1$ have in the solution ϕ^b. There is no b_n appearing in the formula, since symmetry and efficiency already determine how $v(N)$ should be split.

So, if we think of $v(S)$ as the amount claimed by the coalition S, then it is reasonable to assume—of the index (or payoff) ϕ^b—that each b_s be non-negative. For example, consider the vector $z = (1, 0, \ldots, 0, -1) \in \mathbb{R}^n$ and the corresponding game z^j. Let us spell out what this game does:

$$z^j(S) = \begin{cases} 1 & \text{if } |S| = j, 1 \in S \quad \text{and} \quad n \notin S \\ -1 & \text{if } |S| = j, 1 \notin S \quad \text{and} \quad n \in S \\ 0 & \text{otherwise.} \end{cases}$$

Now, Proposition 1 tells us that

$$\phi^b(z^j) = b_j z = b_j(1, 0, \ldots, 0, -1).$$

Thus, in this simple example we realize that a player that contributes positively to coalitions of cardinality j (player 1) receives a positive payoff if and only if $b_j > 0$.

In general, for an arbitrary game v, and players i and j, we have that

$$\begin{aligned}\phi_i^b(v) - \phi_j^b(v) &= \frac{n-1}{n}\left[\sum_{\substack{S\ni i\\ S\not\ni j}} \frac{b_s}{\binom{n-2}{s-1}} v(S) - \sum_{\substack{S\ni j\\ S\not\ni i}} \frac{b_s}{\binom{n-2}{s-1}} v(S)\right] \\ &= \frac{n-1}{n} \sum_{R\not\ni i,j} \frac{b_{r+1}}{\binom{n-2}{r}} \left[v(R\cup\{i\}) - v(R\cup\{j\})\right].\end{aligned}$$

So, if the total participation of player i is larger than that of j—in the sense, for example, that $v([S\backslash\{i,j\}]\cup\{i\}) \geq v([S\backslash\{i,j\}]\cup\{j\})$ for every S—, then the sign of the b_s's control the relative rankings of i and j.

Remark 3 Consider two games, v, w, such that $v(S) = w(S)$, for every coalition $S \neq S_0$ and $v(S_0) > w(S_0)$. Then, requiring that $\phi_i^b(v) \geq \phi_i^b(w)$, for every $i \in S_0$, is equivalent to asking that $b_{s_0} \geq 0$, where s_0 is the cardinality of S_0.

Therefore, following the above discussion, we restrict our attention to indices ϕ^b with b in the positive orthant, $\mathbb{R}_+^{n-1} = \{b = (b_1, b_2, \ldots, b_{n-1}) \mid b_j \geq 0\}$.

Definition 7 Let $[\psi]$ be an LS ranking, $\phi^b \in [\psi]$ an LSE index. The ranking $[\psi]$ is said to be "positive" if $b \neq 0$ and $b \in \mathbb{R}_+^{n-1}$.

Remark 4 Recall that the LSE-index corresponding to the zero vector—i.e. ϕ^0—is the egalitarian index: $v \mapsto \frac{v(N)}{n}(1, 1, \ldots, 1)$.

Also notice that positivity is well defined, since if $\phi^a \in [\psi]$ is another LSE-index, then according to Theorem 2, $a = rb$ for a positive r. Thus positivity is independent of the chosen LSE representative.

6.2 Inessential game property

Recall that a game $v \in G$ is said to be additive if $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. The next lemma is well known.

Lemma 2 *For every vector $x \in \mathbb{R}^n$ define a game $\hat{x}$ such that $\hat{x}(S) = \sum_{i \in S} x_i$. Then $\hat{x}$ is an additive game; moreover, every additive game is of this form.*

Sometimes it is said that additive games solve themselves, in the sense that a natural payoff for $\hat{x}$ is x itself. This motivates the following—also well known—definition.

Definition 8 An index $\varphi : G \to \mathbb{R}^n$ satisfies the inessential game condition, if

$$\varphi(\hat{x}) = x$$

for every vector $x \in \mathbb{R}^n$.

The next lemma identifies which LSE-indices satisfy the inessential game condition.

Lemma 3 *Let ϕ^b be the LSE-index corresponding to the vector $b = (b_1, \ldots, b_{n-1})$. Then, for every $x \in \mathbb{R}^n$,*

$$\phi^b(\hat{x}) = x \quad \textit{if and only if} \quad \sum_{k=1}^{n-1} b_k = 1.$$

Proof Let x be an arbitrary vector in $\mathbb{R}^n$ and let ι denote the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$. Then, as was already mentioned in 4.1, x can be uniquely written as: $x = \frac{\hat{x}(N)}{n}\iota + z$, where z is a vector whose coordinates add up to zero. Then, $\hat{x} = \frac{\hat{x}(N)}{n}\hat{\iota} + \hat{z}$. Observe that $\hat{\iota}$ is simply the game that assigns to every coalition S its cardinality $|S|$. By symmetry and efficiency of ϕ^b, we have that $\phi^b(\hat{\iota}) = \iota$. Observe, also, that $\hat{z} = \sum_{j=1}^{n-1} z^j$—refer to the definition of z^j in 4.1.

Therefore, from Proposition 1

$$\begin{aligned}
\phi^b(\hat{x}) &= \frac{\hat{x}(N)}{n}\phi^b(\hat{\iota}) + \phi^b\left(\sum\nolimits_{j=1}^{n-1} z^j\right) \\
&= \frac{\hat{x}(N)}{n}\iota + \sum\nolimits_{j=1}^{n-1} \phi^b(z^j) \\
&= \frac{\hat{x}(N)}{n}\iota + \sum\nolimits_{j=1}^{n-1} b_j z = \frac{\hat{x}(N)}{n}\iota + \left[\sum\nolimits_{j=1}^{n-1} b_j\right] z.
\end{aligned}$$

The Lemma follows. □

Notice that, except for the egalitarian index, every index ϕ^b, with b in the positive orthant, may be "rescaled" to satisfy the hypotheses of the Lemma: If $b = (b_1, \ldots, b_{n-1}) \neq 0$ is in the positive orthant, then $\sum_i b_i > 0$ and $b' = \frac{1}{\sum_i b_i} b$ satisfies the hypotheses of the Lemma.

6.3 Replicative indices

Definition 9 Let ϕ^b be the LSE-index corresponding to the vector $0 \neq b \in \mathbb{R}^{n-1}$. Then we say that ϕ^b is replicative if

1. $b_i \geq 0$ for every $i : 1, \ldots, n-1$; and
2. $\sum_{i=1}^{n-1} b_i = 1$

In other words, a replicative LSE-index is one that satisfies the inessential game condition and assigns non-negative payoffs to players with non-negative participation in coalitions of arbitrary cardinality.

The next result follows directly from the definition of replicative index and the results of Sect. 5.

Theorem 4 *In each "positive" LS ranking, there is a unique replicative LSE index. Therefore, there is a 1-1 correspondence between positive linear symmetric rankings and replicative LSE indices.*

Proof To each LS ranking $[\psi]$, corresponds a 1-parameter family of LSE-indices ϕ^{rb}, $r > 0$, $b \in \mathbb{R}^{n-1}$. If the ranking is positive, then all $b_j \geq 0$ and $\sum_j b_j > 0$. Thus, $\phi^{\frac{b}{\sum_j b_j}}$ is the unique replicative index in $[\psi]$. □

7 A graphic description of positive rankings

Here we classify all positive LS-rankings.

Recall that the standard simplex $\Sigma \subset \mathbb{R}^{n-1}$ is the set

$$\Sigma = \left\{ b = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1} \mid b_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^{n-1} b_j = 1 \right\}.$$

Notice that the b_k's are the barycentric coordinates of b with respect to the vertices of Σ.

Theorem 5 *There is a 1-1 correspondence between every positive LS ranking, for n players, and the points of the standard simplex Σ in $\mathbb{R}^{n-1}$:*

$$\{\text{Positive LS rankings}\} \overset{1-1}{\longleftrightarrow} \Sigma.$$

Proof According to Theorem 4, every positive LS ranking contains a unique representative, ϕ^b, which is replicative. That is $b = (b_1, b_2, \ldots, b_{n-1})$, where $b_i \geq 0$ for all i, and $\sum_i b_i = 1$. In other words, $b \in \Sigma$. □

Remark 5 The point of the theorem is that the standard simplex Σ parametrizes all linear symmetric rankings that come from indices that assign a positive payoff to players with a positive participation. Such rankings correspond—as we saw in Sect. 5—to semi-rays inside the positive orthant. Each ray intersects Σ in the unique LSE representative which satisfies the inessential game condition.

We also have:

Corollary 2 *The one to one correspondence $b \in \Sigma \mapsto \phi^b$ preserves convex combinations. That is, for every $a, b \in \Sigma$ and $t \in [0, 1]$, then*

$$\phi^{ta+(1-t)b} = t\phi^a + (1-t)\phi^b.$$

Proof LSE indices form an affine space, hence a convex set. Thus, both sides of the equation are LSE indices. To verify that they are the same, according to Proposition 1, it is enough to check equality on games z^j, for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ such that $z_1 + \cdots + z_n = 0$.

$$\begin{aligned}\phi^{ta+(1-t)b}(z^j) &= (ta_j + (1-t)b_j)z = ta_j z + (1-t)b_j z \\ &= t\phi^a(z^j) + (1-t)\phi^b(z^j) \\ &= [t\phi^a + (1-t)\phi^b](z^j)\end{aligned}$$

□

Let us give some examples.

7.1 Shapley's value

Shapley's value is an LSE-index, it is ϕ^b for

$$b = \left(\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right).$$

That is, Shapley's value corresponds to the barycenter of Σ.

7.2 Banzhaf's semi-value

Although Banzhaf's semi-value satisfies three of the four axioms that are commonly used to characterize Shapley value, it does not satisfy efficiency. Nevertheless, from our discussion we know that there is a unique LSE-index B, satisfying the inessential game condition, which gives the same ranking as Banzhaf's semi-value (it can be verified that B coincides with the least squares prenucleolus of Ruiz et al. (1996)). This B corresponds to the point $b \in \Sigma$ with

$$b_j = \frac{\binom{n-2}{j-1}}{2^{n-2}};$$

that is

$$B = \frac{1}{2^{n-2}}\left(1, n-2, \ldots, \binom{n-2}{j-1}, \ldots, n-2, 1\right).$$

For example, for $n = 3, 4, 5, 6$, the barycentric coordinates corresponding to Shapley and Banzhaf rankings are

$$\begin{array}{lll}
Sh = \left(\frac{1}{2}, \frac{1}{2}\right), & B = \left(\frac{1}{2}, \frac{1}{2}\right), & n = 3; \\
Sh = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), & B = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), & n = 4; \\
Sh = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), & B = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right), & n = 5; \\
Sh = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), & B = \left(\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}\right), & n = 6.
\end{array}$$

We first observe that for $n = 3$, the replicative index corresponding to Banzhaf's semi-value is Shapley's value. Thus, these two give the same index ranking (a feature that was first observed in Saari and Sieberg 2001). Observe also, that as soon as $n > 3$, B and Sh are different and, therefore, give different rankings -as noted, also, in Saari and Sieberg (2001).

We see right away from this description that B allocates more weight to players' participation in mid-sized coalitions, whereas Shapley's weights are equally distributed among all cardinalities. This is not surprising considering that Banzhaf semi-value was originally constructed to model voting situations.

Graphically, Sh corresponds to the barycenter of Σ—let us denote, in what follows, by Sh the barycenter of Σ—, whereas B slides towards the middle vertices of Σ.

The following figures depict the situation for $n = 4, 5$.

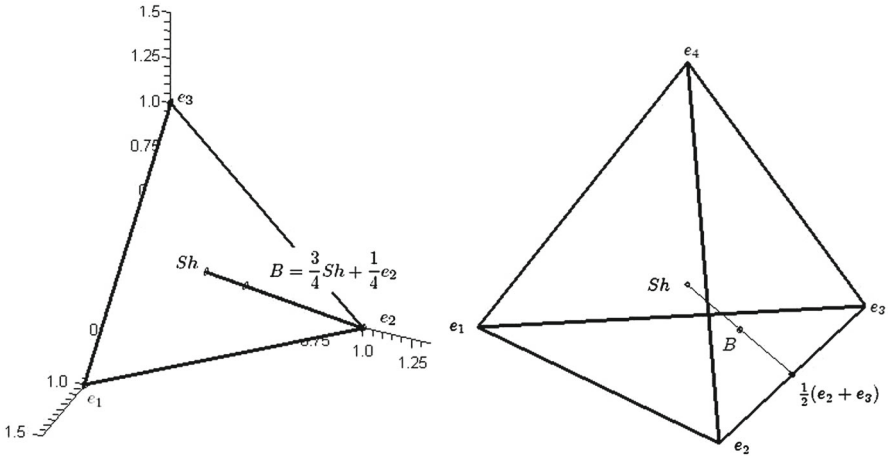


Fig. 1 Banzhaf and Shapley rankings for $n = 4, 5$

7.3 Equal surplus and the vertices of Σ

The equal surplus index (or solution) is the linear, symmetric index given by

$$\varphi_i(v) = v(\{i\}) + \frac{v(N) - \sum_{k\in N} v(\{k\})}{n}.$$

We may interpret this formula as follows: the payoff for player i is what he can obtain by himself plus an equal part of the surplus between what the grand coalition makes and what all individual players can generate separately.

Notice that φ

- is efficient: $\sum_i \varphi_i(v) = \sum_i v(\{i\}) + v(N) - \sum_k v(\{k\}) = v(N)$;
- satisfies the inessential game condition $\varphi_i(\hat{x}) = x_i$; and, in fact
- $\varphi = \phi^b$, with $b = (1, 0, 0, \ldots, 0) = e_1$: choose $z = (1, 0, \ldots, 0, -1) \in \mathbb{R}^n$, $j : 1, \ldots, n-1$, then one easily verifies that

$$b_j = \varphi_1(z^j) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}.$$

Thus, the equal surplus index φ coincides with the LSE-index ϕ^{e_1} corresponding to the first vertex of Σ.

Theorem 6 *Let ϕ^{e_k} be the LSE-index corresponding to the standard basis vector $e_k \in \mathbb{R}^{n-1}$ (ie. the solutions corresponding to the vertices of Σ); in other words, ϕ^{e_k} is the LSE solution where only coalitions of cardinality k matter.*

Then

$$\phi_i^{e_k}(v) = \frac{v(N)}{n} + \frac{n-1}{\binom{n}{k}} \left[\sum_{\substack{|S|=k \\ S \ni i}} \frac{v(S)}{s} - \sum_{\substack{|S|=k \\ S \not\ni i}} \frac{v(S)}{n-s} \right].$$

Corollary 3 *Let ϕ^b be the LSE-index corresponding to the vector $b = (b_1, \ldots, b_{n-1}) \in \Sigma$. Then*

$$\phi^b = \sum_{k=1}^{n-1} b_k \phi^{e_k}.$$

In other words,

$$\phi_i^b(v) = \frac{v(N)}{n} + (n-1) \left[\sum_{S\ni i} \frac{b_s v(S)}{s\binom{n}{s}} - \sum_{S\not\ni i} \frac{b_s v(S)}{(n-s)\binom{n}{s}} \right].$$

Remark 6 If we are strictly interested in ranking, it might be simpler to use the linear symmetric index, determined by $b \in \mathbb{R}^{n-1}$, which is identically zero on symmetric games. If we call this index η^b, then its formula is:

$$\eta_j^b(v) = \sum_{S \ni j} \frac{b_s}{\binom{n-2}{s-1}} v(S).$$

7.4 Consensus, solidarity and a simple index

In Borm et al. (2004) it is proved that the consensus value C is the middle point between the equal surplus index and Shapley value. That is, it corresponds to the middle point between the first vertex e_1 and the barycenter Sh:

$$C = \phi^{\left(\frac{n}{2(n-1)}, \frac{1}{2(n-1)}, \ldots, \frac{1}{2(n-1)}\right)}.$$

The solidarity value, as introduced by Nowak and Radzik (1995), is an LSE-index which is characterized by these three axioms together with the A-null player axiom. It corresponds to ϕ^b with $b = (b_1, \ldots, b_{n-1})$ given by

$$b_j = \frac{1}{(n-1)(j+1)}.$$

In this case we need to normalize it to get a point on Σ; thus, in the sequel, we will call the solidarity index to the index Sol corresponding to the normalized vector $b = (b_1, \ldots, b_{n-1})$ with

$$b_j = \frac{1}{(j+1)\sum_{k=2}^{n} \frac{1}{k}}.$$

Clearly, $b_1 > b_2 > \cdots > b_{n-1} > 0$.

Finally, we may consider the following "simple" index σ:

$$\sigma_i(v) = \sum_{S \ni i} \frac{v(S)}{s} - \sum_{T \not\ni i} \frac{v(T)}{n-t}.$$

Again, after normalization, this index corresponds to the following vector $b \in \Sigma$:

$$b_j = \frac{\binom{n}{j}}{2^n - 2},$$

for $j : 1, \ldots, n-1$.

7.5 A table and figure with the previous examples

		$n=3$	$n=4$	$n=5$
Shapley	Sh	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
Banzhaf	B	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{4},\frac{1}{2},\frac{1}{4})$	$(\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8})$
Equal surplus	EqS	(1,0)	(1,0,0)	(1,0,0,0)
Consensus	C	$(\frac{3}{4},\frac{1}{4})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	$(\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$
Solidarity	Sol	$(\frac{3}{5},\frac{2}{5})$	$(\frac{6}{13},\frac{4}{13},\frac{3}{13})$	$(\frac{30}{77},\frac{20}{77},\frac{15}{77},\frac{12}{77})$
Simple	σ	$(\frac{1}{2},\frac{1}{2})$	$(\frac{2}{7},\frac{3}{7},\frac{2}{7})$	$(\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{6})$

8 Self-duality

From Fig. 2 we notice that the vectors corresponding to Shapley value, Banzhaf semivalue and σ satisfy a certain symmetry, namely: $b_j = b_{n-j}$, for all $j : 1, \ldots, n-1$.

On G there is a duality operator $* : G \to G$ defined by

$$(*v)(S) = v(N) - v(N \backslash S).$$

Definition 10 An index $\phi : G \to \mathbb{R}^n$ is said to be self-dual if

$$\phi(*v) = \phi(v)$$

for every game $v \in G$.

An efficient and self-dual solution gives the same importance to those quantities that coalitions gain and those they miss to gain.

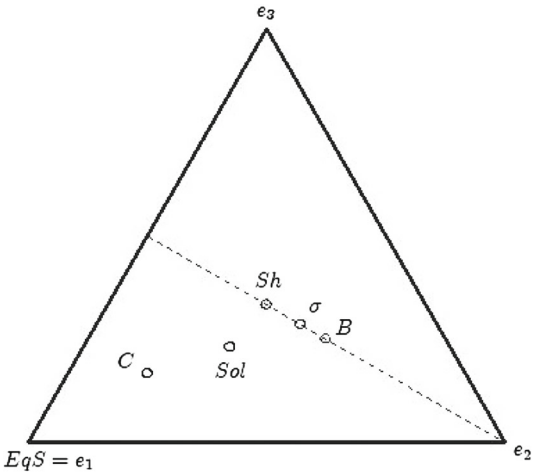


Fig. 2 Graphical representation of the above indices, $n = 4$

Theorem 7 *Let ϕ^b be the LSE-index corresponding to the vector $b = (b_1, \ldots, b_{n-1})$. Then ϕ^b is self-dual if and only if $b_j = b_{n-j}$ for all $j : 1, \ldots, n-1$.*

More generally, given $\phi : G \to \mathbb{R}^n$, we define $\phi^* : G \to \mathbb{R}^n$, the index dual to ϕ, by

$$\phi^*(v) = \phi(*v).$$

Then $\left(\phi^b\right)^* = \phi^{b^*}$, where $b^*_j = b_{n-j}$.

For example, the ENSC (egalitarian non-separable contribution) solution may be written as

$$\mathrm{ENSC}_i(v) = v(N) - v(N\backslash\{i\}) + \frac{1}{n}\left[v(N) - \sum_{k=1}^{n}\left(v(N) - v(N\backslash\{k\})\right)\right]$$

which shows that it is dual to the Equal Surplus solution ϕ^{e_1}. Therefore, ENSC $= \phi^{e_{n-1}}$.

9 Semi-values and ranking

An index ψ is said to be null (or satisfy the nullity axiom) if $\psi_i(v) = 0$ for every game for which i is a null player. Let us consider indices ψ that are linear, symmetric and null. It is well known that every such index is of the form

$$\psi^r_i(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S)\right]$$

for arbitrary numbers $r_0, r_1, \ldots, r_{n-1}$ (more precisely, there is a 1-1 correspondence between vectors $r = (r_0, \ldots, r_{n-1}) \in \mathbb{R}^n$ and linear, symmetric and null solutions ψ^r, given by the preceding formula).

Moreover, we will say that ψ^r is a semi-value if it further satisfies the inessential game condition (this coincides with the usual definition of semivalue—as in Dubey et al. (1981), for example—except for the fact that we don't require $r_s \geq 0$). Similar to Lemma 3, the inessential game condition can be phrased in terms of the r_s as:

$$\sum_{s=0}^{n-1} \binom{n-1}{s} r_s = 1.$$

Define a linear map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\mathcal{E}(r_0, r_1, \ldots, r_{n-1}) = (b_1, b_2, \ldots, b_{n-1})$, via

$$b_s = \binom{n-2}{s-1}(r_{s-1} + r_s);$$

for every $s : 1, \ldots, n-1$.

Proposition 3 *ψ^r and ϕ^b agree up to a constant if and only if $\mathcal{E}(r) = b$.*

Proof Assume that ψ^r and ϕ^b agree up to a constant. Then there exists a function $k : G \to \mathbb{R}$ such that $\psi_i^r(v) - \phi_i^b(v) = k(v)$, for every game v and every player i.

Take $z = (1, 0, 0, \ldots, 0, -1) \in \mathbb{R}^n$ and recall that we have defined a game z^s via the rule

$$z^s(T) = \begin{cases} \sum_{k\in T} z_k & \text{if } |T| = s \\ 0 & \text{if } |T| \neq s. \end{cases}.$$

Now, first of all, adding over all players

$$nk(z^s) = \sum_i \psi_i^r(z^s) - \sum_i \phi_i^b(z^s) = 0$$

since, according to Proposition 1, both $\psi^r(z^s)$ and $\phi^b(z^s)$ are multiples of z. Hence, $\psi^r(z^s) = \phi^b(z^s)$ for every $s : 1, \ldots, n-1$. Moreover, again from Proposition 1, we know that $\phi^b(z^s) = b_s z$. We compute $\psi_1^r(z^s)$:

$$\begin{aligned} \psi_1^r(z^s) &= \sum_{T \not\ni 1} r_t \left[z^s(T \cup \{1\}) - z^s(T) \right] \\ &= \sum_{\substack{T \not\ni 1, n \\ |T| = s-1}} r_{s-1} + \sum_{\substack{T \not\ni 1 \\ T \ni n \\ |T| = s}} r_s = \binom{n-2}{s-1} (r_{s-1} + r_s). \end{aligned}$$

This shows that $\mathcal{E}(r) = b$.

Conversely, let $r \in \mathbb{R}^n$ and $b \in \mathbb{R}^{n-1}$ such that $\mathcal{E}(r) = b$. Let $a \in \mathbb{R}^{n-1}$ be such that ϕ^a is the LSE-index that agrees up to a constant with ψ^r (Theorem 1). The same computation as above shows that for $z = (1, 0, 0, \ldots, 0, -1)$, and every $s : 1, \ldots, n-1$, $\psi^r(z^s) = \phi^a(z^s)$. But

$$a_s = \phi_1^a(z^s) = \psi_1^r(z^s) = \binom{n-2}{s-1} (r_{s-1} + r_s) z = b_s;$$

therefore $a = b$ and $\phi^a = \phi^b$. □

Proposition 4 *Let $b \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}^n$ such that $\mathcal{E}(r) = b$. Then ψ^r satisfies the inessential game condition (i.e. it is a semi-value) if and only if ϕ^b does also.*

Proof Consider then:

$$\begin{aligned} \sum_{s=1}^{n-1} b_s &= \sum_{s=1}^{n-1} \binom{n-2}{s-1} (r_{s-1} + r_s) \\ &= r_0 + \sum_{s=1}^{n-1} \left[\binom{n-2}{s-1} + \binom{n-2}{s} \right] r_s \end{aligned}$$

$$= r_0 + \sum_{s=1}^{n-1} \binom{n-1}{s} r_s = \sum_{s=0}^{n-1} \binom{n-1}{s} r_s.$$

Now, the inessential game condition for ϕ^b (respectively, ψ^r) is equivalent to the left hand sum above (respectively, right hand sum) being equal to one. □

The linear map $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is clearly onto. Thus,

Theorem 8 *Every LS ranking contains a linear, symmetric and null index. Moreover, as long as the LS ranking is in the direction of a vector b (Proposition 2 or Theorem 3), with* $\sum b_j > 0$*, it will contain a semi-value.*

In particular, every positive LS ranking contains a semi-value.

Proof Given any LS ranking $[\varphi]$, pick an LSE representative $\phi^b \in [\varphi]$. Since $\mathcal{E}$ is onto, there is $r \in \mathbb{R}^n$ such that $\mathcal{E}(r) = b$. Now, Proposition 3 tells us that $\psi^r \sim \phi^b$ and therefore $\psi^r \in [\varphi]$.

If $\lambda = \sum b_j > 0$, then $\frac{1}{\lambda}\phi^b \sim \phi^b$, but as we know (Lemma 1) $\frac{1}{\lambda}\phi^b \sim \phi^{\frac{b}{\lambda}}$. By linearity, $\mathcal{E}(\frac{r}{\lambda}) = \frac{b}{\lambda}$, therefore $\psi^{\frac{r}{\lambda}} \sim \phi^{\frac{b}{\lambda}} \sim \phi^b \sim \varphi$. Clearly $\phi^{\frac{b}{\lambda}}$, and therefore $\psi^{\frac{r}{\lambda}}$, satisfies the inessential game condition.

Positive LS rankings are characterized—Theorem 4—by containing an LSE index ϕ^b with $b_j \geq 0$, for all j, and $\sum b_j = 1$; therefore they satisfy the hypotheses of the theorem. □

In terms of the graphical description for positive LS rankings, we see that on top of every $b \in \Sigma$ there is a line, $\mathcal{E}^{-1}(b) = \{r \in \mathbb{R}^n \mid \mathcal{E}(r) = b\}$, corresponding to all semi-values ψ^r that agree with ϕ^b up to a constant. One can make explicit this line:

$$\mathcal{E}^{-1}(b) = \left\{ (r_0, r_1, \ldots, r_{n-1}) \mid r_s = (-1)^s \left[r_0 + \sum_{j=1}^{s} \frac{(-1)^j b_j}{\binom{n-2}{j-1}} \right],\ s : 1, \ldots, n-1; r_0 \in \mathbb{R} \right\}.$$

Example. Recall that the barycenter of Σ, $Sh = (\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$, corresponds to Shapley's value. In this case, from the above formula and a little manipulation, $r \in \mathcal{E}^{-1}(Sh)$ if and only if

$$r_s = (-1)^s \left(r_0 - \frac{1}{n} \right) + \frac{(n-s-1)!s!}{n!}$$

for $s : 1, \ldots, n-1$.

Recall that Shapley's is the only efficient semi-value, and that its expression in terms of the marginal contributions of the players, is:

$$Sh_i(v) = \sum_{S \not\ni i} \frac{(n-s-1)!s!}{n!} \left[v(S \cup \{i\}) - v(S) \right].$$

Therefore, for $r \in \mathcal{E}^{-1}(Sh)$, one may compute that (for every player i and every game $v \in G$)

$$\psi_i^r(v) = Sh_i(v) + \left(r_0 - \frac{1}{n}\right) \sum_{S \subset N} (-1)^s v(S), \quad r_0 \in \mathbb{R},$$

is the family of semi-values that agree up to a constant with Shapley's.

As noted above, in general, for arbitrary $b \in \mathbb{R}^{n-1}$, $r \in \mathcal{E}^{-1}(b)$ if and only if

$$r_s = (-1)^s \left[r_0 + \sum_{j=1}^{s} \frac{(-1)^j b_j}{\binom{n-2}{j-1}} \right]$$

for all $s : 1, \ldots, n-1$. It is not clear to us, from this formula, which semi-value (if any) in $\mathcal{E}^{-1}(b)$ ought to be preferred over the rest.

9.1 Relationship between the simplex Σ and Saari–Sieberg's simplex

As mentioned in the introduction, Saari and Sieberg (2001) considers -mainly- power indices of the form $\psi_i^r(v) = \sum_{S \not\ni i} r_s \left[v(S \cup \{i\}) - v(S) \right]$, with $\sum_{s=0}^{n-1} \binom{n-1}{s} r_s = 1$ and $r_s \geq 0$, i.e., "positive" semi-values. They set

$$\hat{\lambda}_{s+1} = \binom{n-1}{s} r_s$$

and consider the simplex

$$\Delta = \left\{ (\hat{\lambda}_1, \ldots, \hat{\lambda}_n) \mid \sum_{s=1}^{n} \hat{\lambda}_s = 1, \hat{\lambda}_s \geq 0 \right\} \subset \mathbb{R}^n$$

which parametrizes all positive semi-values.

From the definition of $\mathcal{E}$ one computes the linear map L taking $(\hat{\lambda}_1, \ldots, \hat{\lambda}_n) \mapsto (b_1, \ldots, b_{n-1})$. It is given by

$$b_s = \frac{n-s}{n-1} \hat{\lambda}_s + \frac{s}{n-1} \hat{\lambda}_{s+1}.$$

Proposition 4 above shows that $L(\Delta) \subset \Sigma$. In fact, $L(\Delta)$ is the convex hull of the points $\{L(\widehat{e}_j)\}$, and

$$L(\widehat{e}_j) = \begin{cases} e_1 & \text{if } j = 1 \\ \frac{j-1}{n-1} e_{j-1} + \frac{n-j}{n-1} e_j & \text{if } 1 < j < n \\ e_{n-1} & \text{if } j = n \end{cases}.$$

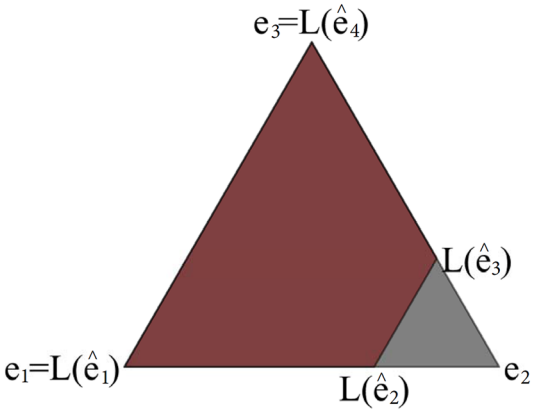


Fig. 3 $L(\Delta)$—in *red*—properly contained inside of $\Sigma, n = 4$

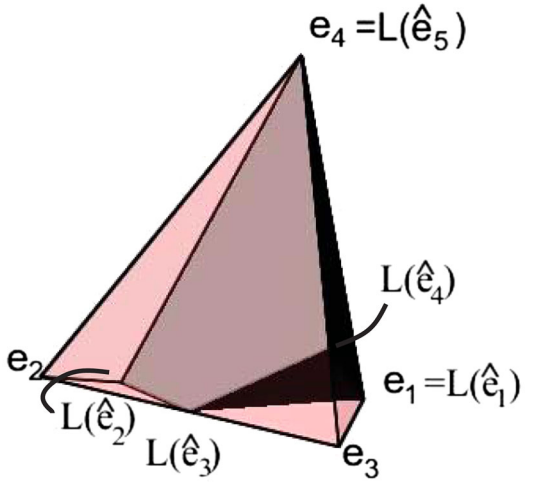


Fig. 4 $L(\Delta)$ properly contained inside of $\Sigma, n = 5$

From this, it also follows (perhaps unexpectedly), that as soon as $n > 3$, $L(\Delta) \neq \Sigma$. In other words, *even though every positive ranking contains a semi-value, it does not necessarily contain a positive semi-value*. Thus, some potentially interesting positive rankings are missing from Δ.

Example. Take $n = 4$. Then $L(\widehat{e_1}) = e_1$, $L(\widehat{e_2}) = \frac{1}{3}e_1 + \frac{2}{3}e_2$, $L(\widehat{e_3}) = \frac{2}{3}e_2 + \frac{1}{3}e_3$ and $L(\widehat{e_4}) = e_3$. Since $L(\Delta)$ is the convex hull of the $L(\widehat{e_j})$, we see that, for example, $e_2 \in \Sigma$ is not in the image of L. (Fig. 3, see also Fig. 4 for $n = 5$) In fact, we know from the previous section that

$$\mathcal{E}^{-1}(e_2) = \left\{ r = \left(r_0, -r_0, r_0 + \frac{1}{2}, -r_0 - \frac{1}{2} \right) \mid r_0 \in \mathbb{R} \right\}$$

which in barycentric $\widehat{\lambda}$ coordinates corresponds to

$$\left\{\hat{\lambda} = \left(r_0, -3r_0, 3\left(r_0 + \frac{1}{2}\right), -r_0 - \frac{1}{2}\right) \mid r_0 \in \mathbb{R}\right\}$$

which clearly does not intersect Δ.

In general, the preimage of a point under L in Δ—a segment, a point or the empty set—, corresponds to positive semi-values giving the same ranking. This 1-dimensional redundancy in ranking was already observed in Saari and Sieberg (2001).

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On the inverse problem for a subclass of linear, symmetric and efficient values of cooperative TU games

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ABSTRACT

In this paper we solve the inverse problem for each linear, symmetric, efficient and regular value (or LSER value for short). That is, given a payoff vector and a LSER value, we find all TU games such that LSER value for them is equal to the given payoff vector. Also, we characterize those linear, symmetric and efficient values that further satisfy the inessential game property.

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1. Introduction

The study of the properties of values for TU games is one of the topics of cooperative game theory since the seminal axiomatic contribution of Shapley in 1953. This is due to the desire to understand the different contexts in which the cooperative game theory can be applied. For instance, there are several studies where the original axioms proposed by Shapley in 1953 are substituted by weak versions or by different axioms that fit certain applications. Other work considers the family of solutions that emerge by taking out one or two of the original axioms. The most popular family of values consist of values that are linear, symmetric, and efficient. This family of values was initially presented by Ruiz et al. [14].

In this paper we study the inverse problem of each linear, symmetric, efficient and regular value (or LSER value for short). By a regular value we mean that each of its parameter is nonzero. This is, given an n-vector of payoffs, say x, to find out all the games in G^N, such that for these TU games the LSER value equals x. Also, we characterize the linear, symmetric and efficient values that satisfy the inessential game property.

Dragan [5] solved the inverse problem for the Shapley value. He proposed a different system of weights, a similar problem was solved for the more general case of the weighted Shapley values, based on some results from linear algebra and the potential of these values due to Hart and Mas-Colell [8,9]. In this case, the solution was successful because he discovered a new algebraic basis for the null space of the linear operator, that he called the potential basis. Later, Dragan [6] uses a similar algebraic basis to solve the inverse problem for semivalues. However, in the case of LSER values such a basis is missing, so that in the present paper, we have to use another approach to solve the inverse problem for these values.

There are several situations where it may be interesting to know the totality of cooperative games that have a common payoff vector.

For example, suppose there is a certain agreement between companies to distribute their costs according to a certain linear, symmetric and efficient value. Moreover, assume such companies wish to diminish such costs for the next semester. Hypothetically, if these agents can for-see such costs (i.e. the payoff vector) for the next period, then it would be useful to know which "game" will produce such a vector. Once they know the collection of those games sharing the given payoff, they will have information that will help them to make decisions (e.g. reinforce certain coalitions).

On the other hand, in cooperative game theory literature there are many different values that have been used in certain concrete situations. On many occasions, such values are linear, symmetric and efficient (LSE), to name a few: Shapley value, solidarity value or the least square prenucleolus. Moreover, the study of the inverse problem for a LSE value could, potentially, be used to give a useful characterization of the value. For example, Béal et al. [2] obtain a characterization of the average tree solution (see Herings et al. [10]) by looking at the inverse problem for this value. Béal et al. [3] provide an alternative characterization of the class of LSER values, and also, they examine the inverse problem for the Shapley value and the equal division value. Furthermore, Béal et al. [4] and Yokote et al. [17] also deal with the inverse problem for the Banzhaf

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value and the Shapley value, respectively. Other examples may be found in Yokote [16] and Béal et al. [1]. Both articles find bases for the kernel of the corresponding value, and a certain invariance axiom is proposed that permits them to characterize the solution.

The inverse problem is also investigated in OR for measuring voting power via simple games. Kurz and Napel [13] provide a recent survey on this question, and Kurz [12] studies the inverse problem of power indices in order to design voting rules for a committee such that a given desired power distribution is met as closely as possible.

This paper is divided in five sections. In Section 2 we give the notation and definitions necessary to explain our work. In Section 3 we present our results and in Section 4 we characterize all LSE values that satisfy the inessential game property. Section 5 has our conclusions.

2. Preliminaries

We recall briefly some concepts of linear algebra. Given a real vector space V its neutral element will be denoted by $\mathbf{0}_V$ and its dimension by $dim(V)$. If the vector space V is the direct sum of the vector spaces V^1 and V^2, that is, $V = V^1 + V^2$ and $V^1 \cap V^2 = \mathbf{0}_V$, we will write $V = V^1 \bigoplus V^2$. If $g : V \to W$ is a linear transformation from the vector space V to W, then we will denote by $null(g)$ its null space and by $g(V)$ its range. Let X be a subset of V, $span(X)$ will denote the smallest vector subspace containing X.

Let N be a fixed finite set, with $|N| = n$, a set of n players. A cooperative game with transferable utility or simply a *TU game* on N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The set of TU games v on N, denoted by G^N, forms a linear space such that $dim(G^N) = 2^n - 1$. For each coalition $S \subseteq N$, $v(S)$ describes the worth of the coalition S when its members cooperate. For any two TU games v and w in G^N and any $\alpha \in \mathbb{R}$, the TU game $\alpha v + w$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. Given a payoff vector $z \in \mathbb{R}^n$ and $S \subseteq N$ we denote $z(S)$ by $z(S) = \sum_{i \in S} z_i$.

Shapley [15] proposes an algebraic basis for G^N composed by the collection of unanimity TU games $\{u_S\}_{S \in 2^N \setminus \emptyset}$, where for all $T \subseteq N$

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

A TU game $v \in G^N$ is called *inessential* if for each $\emptyset \neq S \subseteq N$, $v(S) = \sum_{i \in S} v(\{i\})$. The set of inessential TU games, denoted by A, is a subspace of G^N with $dim(A) = n$. An algebraic basis for A is the collection of unanimity TU games $\{u_{\{i\}}\}_{i \in N}$.

By a *value* in G^N we mean a function $\Phi : G^N \to \mathbb{R}^n$. Let $v \in G^N$ be a TU game, and let $i \in N$ be a player, then $\Phi_i(v)$ is called the payoff of player i in v according to Φ.

We consider the following properties for a value Φ in G^N:

- Efficiency: $\sum_{i \in N} \Phi_i(v) = v(N)$, for all $v \in G^N$.
- Linearity: $\Phi(\alpha v + w) = \alpha \Phi(v) + \Phi(w)$, for all $\alpha \in \mathbb{R}$ and all pairs of TU games $v, w \in G^N$.
- Symmetry: $\Phi(\theta^* v) = \theta \cdot \Phi(v)$, for all permutations θ of N. Recall that the TU game $\theta^* v$ is defined as $S \to v(\theta^{-1} S)$ and $\theta \cdot (y_1, \ldots, y_n) = (y_{\theta(1)}, \ldots, y_{\theta(n)})$, for all vectors $(y_1, \ldots, y_n) \in \mathbb{R}^n$.

Ruiz et al. [14] provide a characterization of all linear, symmetric and efficient values (LSE). This family of values was rediscovered later by Hernández-Lamoneda et al. [11]. In the latter reference a $1-1$ correspondence is established between $(n-1)-$tuples $b = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$ and the LSE values for n players. The correspondence is also obtained in Driessen and Radzik [7].

The correspondence is given by $b \to \varphi^b$, where

$$\varphi_i^b(v) = \frac{v(N)}{n} + \sum_{S \ni i : S \neq N} \frac{b_{|S|}}{\binom{n-1}{|S|-1}} v(S) - \sum_{S \not\ni i} \frac{b_{|S|}}{\binom{n-1}{|S|}} v(S). \tag{1}$$

Expression (1) has the following interpretation: first, $v(N)$ is divided in an equal way among the players, that is, each player receives $\frac{v(N)}{n}$. Then, we need to make compensations between agents according to their different participation in the game. This compensation is made via transfers among the players such that the amount $v(N)$ is preserved. These transfers are made following a simple rule: for each coalition $S \neq N$, the players in S receive from the players in $N \setminus S$, an equal division of $\frac{n b_{|S|} v(S)}{\binom{n}{|S|}}$. Similarly, each player in $N \setminus S$, pays out an equal division of the previous amount. Thus, the final payment that each agent receives is given by (1).

Example 1. We consider a TU game $v \in G^{\{1,2,3\}}$ and a LSE value φ^b, with $b = (1, 1)$, as follows:

$$\begin{aligned}
\varphi_1^b(v) &= v(\{1\}) - v(\{2,3\}) + \frac{1}{2}[v(\{1,2\}) + v(\{1,3\})] \\
&\quad - \frac{1}{2}[v(\{2\}) + v(\{3\})] + \frac{1}{3} v(\{1,2,3\}), \\
\varphi_2^b(v) &= v(\{2\}) - v(\{1,3\}) + \frac{1}{2}[v(\{1,2\}) + v(\{2,3\})] \\
&\quad - \frac{1}{2}[v(\{1\}) + v(\{3\})] + \frac{1}{3} v(\{1,2,3\}), \\
\varphi_3^b(v) &= v(\{3\}) - v(\{1,2\}) + \frac{1}{2}[v(\{1,3\}) + v(\{2,3\})] \\
&\quad - \frac{1}{2}[v(\{1\}) + v(\{2\})] + \frac{1}{3} v(\{1,2,3\}).
\end{aligned}$$

If we calculate the value for the following TU games $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 2$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 3$,

and

$$w(S) = \begin{cases} 1 & \text{if } 3 \in S \text{ and } S \neq \{1,2,3\}, \\ 3 & \text{if } S = \{1,2,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain that $\varphi^b(v) = \varphi^b(w) = (0, 0, 3)$.

The above example shows two TU games such that for these TU games the value φ^b equals $(0, 0, 3)$. In the next section we aim to find all games $v \in G^N$ such that $\varphi^b(v) = x$, where x is a vector of payments previously given and $b_l \neq 0$ for all $l = 1, \ldots, n-1$. We will say that φ^b is a *regular value* if $b_l \neq 0$ for all $l = 1, \ldots, n-1$. In addition, we will write LSER value instead of linear, symmetric, efficient and regular value.

3. The null space of LSER value

In this section, we find an algebraic basis of the null space of each LSER value φ^b. To do this, for each $b \in \mathbb{R}^{n-1}$, with $b_l \neq 0$ for all $l = 1, \ldots, n-1$, we define the following set of games $\mathcal{W}^b$, $\mathcal{W}^b_* \subseteq G^N$ as follows:

$\mathcal{W}^b = \{w^b_S : S \subseteq N, 2 \leq |S|\}$, with

$$w^b_N(T) = \begin{cases} 1 & \text{if } |T| = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and for all $S \subseteq N$ such that $2 \leq |S| \leq n-1$,

$$w^b_S(T) = \begin{cases} 1 & \text{if } |T| = 1 \text{ and } T \not\subseteq S, \\ \dfrac{b_1}{b_{|S|}} \dbinom{n-2}{|S|-1} & \text{if } T = S, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

And $\mathcal{W}^b_* = \{w^b_{\{i\}} : i \in N\} \subseteq G^N$, where for all $i \in N$

$$w^b_{\{i\}}(T) = \begin{cases} \dfrac{1}{b_{|T|}} & \text{if } i \in T \text{ and } T \neq N, \\ n & \text{if } T = N, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Now, we are ready to state the main theorem:

Theorem 1. *Let $\varphi^b : G^N \to \mathbb{R}^n$ be a LSER value and $x \in \mathbb{R}^n$ be a payoff vector. Then, $\varphi^b(v) = x$ if and only if*

$$v = \sum_{i \in N} \frac{x_i}{n} w^b_{\{i\}} + \sum_{\substack{S \subseteq N \\ |S| \geq 2}} a_S w^b_S, \tag{5}$$

where the a_S are arbitrary constants.

In the proof of this theorem we need some results.

Given $x \in \mathbb{R}^n$ we define the TU game x^t, for each $t = 1, \ldots, n$, as follows

$$x^t(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = t, \\ 0 & \text{otherwise,} \end{cases}$$

for each $S \subseteq N$.

Proposition 2. *Let $\varphi^b : G^N \to \mathbb{R}^n$ be a LSE value. If $z \in \mathbb{R}^n$ is such that $z(N) = 0$, then $\varphi^b(z^t) = \frac{n}{n-1} b_t z$ for all $t = 1, \ldots, n-1$.*

Proof. Let $z \in \mathbb{R}^n$ with $z(N) = 0$, $i \in N$ and $t \in \{1, \ldots, n-1\}$,

$$\begin{aligned} \varphi^b_i(z^t) &= \frac{z^t(N)}{n} + \sum_{S \ni i} \frac{b_{|S|}}{\binom{n-1}{|S|-1}} z^t(S) - \sum_{S \not\ni i} \frac{b_{|S|}}{\binom{n-1}{|S|}} z^t(S) \\ &= \frac{b_t z_i}{\binom{n-1}{t-1}} \times \binom{n-1}{t-1} + \sum_{k \neq i} z_k \left[\frac{b_t \binom{n-2}{t-2}}{\binom{n-1}{t-1}} - \frac{b_t \binom{n-2}{t-1}}{\binom{n-1}{t}} \right] \\ &= b_t z_i + \sum_{k \neq i} z_k b_t \left[\frac{t-1}{n-1} - \frac{t}{n-1} \right] \\ &= \frac{n}{n-1} b_t z_i + \sum_{k \in N} z_k b_t \frac{-1}{n-1} \\ &= \frac{n}{n-1} b_t z_i. \end{aligned}$$

As $i \in N$ and $t \in \{1, \ldots, n-1\}$ are arbitrary, then $\varphi^b(z^t) = \frac{b_t n}{n-1} z$ for all $t = 1, \ldots, n-1$. □

In the next proposition we characterize the LSE value, φ^b, whose null space has dimension $2^n - n - 1$.

Proposition 3. *Let $\varphi^b : G^N \to \mathbb{R}^n$ be a LSE value such that $b \neq \mathbf{0}_{\mathbb{R}^n}$. Then $dim(null(\varphi^b)) = 2^n - n - 1$.*

Proof. Suppose that $\varphi^b : G^N \to \mathbb{R}^n$ is such that $b \neq \mathbf{0}_{\mathbb{R}^n}$. Let $x \in \mathbb{R}^n$ and $l \in \{1, \ldots, n-1\}$ such that $b_l \neq 0$. We consider the game $\hat{x} = \frac{x(N)}{n} \sum_{t=1}^n t c_t + \lambda z^l$, where $\lambda = \frac{n-1}{n b_l}$, $z_i = x_i - \frac{x(N)}{n}$ and c_t is defined as $c_t(S) = 1$ if $|S| = t$ and $c_t(S) = 0$ if $|S| \neq t$. Given that the games c_t are symmetric, then by Proposition 2 and that φ^b is linear, symmetric and efficient we have

$$\varphi^b(\hat{x}) = \frac{x(N)}{n} \iota + \frac{n\lambda}{n-1} b_l z = x,$$

where $\iota = (1, \ldots, 1) \in \mathbb{R}^n$. Hence, $x \in \mathbb{R}^n$ is arbitrary, then φ^b is surjective and therefore, $dim(null(\varphi^b)) = 2^n - n - 1$. □

In the next proposition we show a generator set of the null space of each LSER value φ^b.

Proposition 4. *For each LSER value $\varphi^b : G^N \to \mathbb{R}^n$, it holds that $\mathcal{W}^b$ is an algebraic basis for $null(\varphi^b)$.*

Proof. Suppose that $\sum_{S \subseteq N : |S| \geq 2} \delta_S w^b_S = \mathbf{0}_{G^N}$ and let $T \subseteq N$ such that $2 \leq |T| \leq n-1$. If we evaluate $\sum_{S \subseteq N : |S| \geq 2} \delta_S w^b_S$ in T, then we obtain that $\sum_{S \subseteq N : |S| \geq 2} \delta_S w^b_S(T) = \delta_T$. Combining the previous equality and that $\sum_{S \subseteq N : |S| \geq 2} \delta_S w^b_S$ evaluated in any coalition is zero we have $\delta_S = 0$ for all $S \subseteq N$ with $2 \leq |S| \leq n-1$. Given that

$$0 = \left(\sum_{S \subseteq N : |S| \geq 2} \delta_S w^b_S \right)(\{j\}) = \sum_{\substack{S \subseteq N \setminus \{j\} \\ 2 \leq |S| \leq n-1}} \delta_S + \delta_N, \quad \text{with } j \in N,$$

then $\delta_N = 0$. It follows that $\mathcal{W}^b$ is linearly independent, hence, $dim(span(\mathcal{W}^b)) = 2^n - n - 1$.

Now, we prove that $span(\mathcal{W}^b) \subseteq null(\varphi^b)$. To do this, let $w^b_S \in \mathcal{W}^b$ and $i \in N$. If $i \in S$, then

$$\varphi^b_i(w^b_S) = \frac{b_{|S|}}{\binom{n-1}{|S|-1}} \frac{b_1 \binom{n-2}{|S|-1}}{b_{|S|}} - \frac{(n - |S|) b_1}{\binom{n-1}{1}} = 0.$$

If $i \notin S$, then

$$\varphi^b_i(w^b_S) = b_1 - \frac{(n - |S| - 1) b_1}{n-1} - \frac{b_1}{b_{|S|}} \times \frac{|S| b_{|S|}}{n-1} = 0.$$

This implies that $w^b_S \in null(\varphi^b)$. As w^b_S was chosen arbitrarily, then $\mathcal{W}^b \subseteq null(\varphi^b)$. By Proposition 3 the $dim(null(\varphi^b)) = 2^n - n - 1$. Combining the previous equality with $dim(span(\mathcal{W}^b)) = 2^n - n - 1$ and that $\mathcal{W}^b \subseteq null(\varphi^b)$, we have $null(\varphi^b) = span(\mathcal{W}^b)$. □

Lemma 5. *Let $Y \subseteq G^N$ such that $dim(Y) = n$ and let $\Phi : G^N \to \mathbb{R}^n$ be a linear value in G^N. Then, $G^N = Y \bigoplus null(\Phi)$ if and only if $\Phi(Y) = \mathbb{R}^n$.*

Proof. We assume that $G^N = Y \bigoplus null(\Phi)$. Given that $dim(Y) = n$, then $dim(null(\Phi)) = 2^n - n - 1$. Hence, Φ is surjective. This implies that $\Phi(Y) = \mathbb{R}^n$.

Reciprocally, if $\Phi(Y) = \mathbb{R}^n$, then Φ is surjective. Therefore, $dim(null(\Phi)) = 2^n - n - 1$. As $dim(Y) = n$, then Φ is an isomorphism if it is restricted to vector space Y. Let $v \in Y \cap null(\Phi)$, then $\Phi(v) = \mathbf{0}_{\mathbb{R}^n}$. Given that Φ is an isomorphism if it is restricted to Y and $v \in Y$ we have $v = \mathbf{0}_{G^N}$. This implies that $Y \cap null(\Phi) = \{\mathbf{0}_{G^N}\}$. The previous equality and that $dim(null(\Phi)) + dim(Y) = 2^n - 1$ implies that $G^N = Y \bigoplus null(\Phi)$. □

Proposition 6. *For each LSER value $\varphi^b : G^N \to \mathbb{R}^n$, it holds that $G^N = span(\mathcal{W}^b_*) \bigoplus span(\mathcal{W}^b)$ and*

$$\varphi^b_i(w^b_{\{j\}}) = \begin{cases} n & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. Let $i, j \in N$. If $i = j$, then

$$\varphi^b_i(w^b_{\{i\}}) = 1 + \sum_{S \ni i : S \neq N} \frac{1}{\binom{n-1}{|S|-1}} = \sum_{l=1}^n \frac{1}{\binom{n-1}{l-1}} \binom{n-1}{l-1} = n.$$

If $i \neq j$, then

$$\begin{aligned} \varphi^b_i(w^b_{\{j\}}) &= \sum_{\substack{S \ni i \\ S \ni j}} \frac{1}{\binom{n-1}{|S|-1}} - \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \ni j}} \frac{1}{\binom{n-1}{|S|}} \\ &= \sum_{\substack{S \ni i \\ S \ni j}} \frac{1}{\binom{n-1}{|S|-1}} - \sum_{\substack{S \ni i \\ S \ni j}} \frac{1}{\binom{n-1}{|S|-1}} \\ &= 0. \end{aligned}$$

It is clear that the set of games $\mathcal{W}^b_*$ is linearly independent, and therefore, $dim(span(\mathcal{W}^b_*)) = n$. Let $x \in \mathbb{R}^n$ and we define $v^x \in G^N$ as $v^x = \sum_{i\in N} \frac{x_i}{n} w^b_{\{i\}}$. From the linearity of φ^b and (6) we obtain that $\varphi^b(v^x) = x$. This equality plus that $x \in \mathbb{R}^n$ was chosen arbitrarily implies that $\varphi^b(\mathcal{W}^b_*) = \mathbb{R}^n$. By Lemma 5 we obtain that $G^N = span(\mathcal{W}^b_*) \bigoplus null(\varphi^b)$. By Proposition 3 we have $null(\varphi^b) = span(\mathcal{W}^b)$. The above equality implies that

$$G^N = span(\mathcal{W}^b_*) \bigoplus span(\mathcal{W}^b). \quad \square$$

Proof of Theorem 1. We consider $v \in G^N$. By Proposition 6 there exist $v_1 \in span(\mathcal{W}^b_*)$ and $v_2 \in span(\mathcal{W}^b) = null(\varphi^b)$ such that v is uniquely expressed as $v = v_1 + v_2$. From the linearity of φ^b we have $\varphi^b(v) = \varphi^b(v_1) + \varphi^b(v_2) = \varphi^b(v_1)$. Given that, there exist unique constants $a_{\{i\}}$ such that $v_1 = \sum_{i\in N} a_{\{i\}} w^b_{\{i\}}$, then by Proposition 6 we conclude that

$$\varphi^b(v) = \sum_{i\in N} a_{\{i\}} \varphi^b(w^b_{\{i\}}) = (na_{\{1\}}, \ldots, na_{\{n\}}). \tag{7}$$

From (7) it follows $\varphi^b(v) = x$ if and only if $v = \sum_{i\in N} \frac{x_i}{n} w^b_{\{i\}} + \sum_{S\subseteq N:|S|\geq 2} a_S w^b_S$, where the a_S are arbitrary constants. $\square$

Example 2. To illustrate the results, we return to Example 1 to solve the inverse problem of value φ^b with $b = (1, 1)$. We consider the payoff vector $x = (0, 0, 3)$. We want to find all games for which the value φ^b gives x as a result. Computing the algebraic basis of the null space formed by the games $w^b_S \in \mathcal{W}^b$, with $S \subseteq N$, $2 \leq |S| \leq n$, and using formula (5), we obtain the solution of the inverse problem

$$\begin{aligned}
v(\{1\}) &= a_{\{2,3\}} + a_{\{1,2,3\}},\\
v(\{2\}) &= a_{\{1,3\}} + a_{\{1,2,3\}},\\
v(\{3\}) &= 1 + a_{\{1,2\}} + a_{\{1,2,3\}},\\
v(\{1,2\}) &= a_{\{1,2\}},\\
v(\{1,3\}) &= 1 + a_{\{1,3\}},\\
v(\{2,3\}) &= 1 + a_{\{2,3\}},
\end{aligned}$$

and $v(\{1, 2, 3\}) = 3$; the set depends on four parameters, and in general on $2^n - n - 1$ parameters. It is easy to verify that the games from Example 1 are obtained for the following value of the parameters:

- $a_{\{1,2\}} = 3, a_{\{2,3\}} = a_{\{1,3\}} = 2$ and $a_{\{1,2,3\}} = -2$,
- $a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}} = a_{\{1,2,3\}} = 0$.

4. Inessential game property

Now, assume that we have a game $v \in G^N$ such that $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$, that is, the worth of any coalition is equal to the sum that each member of the coalition can get by himself. It is reasonable to assume that the payment of any player in the game must be precisely what he can get by himself. This property is known in the literature as inessential game property. In this section we characterize all LSE values, φ^b, that satisfy this property.

Inessential game property.
$\Phi(v) = (v(\{1\}), \ldots, v(\{n\}))$, for all $v \in A$.

Proposition 7. *Let $\varphi^b : G^N \to \mathbb{R}^n$ be a LSE value. φ^b satisfies the inessential game property if and only if $\sum_{l=1}^{n-1} b_l = \frac{n-1}{n}$.*

Proof. We suppose that $\varphi^b : G^N \to \mathbb{R}^n$ is such that $\sum_{l=1}^{n-1} b_l = \frac{n-1}{n}$. Let $w \in A$ and $x \in \mathbb{R}^n$ such that $w = \sum_{i\in N} x_i u_{\{i\}}$. It is easy to verify that $w = \hat{x}$, where $\hat{x} = \frac{x(N)}{n} \sum_{t=1}^{n} tc_t + \sum_{t=1}^{n-1} z^t$ with z and c_t as in the proof of Proposition 3. Given that the games c_t are symmetric, then by Proposition 2 and that φ^b is a LSE value we obtain that

$$\varphi^b(w) = \varphi^b(\hat{x}) = \frac{x(N)}{n}\iota + \frac{n}{n-1}\sum_{l=1}^{n-1} b_l z, \tag{8}$$

where $\iota = (1, \ldots, 1) \in \mathbb{R}^n$. From (8) we obtain the results. $\square$

5. Conclusions

In this paper we provide an algebraic basis of the null space of the LSE value φ^b, for all $b \in \mathbb{R}^{n-1}$ with $b_l \neq 0$ for all $l = 1, \ldots, n-1$. This allows us to decompose the space of games, G^N, as a direct sum of the null space of φ^b and its respective complement. Using this decomposition we obtain the solution of the inverse problem of φ^b.

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A SIMPLE FAMILY OF SOLUTIONS FOR FOREST GAMES

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(Communicated by Ignacio Garcia-Jurado)

Abstract. In this paper we study TU-games where the cooperation structure among the players is modeled by a forest. Using the classical component efficiency axiom and a generalized version of the component fairness axiom we obtain a family of solutions. We show that every solution in this family is based on a process of transfers among the players, and the average tree solution belongs to the family. Finally, we obtain a solution based on the degree of the nodes and we study a set of properties satisfied by this family.

1. **Introduction.** The transferable utility cooperative games (TU-games) are an effective tool to solve allocation problems among several players when they are willing to cooperate. Since the seminal work of Lloyd S. Shapley [10], the concepts of solutions, stability and properties of TU-games have been a topic of research with extensive developing theories and applied contributions. An interesting situation in the study of TU-games is to consider that a cooperation structure exists among the players, because of their personal interests or conflicts, before the game is played. Myerson [9] studied the situation when the cooperation structure is given by a graph. Under this model, the set of nodes represent the set of players and the edges represent direct cooperation links between players. We assume that the graph is a forest, i.e., each component of the graph is a tree. A forest game is a pair consisting of a TU-game and a forest on the agent set.

Herings et al. [6] introduce the Average Tree solution for the class of forest games, which can be characterized by two axioms: component efficiency and component fairness. Component efficiency axiom asserts that the members of a component ought to allocate to themselves the total worth available to them. When a link is

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severed in a component of a forest, this component breaks up into two new components. Component fairness axiom asserts that deleting a link in a component yields for both resulting two components the same average change in payoff, where the average is taken over the players in each component. Combining component efficiency with component fairness axioms implies that the members of each component receive the sum of two parts: the worth of their component and a share of the surplus generated by the deletion of the link. This share is precisely the relative size of this component. Later, Béal et al. [4] proposed a generalization of the axiom of Herings et al. [6]. Béal et al. [4] used certain constants for each possible component which can be formed. Béal et al. [4] characterized a family of solutions for cycle-free graph games. In our work, we propose a modification to the generalized axiom of Béal et al. [4] and we obtain, in the same way, a family of solutions with a simple mathematical formulation with an interpretation based on transfers. Given specific values for those constants, we obtain the average tree solution of Herings et al. [6] and the family of solutions given by Béal et al. [4]. It should be noted that we provide simple expressions for both solutions and we easily interpret them using transfers between players.

This article is organized as follows: In Section 2 we present the definitions and the notation that we use in the rest of the document and we show the Average Tree solution. In Section 3 we present a generalized version of the component fairness axiom and we obtain a family of solutions for cycle-graph games.. We also present the relation of our family of solutions with the family of solutions proposed by Béal et al. [4]. In Section 4 we obtain a new formulation for the Average Tree solution and we present a solution based on the degree of the nodes. Finally, in Section 5 we study a set of properties satisfied by this family of solutions we are proposing.

2. **Notation and preliminaries.**

2.1. **TU-games.** Let $N := \{1, \ldots, n\}$ be a finite set of agents. A cooperative game with transferable utility (TU-game) is a pair (N, v) where $v : 2^N \to \mathbb{R}$ is a characteristic function defined on the power set of N satisfying $v(\emptyset) = 0$. An element $i \in N$ is called a *player* and every nonempty subset S of N, a *coalition*. The real number $v(S)$ is called the *worth* of the coalition S, and it is interpreted as the total payoff that the coalition S can obtain because of its members. When there is no ambiguity with respect to the set of players N, we identify the TU-game (N, v) with its characteristic function v. Let G^N be the set of all TU-games with the set of players N. For every $v \in G^N$ and $S \subseteq N$, the *unanimity game* u_S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ in other case. Now, we define, for every $S \subseteq N$, the TU-game (S, v_S), where $v_S : 2^S \to \mathbb{R}$ is defined as $v_S(T) = v(T)$ for all $T \subseteq S$. If $v \in G^N$ satisfies the condition $v(S \cup T) \geq v(S) + v(T)$ when $S \cap T = \emptyset$, we shall say v is a *superadditive game*. A *payoff vector* $x \in \mathbb{R}^n$ for $v \in G^N$ is a vector giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A *solution* on G^N is a mapping ϕ that assigns to each TU-game $v \in G^N$ a payoff vector $\phi(v) \in \mathbb{R}^n$.

2.2. **Graphs.** Let $N := \{1, \ldots, n\}$ be a set of nodes. A graph of N is defined as a set of unordered pairs of different elements of N. We refer to each of these unordered pairs as an *edge*. We denote the complete graph of N as $g^N := \{\{i, j\} \mid i, j, \in N, i \neq j\}$ and the set of all possible graphs with nodes in N by $GR^N := \{g \mid g \subseteq g^N\}$. For $g \in GR^N$, $n(g)$ represents the set of nodes on the edges of g and $|n(g)|$ the cardinality of $n(g)$. For $S \subseteq N$, we define by $g(S) := \{\{i, j\} \in g \mid \{i, j\} \subseteq S\}$.

Given a graph $g \in GR^N$, a *path* in g is a sequence of nodes $i_1, \ldots, i_k$ such that $\{\{i_j, i_{j+1}\}\}_{j=1}^{k-1} \subseteq g$. A path is called a *cycle* if all the nodes of the path are different, except $i_1 = i_k$. Two nodes i, j are *connected* in g if $i = j$ or there is a path in g where $i_1 = i, i_k = j$. A node is *isolated* if it is not connected with another node except itself. The set of isolated nodes on g is denoted as $I(g)$. Thus, we say that g is *connected* if any two nodes in $n(g)$ are connected. An isolated node is considered a connected one (just with itself). A graph $g' \in GR$ is called a *subgraph* of g if $g' \subseteq g$. A subgraph g' of g is known as a *component* of g if g' is connected and each pair of nodes $i \in n(g'), j \in n(g) \backslash n(g')$ are not connected. The set of components of g is denoted by $C(g)$, and C_i denotes the component in $C(g)$ that contained the node $i \in n(g)$. Furthermore, the set of nodes in the same subgraph that i will be represented as $N_i := n(C_i)$. The set of all connected subgraphs in g is denoted by $\widehat{C}(g)$.

A graph $g \in GR^N$ is a *tree* if it is connected and without cycles. We denote the set of all trees in GR^N as T^N. It should be noted that $T^N \subset GR^N$. A graph $g \in GR^N$ such that $g' \in C(g)$ is a tree, it is known as a *forest* or a *cycle-free graph*. The set of all cycle-free graphs will be denoted as F^N. An *orientation* of a graph $g \in GR^N$ is a directed graph d obtained from g by replacing each link $\{i, j\}$ by a directed link: either (i, j) or (j, i).

2.3. **Forest games.** A *graph game* on N is a pair (v, g) such that $v \in G^N$ and $g \in GR^N$ representing a cooperation structure among the players. In this case, each edge on g represents a relation of bidirectional communication between two players. In a graph game, only the connected players are able to cooperate. In what follows, we only focus on the set of all forest games, i.e. the set of all pairs (v, g) where $v \in G^N$ and $g \in F^N$. We denote as FG^N the subset of graph games where the graph is a forest, and we call it *forest game*. A solution on FG^N is a map φ that assigns to each $(v, g) \in FG^N$ a payoff vector $\varphi(v, g) \in \mathbb{R}^n$.

On the class of forest games, FG^N, Herings et al. [6] introduce a new allocation rule called the *Average Tree solution*, that can be characterized by component efficiency and component fairness axioms.

Axiom 1. (Component efficiency, Myerson [9]) A solution $\varphi : FG^N \to \mathbb{R}^n$ is *efficient by components*, if

$$\sum_{i \in n(C)} \varphi_i(v, g) = v(n(C)), \qquad \forall C \in C(g). \tag{1}$$

for every $g \in F^N$.

This axiom requires that, for each component $C \in C(g)$, the solution allocates the amount $v(n(C))$ among the players in $n(C)$.

Let $C \in C(g)$ be a component of a graph. Removing the edge $\{i, j\} \in C$ causes the split of the component C into two trees. For an edge $\{i, j\} \in g$, we denote by $C_{ij} := \{C \in C(g \backslash \{i, j\}) \mid i \in n(C)\}$ the component in $g \backslash \{i, j\}$ containing the node i after removing the link $\{i, j\}$ and by $N_{ij} := n(C_{ij})$ the set of nodes in the same component as the node i after the removal of the edge $\{i, j\}$.

Axiom 2. (Component fairness, Herings et al. [6]) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies the *component fairness* axiom if

$$\frac{1}{|N_{ij}|}\sum_{h\in N_{ij}}[\varphi_h(v,g)-\varphi_h(v,g')]=\frac{1}{|N_{ji}|}\sum_{h\in N_{ji}}[\varphi_h(v,g)-\varphi_h(v,g')], \qquad (2)$$

for all $g \in F^N$ and for every $\{i,j\} \in g$, where $g' = g\backslash\{i,j\}$.

According to this axiom, if we remove the edge $\{i,j\}$ of g, the average loss (or benefit) in the payoffs of the players in N_{ij} and N_{ji} must be equal.

The Average Tree solution can be expressed as the average of marginal contribution vectors, which require some definitions to be described. We follow Herings et al. [6]. By a rooted tree t_r on the subgraph $g(n(C))$, we mean an orientation that arises from a component C of a forest g by selecting an agent $r \in n(C)$, called the *root*, and directing all links of $g(n(C))$ away from r. For a given tree $g(n(C))$, each agent $r \in C$ is the root of exactly one rooted tree t_r in $g(n(C))$. Notice that for any rooted tree t_r on C, any agent $k \in n(C) \setminus \{r\}$, there is exactly one directed link (j,k); agent j is the unique *predecessor* of k and k is a *successor* of j in t_r. Denote by $s_r(j)$ the possibly empty set of successors of agent j in t_r. An agent k is a *subordinate* of j in t_r if there is a directed path from j to k. The set $S_r(j)$ denotes the union of the set of all subordinates of j in t_r and $\{j\}$.

Let us consider a cycle-free graph game $(v,g) \in FG^N$, a component $C \in C(g)$, and a root node $r \in n(C)$. Consider the marginal contribution vector $m^r(v,g)$ on $\mathbb{R}^{|n(C)|}$ defined as follows:

$$m_i^r(v,g)=v(S_r(i))-\sum_{j\in s_r(i)}v(S_r(j)), \qquad \forall i\in n(C). \qquad (3)$$

The marginal contribution $m_i^r(v,g)$ of $i \in n(C)$ in t_r is thus equal to the worth of the coalition consisting of agent i and all his subordinates in t_r minus the sum of the worths of the coalitions consisting of any successor of i and all subordinates of this successor in t_r.

The Average Tree solution is the allocation rule AT which assigns to each $(v,g) \in FG^N$ the payoff vector in which agent i in a component $C \in C(g)$ gets the average over $r \in n(C)$ of $m_i^r(v,g)$:

$$AT_i(v,g)=\frac{1}{|n(C)|}\sum_{r\in n(C)}m_i^r(v,g), \qquad \forall C\in C(g),\forall i\in n(C). \qquad (4)$$

For further developments on the Average Tree solution, see Herings et al. [7] and Baron et al. [2].

3. A family of solutions.

In this section we present a generalization of the component fairness axiom. With this axiom plus component efficiency, we will obtain an expression for a family of solutions for forest games. We represent the set of the real number strictly greater than zero as $\mathbb{R}_{++}$.

Axiom 3. (Generalized component fairness) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies the *generalized component fairness* axiom if

$$\frac{1}{\alpha_{ij}}\sum_{h\in N_{ij}}[\varphi_h(v,g)-\varphi_h(v,g')]=\frac{1}{\alpha_{ji}}\sum_{h\in N_{ji}}[\varphi_h(v,g)-\varphi_h(v,g')], \qquad (5)$$

for all $g \in F^N$, for every $\{i,j\} \in g$ where $g' = g\backslash\{i,j\}$ and $\alpha_{ij}, \alpha_{ji} \in \mathbb{R}_{++}$ for all $\{i,j\} \in g$.

When an edge $\{i,j\}$ is removed, two new trees arise. The constant α_{ij} is associated with the tree containing the node i after removing that edge and similarly for α_{ji}. These constants indicate how the change of the payoff for the players in each component is weighted due to the removal of the edge $\{i,j\}$.

In the same direction of Béal et al. [4] we can relate the generalized component fairness axiom to a solution for a bargaining problem, where the components C_{ij} and C_{ji} have to bargain about the consequences of creating the edge $\{i,j\}$. So, if we consider solutions satisfying the component efficiency axiom, a very natural option for being the disagreement point is $(v(N_{ij}), v(N_{ji}))$. Also, we assume that the TU-game is superadditive, that is, $v(N_i) \geq v(N_{ij}) + v(N_{ji})$ for every edge $\{i,j\} \in g$. Thus, the set of feasible agreements is given by $\{(x,y) \in \mathbb{R}^2 \mid x+y \leq v(N_i)\}$. If the bargaining power of the coalitions N_{ij} and N_{ji} during the negotiation is given by $\alpha_{ij}/(\alpha_{ij}+\alpha_{ji})$ and $\alpha_{ji}/(\alpha_{ij}+\alpha_{ji})$ respectively, the Asymmetric Nash bargaining solution, Binmore et al. [5], for this bargaining problem is precisely given by our generalized component fairness axiom.

Theorem 3.1. *Given $(v,g) \in FG^N$ and $\alpha_{ij}, \alpha_{ji} \in \mathbb{R}_{++}$ for every $\{i,j\} \in g$, there is a unique solution $\varphi(v,g)$ that satisfies the component efficiency and the generalized component fairness axioms.*

Proof. Let (v,g) be a forest game and $\varphi : FG^N \to \mathbb{R}^n$ be a solution which satisfies the component efficiency and the generalized component fairness axioms. Let $\{i,j\}$ be an edge on g. So,

$$\alpha_{ji}\left[\sum_{h\in N_{ij}} \varphi_h(v,g) - v(N_{ij})\right] = \alpha_{ij}\left[\sum_{h\in N_{ji}} \varphi_h(v,g) - v(N_{ji})\right].$$

Notice that for every $C \in C(g)$, we have

$$\sum_{i\in n(C)} \sum_{\{i,j\}\in g} \frac{\alpha_{ji}v(N_{ij}) - \alpha_{ij}v(N_{ji})}{\alpha_{ij}+\alpha_{ji}} = 0.$$

Now, we consider that for every edge $\{i,j\} \in g$

$$\sum_{h\in N_{ij}} \varphi_h(v,g) = \frac{\alpha_{ij}}{\alpha_{ij}+\alpha_{ji}} v(N_i) + \left[\frac{\alpha_{ji}v(N_{ij}) - \alpha_{ij}v(N_{ji})}{\alpha_{ij}+\alpha_{ji}}\right]. \tag{6}$$

The above expression indicates the sum of the solutions for all players on N_{ij}. Removing the other edges incident to the node i, $\{h,i\} \in g$, $h \neq j$, we obtain similar expressions for each one of the nodes on N_{hi}. Now, subtracting these expressions from (6), we have

$$\varphi_i(v,g) = \left(\frac{\alpha_{ij}}{\alpha_{ij}+\alpha_{ji}} - \sum_{\{i,h\}\in g, h\neq j} \frac{\alpha_{hi}}{\alpha_{ih}+\alpha_{hi}}\right) v(N_i) + \sum_{\{i,h\}\in g} \left[\frac{\alpha_{hi}v(N_{ih}) - \alpha_{ih}v(N_{hi})}{\alpha_{ih}+\alpha_{hi}}\right].$$

Again, using algebraical manipulation, the next condition follows,

$$\varphi_i(v,g) = \left(1 - \sum_{\{i,h\}\in g} \frac{\alpha_{hi}}{\alpha_{ih}+\alpha_{hi}}\right) v(N_i) + \sum_{\{i,h\}\in g} \left[\frac{\alpha_{hi}v(N_{ih}) - \alpha_{ih}v(N_{hi})}{\alpha_{hi}+\alpha_{ih}}\right]. \quad (7)$$

So, given a set of constants $\alpha_{ij}, \alpha_{ji} \in \mathbb{R}_{++}$ for each $\{i,j\} \in g$, the previous solution is determined uniquely. Then, the proof ends. □

Béal et al. [4] consider a weak version of generalized component fairness axiom, called the weighted component axiom, with the restriction $\beta_{ij}+\beta_{ji} = 1$ for $\beta_{ij}, \beta_{ji} \in \mathbb{R}_{++}$ for each $\{i,j\} \in g$. They obtained a family of solutions with the following expression:

$$\varphi_i^{\beta}(v,g) = v(N_i) - \sum_{\{i,j\}\in g} v(N_{ij}) - \sum_{\{i,j\}\in g} \beta_{ij}[v(N_i) - v(N_{ij}) - v(N_{ji})], \quad (8)$$

for all $C \in C(g)$, and for all $i \in n(C)$.

Taking into account expressions (7) and (8), any specification of the generalized component fairness axiom given by a pair $\alpha_{ij}, \alpha_{ji} \in \mathbb{R}_{++}$ for each $\{i,j\} \in g$ induces the system of weights β as considered in Béal et al. [4] such that $\beta_{ji} = \frac{\alpha_{ji}}{\alpha_{ji}+\alpha_{ij}}$ and $\beta_{ij} = \frac{\alpha_{ij}}{\alpha_{ij}+\alpha_{ji}}$. These expressions explain why an instance of weighted component fairness with some null weight cannot be obtained by a specification of generalized component fairness. As a consequence Theorem 3.1 provides a slightly more general form the expression for a family of solutions.

For every $g \in F^N$ we denote as $\mathfrak{F}(g)$ the set of solutions that satisfy the efficiency component and generalized component fairness axioms, with a set of constants $\alpha_{ij}, \alpha_{ji} \in \mathbb{R}_{++}$ for each $\{i,j\} \in g$.

Expression (7) indicates how the payoff allocation process is done according to a solution for forest games that satisfies the component efficiency and the generalized component fairness axioms: First, the worth of the component containing the node i is divided proportionally amongst the agents in that component according to the weights $1 - \sum_{\{i,h\}\in g} \frac{\alpha_{hi}}{\alpha_{ih}+\alpha_{hi}}$. After that, there are several transfers between the node i and every node h connected directly to the node i: player i receives $\alpha_{hi}v(N_{ih})/(\alpha_{hi}+\alpha_{ih})$ from player h, but Player h receives $\alpha_{ih}v(N_{hi})/(\alpha_{hi}+\alpha_{ih})$ from player i. After all these transfers, the final payoff for the player i in the forest game (v,g) is given by (7). So, we can obtain new solutions based on a natural way for calculating the weights α_{ij} and α_{ji} characterized uniquely via the component efficiency axiom and the component fairness axiom with these weights on Equation (5).

4. **Particular cases.**

Proposition 1. *For every $(v,g) \in FG^N$, the Average Tree solution belongs to $\mathfrak{F}(g)$. Moreover, this solution has the next formulation:*

$$\varphi_i(v,g) = \frac{v(N_i)}{|N_i|} + \sum_{\{i,j\}\in g} \left[\frac{|N_{ji}|v(N_{ij}) - |N_{ij}|v(N_{ji})}{|N_i|}\right], \quad (9)$$

for every $i \in N$.

Proof. For every $(v,g) \in FG^N$, it is clear that the Average Tree solution belongs to $\mathfrak{F}(g)$ taking $\alpha_{ij} = |N_{ij}|$ for each edge $\{i,j\} \in g$. Then, we apply the results of Theorem 3.1. We consider that for every edge $\{i,j\} \in g$

$$\sum_{h \in N_{ij}} \varphi_h(v,g) = \frac{|N_{ij}|v(N_i)}{|N_{ij}| + |N_{ji}|} + \frac{|N_{ji}|v(N_{ij}) - |N_{ij}|v(N_{ji})}{|N_{ij}| + |N_{ji}|}. \tag{10}$$

Because of the component efficiency axiom, we have

$$v(N_i) = \sum_{h \in N_{ij}} \varphi_h(v,g) + \sum_{h \in N_{ji}} \varphi_h(v,g).$$

Replacing the last expression in (10) and, applying algebraical manipulation, we obtain (4). □

Expression (9) provides an alternative formulation for the Average Tree solution given by (4). In our expression we observed how the payoffs are allocated among the players. First, the worth of each component is divided equally among the players in the component. Then, several transfers are made between each pair of connected nodes. For a player $i \in N$, and considering that $|N_i| \neq 1$, we eliminate the incident edges on i, $\{i,j\}$, one-by-one. The player j gives $|N_{ij}|/|N_i|$ of the worth of his resulting component, $v(N_{ji})$, to the player i. Also, the player i gives $|N_{ji}|/|N_i|$ of $v(N_{ij})$ to the player j. So, the payoff given in (9) is the result of this process.

By proposing different values for the constants α_{ij}, α_{ji} for each edge $\{i,j\} \in g$ in (7), we obtain alternative solutions.

Axiom 4. (e-component fairness, Béal et al. [3]) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies the *e-component fairness* axiom if, for any $g \in F^N$ and for any edge $\{i,j\} \in g$, the next expression holds:

$$\sum_{h \in N_{ij}} [\varphi_h(g) - \varphi_h(g')] = \sum_{h \in N_{ji}} [\varphi_h(g) - \varphi_h(g')],$$

where $g' = g \backslash \{i,j\}$.

This axiom establishes that the loss (or benefit) that the agents on N_{ij} and N_{ji} obtain due to removal of the edge $\{i,j\}$ must be equal for each one of these sets. The difference with the component fairness axiom is that this axiom is considering the average loss (or benefit) of the component after the removal of the link, and the equal treatment axiom takes into account the total loss (or benefit) of the component after the same removal.

Given a graph $g \in GR^N$ and for every node $i \in N$, we define the *degree* of i, $\deg_i(g) := |\{j \in N \mid \{i,j\} \in g\}|$, as the number of nodes in g connected with i.

Theorem 4.1. *There is a unique solution $\varphi : FG^N \to \mathbb{R}^n$ that satisfies component efficiency and equal treatment by component axioms. Moreover, this solution has the next formulation:*

$$\varphi_i(v,g) = \left(1 - \frac{1}{2}\deg_i(g)\right) v(N_i) + \frac{1}{2} \sum_{\{i,j\} \in g} [v(N_{ij}) - v(N_{ji})], \tag{11}$$

for every $i \in N$, $g \in F^N$.

Proof. It follows immediately if we take $\alpha_{ij} = \alpha_{ji} = c$ in expression (5). □

The solution given in the expression (11) is called as the *Average orientation solution* in Béal [3]. This solution provides a process to allocate payoffs to the players in cycle-free graph games. First, the worth of each component $C \in C(g)$ is divided among the nodes that formed it, where player $i \in n(C)$ receives the proportion $(1-\deg_i(g)/2)$ of $v(N_i)$. Then, a series of transfers between each pair of connected nodes are made. Player i pays $v(N_{ji})/2$ to the player j for each $\{i,j\} \in g$. Also, the player i also receives $v(N_{ij})/2$ from the player j. After all these transfers, the resulting payoff is indicated by (11). Notice, again, that we obtain a different expression for this value in comparison with the one obtained by Béa et al. [4].

After some algebraical manipulation, Solution (11) can be written as follows:

$$\varphi_i(v,g) = v(N_i) - \sum_{\{i,j\}\in g} \left[\frac{v(N_i) - v(N_{ij}) + v(N_{ji})}{2}\right], \qquad \forall i \in N.$$

So, if we assume that the TU-game is superadditive, the process for allocating payoffs according to (11) can be interpreted as follows: First, every agent i on a component $C \in C(g)$ receives $v(N_i)$. After that, for every edge $\{i,j\} \in C$, the components C_{ij} and C_{ji} solve a two-person contested garment problem with claims $v(N_{ij})$ and $v(N_{ji})$ respectively over the state $v(N_i)$, according to the talmudic recommendation. Then, player i discounts for his payoff the solution of this problem for the component C_{ij}. After all these discounts, the final payoff for player i is given by (11). For further information about the contested garment problem and the talmudic solution please refer to Aumann and Maschler [1].

With this last result, we finalize our study of the family of solutions $\mathfrak{F}(g)$ for cycle-free graph games, with $g \in \mathfrak{F}(g)$. The result given in (7) provides a perspective about how to generate new solutions for these kind of games: It is enough to provide a different and reasonable way about how the change in payoffs of components must be due to the disconnection of two nodes, and this fact must define the set of constants α_{ij}, α_{ji} for each $\{i,j\} \in g$ uniquely.

5. **Additional properties.** In this section we showed that every member of the family of solutions $\mathfrak{F}(g)$ satisfies others well-known properties inside of the cooperative game theory.

The set of TU-games, G^N, is a vector space with the sum and the scalar product defined in the usual way: for $v_1, v_2 \in G^N$ and $\lambda \in \mathbb{R}$, $(v_1+v_2)(S) := v_1(S) + v_2(S)$ and $(\lambda v_1)(S) := \lambda v_1(S)$ for each $S \subseteq N$.

Axiom 5. (Linearity in G^N) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies *the linearity in* G^N axiom if given $v_1, v_2 \in G^N$, $g \in F^N$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\varphi(\lambda_1 v_1 + \lambda_2 v_2, g) = \lambda_1 \varphi(v_1, g) + \lambda_2 \varphi(v_2, g).$$

Axiom 6. (Equal treatment of vital players, Jackson [8]) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies the *equal treatment of vital players* axiom if by each $S \subseteq N$ such that $S = n(C)$ for some $C \in C(g)$ or $S = \{j\}$ for some $j \in I(g)$

$$\varphi_i(u_S, g) = \begin{cases} 1/s, & \text{if } i \in S, \\ 0, & \text{in other case.} \end{cases}$$

Suppose that it is necessary the cooperation of all players in a component for obtaining an amount of one unit. Otherwise, they get nothing. The above axiom states that the payoffs for the players in the component must be such that the unit is divided equally among them.

Let $v \in G^N$ and $S \subseteq N$. For every $g \in GR^N$, we denote as $g_S := g \cap g^S$ the subgraph in g restricted to the nodes on S.

Axiom 7. (Decomposition in components, van den Nouweland [11]) A solution $\varphi : FG^N \to \mathbb{R}^n$ satisfies the *decomposition in components* axiom if for every $S \subseteq N$ such that $S = n(C)$ for some $C \in C(g)$, or $S = \{j\}$ for every $j \in I(g)$ the next condition holds

$$\varphi_i(v, g) = \varphi_i(v_S, g_S),$$

for every $i \in S$ and $g \in F^N$.

This property states that the solution for the players in a component does not depend on how the other players are organized on the other components.

Theorem 5.1. *Let $\varphi : FG^N \to \mathbb{R}^n$ be a solution in $\mathfrak{F}(g)$. Then, φ satisfies the (a) Linearity in G^N, (b) Equal treatment of vital players and (c) Decomposition in components axioms.*

Proof. (a) For proving that solution (7) is a linear solution in G^N, we need to calculate Solution (7) substituting the game $(\lambda_1 v_1 + \lambda_2 v_2)$ instead of the game $v \in G^N$. Then, applying the definition of the sum of two games, distributing and arranging terms, the result is obtained.

(b) Let g be a graph in T^N and let $S \subseteq N$ be a coalition such that $S = n(C)$ for some $C \in C(g)$ or $S = \{j\}$ for some $j \in I(g)$; notice that $u_S(N_{ij}) = u_S(N_{ji}) = 0$ for all $\{i, j\} \in g$ and then, the result follows.

(c) It is straightforward to see that solution (7), for a player $i \in N$, only takes into account the subgraphs on the same component that i. Thus, clearly, restricting the graph game on that component, the payoff for the players in that component must be the same.

□

One additional line of research is the following: In Herings et al. [6] a characterization of the Average Tree solution with the linearity Axiom is presented. Is it possible to characterize the family of solutions $\mathfrak{F}(g)$ with an (adjusted) characterization using this property? If it is not possible to obtain such characterization, it could also be interesting to study other appealing properties in this context.

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GAMES WITH NESTED CONSTRAINTS GIVEN BY A LEVEL STRUCTURE

Regresar

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Abstract. In this paper we propose new games that satisfy nested constraints given by a level structure of cooperation. This structure is defined by a family of partitions on the set of players. It is ordered in such a way that each partition is a refinement of the next one. We propose a value for these games by adapting the Shapley value. The value is characterized axiomatically. For this purpose, we introduce a new property called *class balance contributions* by generalizing other properties in the literature. Finally, we introduce a multilinear extension of our games and use it to obtain an expression for calculating the adapted Shapley value.

1. **Introduction.** There are practical situations where a set of agents have mechanisms to reach agreements. Usually those situations can be modeled as transferable utility cooperative games (TU games). One of the main problems is the distribution of payoffs between the agents. When *a priori* coalition agreements exist, such situations are modeled using cooperative games with coalitional structures (that is, a partition of the set of players). Different extensions of the Shapley value for these games have been presented by Aumann and Dreze[2] and Owen[17]. However, sometimes the information of the cooperation structure provided by the coalitional structure is insufficient. Specifically, the coalitional structure may not describe the situations where cooperation arises on several levels (with multiple partitions of the set of players). For this case, Winter[20] introduced games with a *level structure* and presented an extension of *the Owen value* for this structure.

In the classical model of cooperative games, any set of players is a feasible coalition. Indeed, a cooperative game may be modeled as a real-valued function defined on the power set of players. But in some situations, cooperation between players can be restricted by exogenous conditions which prevent the formation of some of these coalitions. Some examples of restricted cooperation are presented in [12] and [14], for communication games; [5], for cooperative games under precedence constraints; [3], for games on convex geometries; and [11], for games with general coalitional structure.

In this paper we present a special restricted cooperation given by a level structure. To give an example, we will present the next situation with a political division

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structure. Consider the problem of allocating the payoff of each city for the maintenance of highways. The segments of highway are generally the responsibility of the entity in which they were built. In this way, the cooperation between the governmental entities can reduce maintenance costs in the entire highway system. Individual entities are encouraged to use federal funds on improving the efficiency and safety of this system. Then, some percent of the construction and maintenance costs of interstate highways in many countries have been paid mainly through fuel taxes, user fees, designated property taxes, and other taxes collected by the federal, state, and local governments. The costs of maintenance and percentages are different and the information is collected by each entity. For this reason, coalitions are only possible between cities within the same county, between counties of the same state and between different states.

A level structure (mentioned in the previous paragraph relating to maintenance of highway system taxes) describes a hierarchy of cooperation between the players as mentioned above, where cities are players. Winter[20] provides a model of games with this structure and proposes a solution called *level structure value*. This value is computed as the player's expected marginal contributions to the coalitions formed by a sequential process. For any order that is consistent with the level structure[1], every player joins his predecessors in the order, with a uniform distribution over these orders. Nevertheless, the values of those marginal contributions can not be calculated from the previous situation. This requires information of coalitions that is missing. For example, when just one city joins its predecessor cities in an order consistent with the political division, we can obtain sets of counties, joining cities in a part of another county, or sets of states along with counties in another state joining some cities in another county. In this paper we focus on solving this problem with restricted information.

Here we present games with a coalitional structure similar to the one presented by Winter[20], but with an important conceptual difference. The cooperation among players is restricted by nested constraints given at the level structure. Thus the domain of the *characteristic function* is not the power set of players. In this structure, the set of players is classified into several levels according to a family of partitions of the set of players, $(\mathcal{P}^0, \mathcal{P}^1, ..., \mathcal{P}^{p+1})$, where each partition is a refinement of the next one. Throughout the paper, $\mathcal{P}^0$ denotes the partition of singletons. The last partition $\mathcal{P}^{p+1}$ stands for the partition of the grand coalition. These will be called trivial partitions. The elements of each partition will be called *classes*. Every class has a set of subclasses which is the set of elements in the previous partition contained in the class. In fact subclasses provide a partition of the class to which they belong. In our model, only subsets of players are considered *feasible coalitions* if they satisfy *nested constraints* in the following sense. A coalition S is the union of subclasses belonging to the same class. The games defined on the set of feasible coalitions will be called *cooperative games with nested structures* or simply *nested games*. The classical situation occurs exactly when the set of players only has the trivial partitions. In this way, we consider the nested constraints as *a rule to form* a coalition and we consider that only *feasible coalitions* can have a worth.

The literature cited above, in which restricted cooperation is studied, propose adaptations of the value of Shapley to solve games in these new structures of cooperation. In them we can observe two different approaches of adaptation. For communication games and games with general coalitional structure, different ways

[1]An order is consistent with the level structure if players of the same class appear successively.

are proposed to extend the characteristic function to the unfeasible coalitions in which the Shapley value can be calculated. On the other hand, the models for cooperative games under precedence constraints and games on convex geometries define processes in which only the worth of feasible coalitions are used. This avoids an artificial definition of the worth on unfeasible coalitions. In this paper we follow the second approach and give a solution for the new type of games defined in our model using only the worth of feasible coalitions. Specifically, we develop an adaptation of the Shapley value for these games. We study the structure of the feasible coalitions and obtain a value calculated with modified marginal contributions. Here a whole subclass joins or leaves the coalition. In the particular case that there is only one partition other than a trivial one, $(\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$, our adaptations of the Shapley value matches the *collective value* presented by Kamijo[10].

There are well-known characterizations for values in games with level structures. They use generalizations or adaptations of *balanced contributions principle* by Myerson[13]. Calvo et al.[4] provided a characterization of the *level structure value* using the property of *balanced contributions*. This property states that for any two subclasses that belong to the same class, the profit or deficit on the total payoff of the members in each subclass should be equal when the other subclass leaves the game. Gómez-Rúa and Vidal-Puga[6] proposed another value for values in games with level structures; they characterize it with *balanced per capita contributions property*. This property states that for any two subclasses that belong to the same class, the average amount that players in each subclass would gain or lose should be equal when the other subclass leaves the game. The average is taken over the cardinality of each subclass.

Now we will introduce a property for nested games by generalizing the *collective balanced contributions property* of Kamijo[10]. We called this the *class balanced contributions property*. This states that for any two subclasses that belong to the same class, the change in the payoff for each one player in each subclass should be equal when the other whole subclass leaves the game.

The *multilinear extension* of Owen[15, sec. 10] is an efficient tool for computing the Shapley value for games with large set of players. Owen et al.[16] modify this method to calculate the *coalitional value* of Owen[17]. They propose a way to generalize this method for the level structure value. Here, we define the *multilinear extension for nested games* and we use it to obtain another expression of the adapted Shapley value.

This paper is organized as follows: In Section 2, we introduce our model by describing the set of feasible coalitions. We show some algebraic properties of this set as a partially ordered set with the inclusion relation. Those properties are the principal tool for giving a representation of a nested game with respect to a basis of unanimity nested games. In Section 3, we determine a Shapley value for nested games by first defining it on the basis of unanimity nested games and then extending it linearly. To interpret the adaptation of the Shapley value, we will define a game for each class where its players will be the subclasses that comprise it. With the weighted Shapley value of these games, where the weights are the cardinalities of the subclasses, we define a distribution among the agents that make up such subclasses. Considering a weight vector to model asymmetry between agents, we can give an adapted weighted Shapley value for nested games. Finally, we define a *multilinear extension* for the nested games. This extension allows us to show a different way to compute the adapted Shapley value.

2. **Model formulation.** In order to present our model of games under nested constraints of cooperation, we need to introduce some general aspects and notations. To describe the level structure, we consider multiple partitions of the set players $N = \{1, 2, ..., n\}$. They have certain characteristics that we will introduce below.

Let P and Q be partitions of N, we can say that P is a refinement of Q, and is denoted by $P \prec Q$, if for all $X \in P$, there is $Y \in Q$ such that $X \subseteq Y$.

A level structure is a sequence $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1, ..., \mathcal{P}^{p+1})$, with $\mathcal{P}^k$ a partition of N, $0 \leq k \leq p+1$, such that: $\mathcal{P}^0 = \{\{1\}, \{2\}, ..., \{n\}\}$, $\mathcal{P}^{p+1} = \{N\}$ and $\mathcal{P}^k \prec \mathcal{P}^{k+1}$, for all $0 \leq k \leq p$.

We call $\mathcal{P}^k$ the *k-th level* of $\mathfrak{P}$. We say that 0-level and $(p+1)$-level are the trivial levels. When there are no different partitions of the trivial levels, we say that $\mathfrak{P}$ is a *trivial* level structure.

We denote $\mathcal{P}^k = \{C_1^k, C_2^k, ..., C_{m_k}^k\}$ and we let $M^k = \{1, 2, ..., m_k\}$. In this way, each element of $\mathcal{P}^k$ will be called a *k-class*. We will consider *the set of indices* of C_t^k defined as follows:

$$M_t^k = \{s \mid C_s^{k-1} \subseteq C_t^k\}. \tag{1}$$

Then, every class C_t^k can be characterized as the joining of the elements in the partition $\mathcal{P}^{k-1}$ that compose it,

$$C_t^k = \bigcup_{s \in M_t^k} C_s^{k-1}.$$

Now we are ready to introduce the model of games under nested constraints given by a level structure. Since this model only considers coalitions of players in sets of $(k-1)$-classes that belong to the same k-class, only these coalitions will be allowed. The set containing these coalitions will be called *the set of feasible coalitions of* C_t^k, which is

$$B_t^k = \left\{ \bigcup_{s \in T} C_s^{k-1} \mid T \subseteq M_t^k \right\}, \tag{2}$$

for $k = 1, ..., p+1$ and $t = 1, ..., m_k$. Then, *the set of feasible coalitions on the level* k is

$$B^k = \bigcup_{t=1}^{m_k} B_t^k.$$

Therefore, the set of all feasible coalitions in a situation of cooperation with nested constraints is the joining of all feasible coalitions in the $(p+1)$-levels,

$$\mathcal{B}_N = \bigcup_{k=1}^{p+1} B^k.$$

We can take $(\mathcal{B}_N, \subseteq)$ as a partially ordered set (or *poset*). Moreover, $\mathcal{B}_N$ is a *finite lattice* with the usual *join* and *meet* operations[2]. It has some interesting properties. For this specify work the following properties are of paramount importance.

Remark 1. By (2) we can ensure that each partially ordered subset $(B_t^k, \subseteq)$ and $(2^{M_t^k}, \subseteq)$ are isomorphic. Indeed, every *interval*

$$[S, T] = \{R \in \mathcal{B}_N \ : \ S \subseteq R \subseteq T\}$$

for $S, T \in B_t^k$, $S \subseteq T$, is *Boole algebra*.

[2]The *least upper bound* of S and T in the poset is S join T. Dually the *greatest lower bound* of S and T is S meet T, [18, p. 285].

Remark 2. Let $S, T \in \mathcal{B}_N$, if $S \subseteq T$ and $S \in B_t^k$ but $T \notin B_t^k$ then

$$[S, T] = [S, C_t^k] \cup [C_t^k, T].$$

This means that $[S, T]$ is not an atomic partially ordered set, due to the join of all coalitions covering S is $C_t^k \subset T$[3].

Below, we provide a situation of nested constraints of cooperation which we will use to exemplify several aspects throughout the paper. This has the political division structure mentioned in the introduction for the problem of cost allocation in the maintenance of highways.

Example 1. Let $N = \{1, 2, 3, 4\}$ be a set of cities in a country. These cities are grouped into counties: $C_1^1 = \{1\}$, $C_2^1 = \{2, 3\}$ and $C_3^1 = \{4\}$. Also, these cities come from different states $C_1^2 = \{1, 2, 3\}$ and $C_2^2 = \{4\}$. Furthermore, we denote the class of the country by $C_1^3 = \{1, 2, 3, 4\}$. Then, the sets of indices that characterize the classes of the upper levels are:

$$M_1^2 = \{1, 2\},\ M_2^2 = \{3\},\ M_1^3 = \{1, 2\}.$$

Then, the set of feasible coalitions is

$$\mathcal{B}_N = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}.$$

The set $\mathcal{B}_N$ defines the feasible coalitions for nested constraints of cooperation (hereafter referred to as *nested structure*), which allows us to give a definition for these types of games analogous to the TU games,

Definition 2.1. We call a *cooperative nested game* to the pair $(v, \mathcal{B}_N)$, where $N = \{1, 2, ..., n\}$ is the set of players, $\mathcal{B}_N$ is a nested structure and $v : \mathcal{B}_N \to \mathbb{R}$ is a characteristic function such that $v(\emptyset) = 0$.

Throughout this paper, the function v is restricted on proper subsets of $\mathcal{B}_N$, which are nested structures too. By abuse of the notation, we use the same letter to designate the function on $\mathcal{B}_N$ and its restriction on the proper subsets.

Example 2. Let $N = \{1, 2, 3, 4\}$ be the set of cities with nested structure presented in Example 1. The characteristic function which models maintenance cost of a highway system is represented in Table 1.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3, 4\}$
$v(S)$	1	1	0	1	2	4	6

TABLE 1. Characteristic function for maintenance cost of a highway system.

By $G_{\mathcal{B}_N}$ we denote the vector space of all cooperative nested games on $\mathcal{B}_N$. For $T \in \mathcal{B}_N$ we define *the nested unanimity game* $u_T \in G_{\mathcal{B}_N}$ by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. In this fashion, $U_{\mathcal{B}_N} = \{u_T | T \in \mathcal{B}_N,\ T \neq \emptyset\}$ is a basis of $G_{\mathcal{B}_N}$.

Every nested game is a linear combination of nested unanimity games,

$$v = \sum_{T \in \mathcal{B}_N} \Delta_T(v) u_T \quad \text{and} \quad v(S) = \sum_{\substack{T \subseteq S \\ T \in \mathcal{B}_N}} \Delta_T(v), \quad \forall S \in \mathcal{B}_N.$$

[3] R cover S if $S \subset R$ and not exist $Q \in \mathcal{B}_N$ such that $S \subset Q \subset R$.

Using *the Möbius inversion formula* [18, p. 303], we have that

$$\Delta_T(v) = \sum_{\substack{R \subset T \\ R \in \mathcal{B}_N}} \mu(R,T) v(R), \quad \forall T \in \mathcal{B}_N,$$

where μ is the *Möbius function*. Hence by Remarks 1 and 2, while applying techniques for computing Möbius function [18, Example 3.8.3, Corollary 3.9.5], $\mu(R,T)$ takes the form

$$\mu(R,T) = \begin{cases} (-1)^{\ell(R,T)} & \text{if } \ R,T \in B_t^k \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell(R,T) := \left|\left\{C_r^{k-1} \ : \ C_r^{k-1} \subseteq T \setminus R\right\}\right|$. Then

$$\Delta_T(v) = \sum_{\substack{R \subset T \\ R,T \in B_t^k}} (-1)^{\ell(R,T)} v(R), \quad \forall T \in \mathcal{B}_N. \tag{3}$$

Following [7], we call $\Delta_T(v)$ the *dividend* of T in the game v.

Now, we are going to define *games of k-classes*[4]. The main idea on these games is to observe the $(k-1)$-class as agents by using the sets of indices in (1).

Definition 2.2. Let $(v, \mathcal{B}_N)$ be a nested game. The *game of classes in* C_t^k of $(v, \mathcal{B}_N)$ is a game $(M_t^k, v_{C_t^k})$ defined by

$$v_{C_t^k}(R) = v\left(\bigcup_{s \in R} C_s^{k-1}\right),$$

for each $R \subseteq M_t^k$.

If there is no ambiguity, we will denote the game $(M_t^k, v_{C_t^k})$ only with its characteristic function $v_{C_t^k}$.

Remark 3. By definition, we can note that if $C_t^k \subseteq C_s^{k+1}$, then the index $t \in M_s^{k+1}$ is a player in $v_{C_s^{k+1}}$ and M_t^k is the grand coalition in $v_{C_t^k}$. In fact,

$$v(C_t^k) = v_{C_t^k}(M_t^k) = v_{C_s^{k+1}}(\{t\}).$$

Example 3. The games of classes for the nested game $(v, \mathcal{B}_N)$ of Example 2 are $(M_1^2, v_{C_1^2})$, $(M_2^2, v_{C_2^2})$ and $(M_1^3, v_{C_1^3})$. They have characteristic functions presented in Tables 2 and 3, respectively.

R	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_{C_1^2}(R)$	1	2	4

R	$\{3\}$
$v_{C_2^2}(R)$	1

TABLE 2. Games of classes for counties as players.

R	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_{C_1^3}(R)$	4	1	6

TABLE 3. Game of classes for states as players.

[4]This game is referred to as a quotient game in [17].

Given a fixed player $i \in N$, we know that it belongs to a class for each level. These classes are involved in different *games of classes* in which player i has a role. We will denote the set of all games of classes in which i is present by

$$J_i := \{v_{C_t^k} \mid i \in C_t^k\}.$$

In order to simplify the notation for the rest of the paper, we will usually denote the index of the k-class containing player i as $[i]^k$, and we will write it simply $[i]$ when no confusion arises, that is, the player i is in class $C_{[i]}^k$, for each k.

3. **Adapted Shapley value.** A solution or value for cooperative nested games is mapping f which assigns to every game $(v, \mathcal{B}_N)$ a vector $f(v, \mathcal{B}_N) \in \mathbb{R}^n$, where its entries represent payoff for each player in N. We will introduce a value for nested games that is an adaptation of the Shapley value.

We provide a value similar to the Shapley value in the sense that the games in the basis $U_{\mathcal{B}_N}$ are assigned the same value as the Shapley value assigned to unanimity classic games

$$\varphi_i(u_T, \mathcal{B}_N) = \begin{cases} \frac{1}{|T|} & \text{if } \ i \in T, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

A linear value that satisfies (4) for every game $v \in G_{\mathcal{B}_N}$ is

$$\varphi_i(v, \mathcal{B}_N) = \sum_{\substack{S \ni i \\ S \in \mathcal{B}_N}} \frac{\Delta_S(v)}{|S|}, \tag{5}$$

where $\Delta_S(v)$ are the dividends of v. Hence from our formula for the dividends, we have that equation (5) which takes the form

$$\varphi_i(v, \mathcal{B}_N) = \sum_{\substack{S \ni i \\ S \in \mathcal{B}_N}} \gamma_S \left(v(S) - v(S_{-i})\right),$$

where $S_{-i} := S \setminus C_{[i]}^{k-1}$, for all $S \in B^k$ and

$$\gamma_S = \sum_{\substack{T \supset S \\ S,T \in B_t^k}} \frac{(-1)^{\ell(S,T)}}{|T|}, \quad S \neq \emptyset.$$

This expression of φ allows us to observe the similarity with the weighted Shapley value[5] for the games in J_i if we take each $C_t^{k-1} \subset C_r^k$ as a player through $(M_r^k, v_{C_r^k})$. Because it is useful to provide an initial interpretation of φ and characterize it, we will give the following expression,

$$\varphi_i(v, \mathcal{B}_N) = v(i) + \sum_{w \in J_i} \frac{Sh_{[i]}^{\lambda}(w) - w([i])}{\lambda_{[i]}} \tag{6}$$

where every weight $(\lambda)_j$ is the cardinality of the class represented by j in the game of classes w. It is straightforward to check that (6) is linear and satisfies (4).

Each term

$$D_i(w) := \frac{Sh_{[i]}^{\lambda}(w) - w([i])}{\lambda_{[i]}}$$

[5]The weighted Shapley value was introduced by Kalai and Samet[9]. By $Sh^{\lambda}(w)$ we denote this value for a game w and a weight vector λ.

has the following interpretation: The game of classes is solved with a weighted Shapley value ($Sh^{\lambda}_{[i]}$), where the weights represent the asymmetry of size in the participant classes. The difference $Sh^{\lambda}_{[i]}(w) - w([i])$ represents the excess or defect of the class between the allocated value and the quantity that can be obtained by itself ($w([i]) = v(C^k_{[i]})$). Finally, an egalitarian distribution of that difference is made among the members of the same class. The distribution in equal shares of that amount provides a solution suggesting a "collective" support in each class. In fact, expression (6) of φ arises as a generalization for the well-known *collective value* presented by Kamijo[10] for coalitional structures. If we take $\mathfrak{P} = (P^0, P^1, P^2)$ which only has one partition different from the trivial partitions, we obtain the simplified expression

$$\varphi_i(v, \mathcal{B}_N) = Sh_i(v) + \frac{Sh^{\lambda}_{[i]}(v_{C^2_{[i]}}) - v(C^1_{[i]})}{|C^1_{[i]}|}, \quad \forall i \in N,$$

and that is equivalent to *the collective value*.

As a next step we will show some interesting properties that the value φ satisfies. For this we should consider $\mathcal{B}_N$ restricted to the coalition S,

$$\mathcal{B}_S := \{T \in \mathcal{B}_N \mid T \subseteq S\}.$$

If we restrict the nested game to a class C^l_r, we have the value φ for a player $i \in C^l_r$. It is given by

$$\varphi_i(v, \mathcal{B}_{C^l_r}) = v(i) + \sum_{k=1}^{l} D_i(v_{C^k_{[i]}}).$$

We continue in this fashion obtaining the next proposition.

Proposition 1. *Let $C^l_r \subseteq C^{l+1}_s$. Given player $i \in C^l_r$, then the value in the restricted structure $\mathcal{B}_{C^{l+1}_s}$ is*

$$\varphi_i(v, \mathcal{B}_{C^{l+1}_s}) = \varphi_i(v, \mathcal{B}_{C^l_r}) + D_i(v_{C^{l+1}_s}).$$

Example 4. To calculate the payoff which assigns solution (6) for player 2 of the game presented in Example 2 we have

$$v(\{2\}) = 1, \ D_2(C^1_2) = \frac{1}{2}, \ D_2(C^2_1) = \frac{1}{3}, \ D_2(C^3_1) = \frac{1}{4}.$$

Furthermore, the final allocation of the value is

$$\varphi(v, \mathcal{B}_N) = \left(1 + \frac{1}{3} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, 1 + \frac{1}{4}\right) = \left(\frac{19}{12}, \frac{25}{12}, \frac{13}{12}, \frac{5}{4}\right).$$

Characterization. Now we will provide an axiomatic characterization for the value φ. We introduce the property that describes efficiency and another property which extends the *balanced contributions property*.

A value f is *efficient* if $\sum_{i \in N} f_i(v, \mathcal{B}) = v(N)$ for each $v \in G_{\mathcal{B}_N}$.

Definition 3.1. A value f satisfies the *Class Balanced Contributions* (CBC) if and only if, for all $C^l_t, C^l_s \subset C^{l+1}_r$ $(l = 0, ..., p)$ we have

$$f_i(v, \mathcal{B}_{C^{l+1}_r}) - f_i(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_s}) = f_j(v, \mathcal{B}_{C^{l+1}_r}) - f_j(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_t}),$$

for every $i \in C^l_t$, $j \in C^l_s$ and all $v \in G_{\mathcal{B}_N}$.

CBC requires that given two classes C_t^l and C_s^l in the same upper class C_r^{l+1}, the change in payoff measured by the value of each player in C_t^l, if C_s^l leaves the game, must be equal to the change in payoff measured by the value to each player in C_s^l if C_t^l leaves the game.

Remark 4. If we have only the trivial levels $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1)$, then $\mathcal{B}_N = 2^N$. Then the CBC property matches with the following property, defined by Myerson[13].

A value f satisfy *Balanced Contributions* if only if

$$f_i(v, 2^N) - f_i(v, 2^{N\setminus\{j\}}) = f_j(v, 2^N) - f_j(v, 2^{N\setminus\{i\}}),$$

for all $i, j \in N$ and all nested game $v \in G_{2^N}$.

Remark 5. When there is only one different level from the trivial levels $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$, the CBC property coincides with the following property presented by Kamijo[10].

A value f satisfy *Collective Balanced Contributions* if only if

$$f_i(v, \mathcal{B}_N) - f_i(v, \mathcal{B}_{N\setminus C_s^1}) = f_j(v, \mathcal{B}_N) - f_j(v, \mathcal{B}_{N\setminus C_t^1}),$$

for every $i \in C_t^1$, $j \in C_s^1$, $t \neq s$, and for all $v \in G_{\mathcal{B}_N}$.

Proposition 2. *φ is efficient and satisfies CBC.*

Proof. It is straightforward to check that φ is efficient. Now we will show that φ satisfies CBC: Let $(v, \mathcal{B}_N) \in G_{\mathcal{B}_N}$, $i, j \in N$ and $C_{[i]}^l, C_{[j]}^l \subset C_r^{l+1}$, for fixed r. By definition of φ, we note that if $l = 0$ then φ is the Shapley value over C_r^1 and clearly it satisfies CBC. Now if $l \geq 1$, by Proposition 1, we have

$$D_i(v_{C_r^{l+1}}) = \varphi_i(v, \mathcal{B}_{C_r^{l+1}}) - \varphi_i(v, \mathcal{B}_{C_r^{l+1}\setminus C_{[i]}^l}).$$

We obtain a similar expression for player j but we must verify that $D_i(v_{C_r^{l+1}}) = D_j(v_{C_r^{l+1}})$. Calculating this expression we obtain

$$\frac{Sh_{[i]}^{\lambda}(v_{C_r^{l+1}}) - Sh_{[i]}^{\lambda}(v_{C_r^{l+1}\setminus C_{[j]}^l})}{\lambda_{[i]}} = \frac{Sh_{[j]}^{\lambda}(v_{C_r^{l+1}}) - Sh_{[j]}^{\lambda}(v_{C_r^{l+1}\setminus C_{[i]}^l})}{\lambda_{[j]}},$$

which was proved by Hart and Mas-Colell[8], p. 604. Thus we conclude that φ satisfies the property of CBC. □

Next we present a characterization of φ using the mentioned properties in Proposition 2.

Theorem 3.2. *There exists a unique value for cooperative games with nested structure that satisfies the properties of* efficiency *and* CBC. *That value is φ.*

Proof. As we showed in Proposition 2, we know that φ is efficient and satisfies CBC. Next, we fix a nontrivial $\mathcal{B}_N$. There is at least a nontrivial level. Let f and g be two efficient solutions satisfying CBC. We will show that $i \in N$, $f_i(v, \mathcal{B}_N) = g_i(v, \mathcal{B}_N)$ for any $v \in G_{\mathcal{B}_N}$.

We will prove the result using induction on the number of levels that define $\mathcal{B}_N$. Consider the level $k = 1$. Since the lower level classes are singletons, we have that the feasible coalitions in $\mathcal{B}_{C_{[i]}^1}$ are

$$B_{[i]}^1 = 2^{C_{[i]}^1}.$$

Then the game in the first level, $(v, \mathcal{B}_{C^1_{[i]}})$, is a classical TU game and CBC is equivalent to the property of *balanced contributions* defined by Myerson[13]. Therefore we can conclude that

$$f_i(v, \mathcal{B}_{C^1_{[i]}}) = g_i(v, \mathcal{B}_{C^1_{[i]}}) = Sh_i(C^1_{[i]}, v).$$

Now we assume that the result holds for levels k, $k \leq l \leq p$. This means

$$f_i(v, \mathcal{B}_{C^k_{[i]}}) = g_i(v, \mathcal{B}_{C^k_{[i]}}),$$

for every class $C^k_{[i]} \in B^k \subseteq \mathcal{B}_N$, with $1 \leq k \leq l$, and we proceed to prove it for level $l+1$.

Let us use induction on the cardinality of set of indices of C^{l+1}_r, denoted by M^{l+1}_r. Consider $i \in C^l_{[i]} \subseteq C^{l+1}_r$. First assume that $|M^{l+1}_r| = 1$, i.e. the only subclass that C^{l+1}_r contains is $C^l_{[i]}$ and feasible coalitions of nested structures $\mathcal{B}_{C^{l+1}_r}$ and $\mathcal{B}_{C^l_{[i]}}$ are equal. Hence

$$(v, \mathcal{B}_{C^{l+1}_r}) = (v, \mathcal{B}_{C^l_{[i]}}),$$

thus by the induction hypothesis we conclude that

$$f_i(v, \mathcal{B}_{C^{l+1}_r}) = f_i(v, \mathcal{B}_{C^l_{[i]}}) = g_i(v, \mathcal{B}_{C^l_r}) = g_i(v, \mathcal{B}_{C^{l+1}_r}).$$

Assuming the result holds for $|M^{l+1}_r| = m$, we will prove that the result holds for $|M^{l+1}_r| = m+1$.

Let $|M^{l+1}_r| = m+1 > 1$, then there exists $j \in C^l_{[j]} \subseteq C^{l+1}_r$, such as $C^l_{[i]} \neq C^l_{[j]}$. Due to the CBC property, we have

$$h_i(v, \mathcal{B}_{C^{l+1}_r}) - h_j(v, \mathcal{B}_{C^{l+1}_r}) = h_i(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[j]}}) - h_j(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[i]}}),$$

for $h = f, g$. Thus by induction hypothesis, we have

$$f_i(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[j]}}) = g_i(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[j]}}),$$

$$f_j(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[i]}}) = g_j(v, \mathcal{B}_{C^{l+1}_r \setminus C^l_{[i]}}).$$

Therefore

$$f_i(v, \mathcal{B}_{C^{l+1}_r}) - g_i(v, \mathcal{B}_{C^{l+1}_r}) = f_j(v, \mathcal{B}_{C^{l+1}_r}) - g_j(v, \mathcal{B}_{C^{l+1}_r}).$$

We may now use efficiency to conclude that if we fix i and take summation over $j \in C^{l+1}_r$, then we have

$$f_i(v, \mathcal{B}_{C^{l+1}_r}) - g_i(v, \mathcal{B}_{C^{l+1}_r}) = 0.$$

Thus we have proved that $f_i(v, \mathcal{B}_{C^{l+1}_r}) = g_i(v, \mathcal{B}_{C^{l+1}_r})$ for all $i \in C^{l+1}_r$.

Finally, applying the induction hypothesis on levels, we conclude that $f_i(v, \mathcal{B}_{C^k_r}) = g_i(v, \mathcal{B}_{C^k_r})$ for each $k \leq p+1$. Specifically if we take $k = p+1$, we obtain

$$f_i(v, \mathcal{B}_N) = g_i(v, \mathcal{B}_N)$$

for all $i \in N$. □

Proposition 3. *(Kamijo[10], Theorem 2) Let $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$ be a level structure. The* collective value *(3) is a unique efficient solution satisfying CBC.*

Kamijo[10] obtained the characterization of the collective value by efficiency, balance contributions and collective balance contributions properties. Then, proof of the above result is straightforward by Remark 4, Remark 5 and Theorem 3.2. Moreover, this work provides a different approach of the result due to the domain of characteristic functions in nested games is $\mathcal{B}_N$. We thus conclude that the collective value only uses the information of coalitions that satisfy the nested constraints when the level structure is $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$.

Remark 6. If we introduce a weight vector to model certain asymmetry between the players, then a generalization of the weighted Shapley value can be obtained analogously to the adaptation of the Shapley value. To obtain a solution that takes into account this asymmetry, we propose the value:

$$\varphi_i^{\omega}(v, \mathcal{B}_N) = v(i) + \omega_i \sum_{w \in J_i} \frac{Sh_{[i]}^{\omega}(w) - w([i])}{\omega_{[i]}},$$

where ω_j is the sum of weights of the members of class what j represents in the game of classes w.

Its characterization is analogous to the Theorem 3.2 by slightly adapting the CBC property. A value f satisfies ω-*Class Balanced Contributions* (ω-CBC) if and only if, for all $C_t^l, C_s^l \subset C_r^{l+1}$ $(l = 0, ..., p)$ we have

$$\frac{f_i(v, \mathcal{B}_{C_r^{l+1}}) - f_i(v, \mathcal{B}_{C_r^{l+1} \setminus C_{[j]}^l})}{\omega_i} = \frac{f_j(v, \mathcal{B}_{C_r^{l+1}}) - f_j(v, \mathcal{B}_{C_r^{l+1} \setminus C_{[i]}^l})}{\omega_j},$$

for every $i \in C_t^l$, $j \in C_s^l$ and all $v \in G_{\mathcal{B}_N}$.

Thus, ω-CBC requires that the contributions of classes are balanced in proportion to the weights of each player.

4. **The multilinear extension of nested games.** As mentioned Owen et al.[16], *the multilinear extension* (MLE) has been shown to be an effective tool for computing the Shapley value for TU games with large players' set. For this reason they modify the method of MLE to calculate the value of Owen[17] for games with coalition structure. Moreover they propose a way to generalize this method for the level structure value of Winter[20]. In this section, we modify MLE to calculate the value φ for nested games.

In accordance with the work of Owen[15, sec. 10] we define *the multilinear extension of a nested game* using the linear decomposition presented in the Section 2 as follows

$$f(v)[q_1, ..., q_n] = \sum_{S \in \mathcal{B}_N} \prod_{j \in S} q_j \Delta_S(v),$$

where $\Delta_S(v)$ for all $S \in \mathcal{B}_N$ are the *dividends* defined in (3).

It is possible to calculate the value φ for a nested game using this extension. The following result shows a new way to do it.

Proposition 4. *Let $(v, \mathcal{B}_N)$, $i \in N$ and $f(v)$ the multilinear extension of v. Then the value φ is given by*

$$\varphi_i(v, \mathcal{B}_N) = \int_0^1 \frac{\partial f(v)}{\partial q_i}[t, .., t]dt.$$

Proof. Calculating the partial derivative of $f(v)$ with respect to q_i we have

$$\frac{\partial f(v)}{\partial q_i}[q_1, ..., q_n] = \sum_{\substack{S \ni i \\ S \in \mathcal{B}_N}} \prod_{\substack{j \in S \\ j \neq i}} q_j \Delta_S(v).$$

Now, evaluating this derivative in the main diagonal $[t, ..., t]$ and integrating with respect to t, we obtain

$$\begin{aligned} \int_0^1 \frac{\partial f(v)}{\partial q_i}[t, ..., t]dt &= \int_0^1 \sum_{\substack{S \ni i \\ S \in \mathcal{B}_N}} t^{|S|-1} \Delta_S(v) dt \\ &= \sum_{\substack{S \ni i \\ S \in \mathcal{B}_N}} \frac{\Delta_S(v)}{|S|} = \varphi(v, \mathcal{B}_N). \end{aligned}$$

□

5. **Conclusions and future work.** The main conclusion of this paper is that the model of games under nested constraints gives rise to a special type of games with domain of feasible coalitions in a new structure composed of Boolean algebras. Due to the properties of such structure, the algebraic extension of the Shapley value which we use for this type of games allowed us to characterize it and obtain a closed expression for that value.

There are a number of directions for future work, in particular for the advancement of applications in the study of more complex situations. On the one hand, it would be interesting to study this model with a communication structure (graph) and propose new values. As a first step in this direction, the study of two-level communication structure models have been presented by Brink et al.[19]. In fact, since the characterization of the values presented here are made with the property of CBC, this suggests a clear pathway to propose an adapted Myerson value for n-level communication structure. On the other hand, another way is to study different solutions for the multilinear extension of the nested games. Hence, it would also be interesting to propose a model for games with fuzzy coalitions under nested structure of cooperation constraints. It would also be interesting to propose adaptations and characterize different solution concepts of Shapley value for games that arise in our model. The power index Banzhaf would be adapted for simple games as the generalization presented by Álvarez-Mozos and Tejada[1]. Other general concepts of solution as the core and the nucleolus would be studied.

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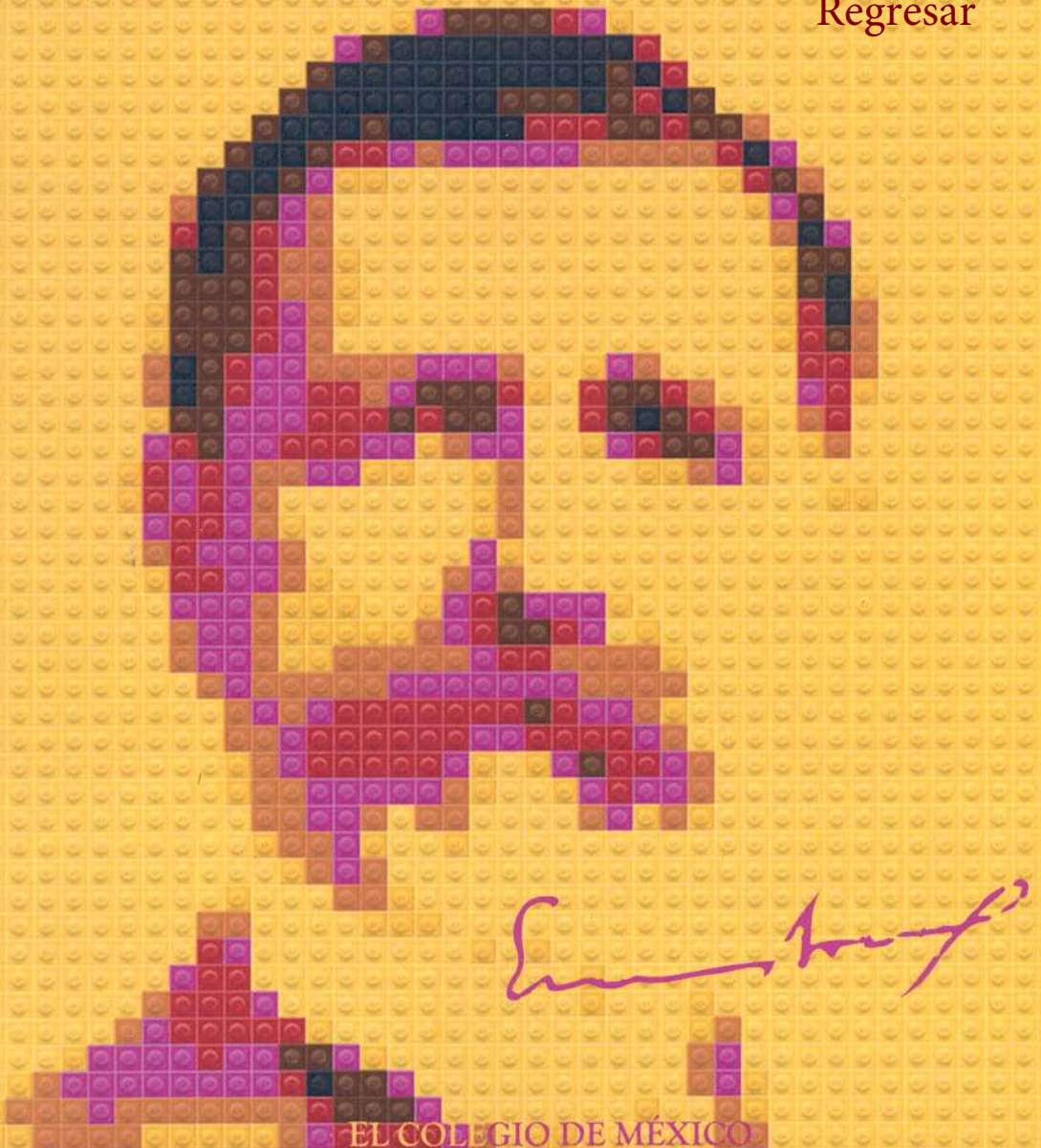
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Émile Borel

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Chapter 10
Additional Properties of the Owen Value

Oliver Juarez-Romero, William Olvera-Lopez, and Francisco Sanchez-Sanchez

Abstract In this work we present two properties of coalitional values for games with coalitional structure. The main goal of the paper is to prove that the Owen value satisfies these properties, which are related to a gain game and a lost game. The satisfaction of these properties provides a greater stability for this value because it is immune to a possible manipulation given by these games.

Keywords Cooperative games • Coalitional structures • Owen value • Shapley value • Stability • Manipulability • Gain and lost games

10.1 Introduction

The organization of agents in coalitional structures is an important fact in many real-world contexts, such as the formation of cartels, trading blocks among nation states, research joint ventures, and political parties. These situations can be modeled through transferable utility games, in which the players are partitioned into coalitions for the purpose of bargaining. All players in the same coalition agree before the game that any cooperation with other players will only by carried out collectively. Given a coalitional structure, the bargaining occurs between coalitions and between players in the same coalition. The main idea is that coalitions play among themselves as individual players in a game among coalitions and then, the profit obtained by each coalition is distributed among its members. Owen (1977) studied the allocation that arises from applying the Shapley value (Shapley (1953)) twice: first, in a game among coalitions and then in a game inside each coalition.

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In this paper we study a kind of stability of the Owen value. Given a game with coalitional structure and a coalitional value, we build two games called the gain and the lost games by association, respectively. These games model the changes in the payoffs when new coalitional structures arise by manipulation of the coalitional value. The main result establishes that if the coalitional value is the Owen value, manipulating does not help: the payoffs that the players receive in the gain and the lost games by association are the same as in the original game. A similar property was studied by Hamiache (2001).

This article is organized as follows: Sect. 10.2 provides the notation that we use in the rest of the document. On Sect. 10.3, we show that the Owen value is immune to the manipulation given in the gain and lost games by association.

10.2 Notation and Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of agents. A *cooperative game* with transferable utility (TU-game) is a pair (N, υ) where $\upsilon : 2^N \to \mathbb{R}$ is a *characteristic function* defined on the power set of N satisfying $\upsilon(\emptyset) = 0$. An element i of N is called a *player*, every nonempty subset S of N a *coalition*, and N the *grand coalition*. The real number $\upsilon(S)$ is called the *worth* of coalition S. $\upsilon(S)$ is interpreted as the total payoff that the coalition S can obtain for its members. Let G^N be the set of all cooperative TU-games with players set N. A *payoff vector* $x \in \mathbb{R}^n$ of $\upsilon \in G^N$ is a vector giving a payoff x_i to any player $i \in N$. A solution on G^N is a map φ that assigns to each $\upsilon \in G^N$ a payoff vector $\varphi(\upsilon) \in \mathbb{R}^n$. A player $i \in N$ is a *null player* in $(N, \upsilon) \in G^N$ if for every $S \subseteq N$ such that $S \ni i$ we have that $\upsilon(S) = \upsilon(S \setminus \{i\})$. Two players $i, j \in N$ are *symmetric* in $(N, \upsilon) \in G^N$ if for every $S \subseteq N$ such that $i \in S$ we have that $\upsilon(S) = \pi\upsilon(\pi S)$, where $\pi(S) = \pi^{-1}(S)$, $\pi\upsilon(T) = \upsilon(\pi(T))$ for all $T \subseteq N$ and $\pi \in S_n$ [1] is a permutation such that $\pi(i) = j$.

A very well-known solution on G^N is the *Shapley value*. The Shapley value can be defined by orders as follows:

$$\mathrm{Sh}_i(N, \upsilon) = \frac{1}{|\, S_n \,|} \sum_{\pi \in S_n} \left[\upsilon(P_i^\pi \cup \{i\}) - \upsilon(P_i^\pi)\right], \qquad \text{for all } i \in N,$$

where P_i^π denotes the set of all predecessors of i in π, that is, $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$. For further information about the Shapley value, please refer Shapley (1953).

For all finite set N, a *coalitional structure* over N is a partition of N, i.e., $B = \{B_1, \ldots B_m\}$ is a coalitional structure if it satisfies that $\bigcup_{1 \le k \le m} B_k = N$ and $B_k \bigcap B_l = \emptyset$ when $k \neq l$. We assume that $B_k \neq \emptyset$ for all k. The sets $B_k \in B$ are called unions or blocks. There are two trivial coalitional structures. The first one, which we denote by B^N, where only the grand coalition is formed, that is, $B^N = \{N\}$,

[1] S_n is the group of permutations of N.

and the second one is the coalitional structure where each union is a singleton and it is denoted by B^n, that is, $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$. We denote by $\mathcal{B}(N)$ the set of all coalitional structures over N. A TU-game (N, υ) with coalitional structure $B \in \mathcal{B}(N)$, is denoted by (B, υ). Let CSG^N denote the family of all TU-games with coalitional structure with player set N.

For every game $(B, \upsilon) \in CSG^N$, with $B = \{B_1, \ldots B_m\}$, the *quotient game* is the TU-game $(M, \upsilon_B) \in G^M$ where $M = \{1, 2, \ldots, m\}$ and $\upsilon_B(T) = \upsilon(\bigcup_{i \in T} B_i)$ for all $T \subseteq M$. (M, υ_B) is the TU-game induced by (B, υ) considering the coalitions of B as players. Notice that for the trivial coalitional structure B^n, we have $(M, \upsilon_B) \equiv (N, \upsilon)$. For all $\{k, l\} \subseteq M$, we say that B_k and B_l are *symmetric* in (B, υ) if k and l are symmetric in the TU-game (M, υ_B). For all $k \in M$, we say that B_k is a *null coalition* if k is a null player in the quotient game (M, υ_B).

Given $B \in \mathcal{B}(N)$, for all $k \in M$ and all $S \subseteq B_k$, we denote by $B\ |_S$ the new coalitional structure defined on $(\cup_{j \neq k} B_j) \cup S$ which appears when the complementary of S in B_k leaves the game, that is,

$$B\ |_S = \{B_1, \ldots, B_{k-1}, S, B_{k+1}, \ldots, B_m\}.$$

A *coalitional value* is a function Φ that assigns to each $(B, \upsilon) \in CSG^N$ a payoff vector $\Phi(B, \upsilon) \in \mathbb{R}^n$. One of the most important coalitional values is the *Owen value* (Owen (1977)). Given $(B, \upsilon) \in CSG^N$ and $k \in M$, for all $S \subseteq B_k$, we denote $\bar{S} = B_k \backslash S$. Owen (1977) defined a game $(M, \upsilon_{B|_S})$ that describes what would happen in the quotient game if union B_k was replaced by S, i.e.,

$$\upsilon_{B|_S}(T) = \upsilon(\cup_{j \in T} B_j \setminus \bar{S}), \qquad \text{for all } T \subseteq M.$$

Next, he defines an internal game (B_k, υ_k) by setting $\upsilon_k(S) = \mathrm{Sh}_k(M, \upsilon_{B|_S})$ for all $S \subseteq B_k$. Thus, $\upsilon_k(S)$ is the payoff to S in $\upsilon_{B|_S}$. The *Owen value* of the game (B, υ) is the payoff vector $\mathrm{Ow}(B, \upsilon) \in \mathbb{R}^n$ defined by

$$\mathrm{Ow}_i(B, \upsilon) = \mathrm{Sh}_i(B_k, \upsilon_k), \qquad \text{for all } k \in M \text{ and for all } i \in B_k.$$

This procedure has the next interpretation: First, the union k plays the quotient game (M, υ_B) among the unions; then, the payoff obtained is shared among its members by playing the internal game (B_k, υ_k). In both levels of bargaining, the payoffs are obtained using the Shapley value.

The Owen value satisfies the *quotient game property*: that is,

$$\sum_{i \in B_k} \mathrm{Ow}_i(B, \upsilon) = \mathrm{Sh}_k(M, \upsilon_B), \quad \text{for all } k \in M.$$

Notice that for the trivial coalition structures B^N and B^n,

$$\mathrm{Ow}(B^N, \upsilon) = \mathrm{Ow}(B^n, \upsilon) = Sh(N, \upsilon).$$

The Owen value can be defined by orders. Let B be a coalitional structure over N and $\pi \in S_n$. We say that π is *admissible respect to* B if for all $\{i,j,k\} \subseteq N$ and $l \in M$ such that $\{i,k\} \subseteq B_l$, if $\pi(i) < \pi(j) < \pi(k)$, then $j \in B_l$. In other words, π is admissible respect to B if players in the same group of B appear successively in π. We denote by $\mathcal{A}(B,N)$ the set of all admissible orders (on N) respect to B. The Owen value is given by the formula

$$\mathrm{Ow}_i(B,\upsilon) = \frac{1}{|\ \mathcal{A}(B,N)\ |} \sum_{\pi \in \mathcal{A}(B,N)} \left[\upsilon(P_i^\pi \cup \{i\}) - \upsilon(P_i^\pi)\right], \qquad \text{for all } i \in N, \tag{10.1}$$

where P_i^π denotes the set of all predecessors of i in π, that is, $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$.

Now, we present the axioms that characterize the Owen value on CSG^N. Let Φ be a coalitional value. We define

$$\Phi(B,\upsilon)[S] = \sum_{i \in S} \Phi_i(B,\upsilon), \qquad \text{for all } S \subseteq N.$$

(1) *Efficiency*: For all $(B,\upsilon) \in CSG^N$, $\Phi(B,\upsilon)[N] = \upsilon(N)$.
(2) *Additivity*: For all $(B,\upsilon),(B',\omega) \in CSG^N$, with $B = B'$, $\Phi(B,\upsilon+\omega) = \Phi(B,\upsilon) + \Phi(B,\omega)$.
(3) *Intracoalitional symmetry*: For all $(B,\upsilon) \in CSG^N$, all $k \in M$ and every $\{i,j\} \subseteq B_k$, if i and j are symmetric players in (N,υ), then $\Phi_i(B,\upsilon) = \Phi_j(B,\upsilon)$.
(4) *Coalitional symmetry:* For all $(B,\upsilon) \in CSG^N$ and all $\{k,l\} \subseteq M$, if B_k and B_l are symmetric players in (B,υ), then $\Phi(B,\upsilon)[B_k] = \Phi(B,\upsilon)[B_l]$.
(5) *Null player axiom:* For all $(B,\upsilon) \in CSG^N$ and all $i \in N$, if i is a null player in (N,υ), then $\Phi_i(B,\upsilon) = 0$.

Theorem 1 (Owen 1977). *A value Φ on CSG^N satisfies efficiency, additivity, intracoalitional symmetry, coalitional symmetry, and null player axiom if and only if Φ is the Owen value.*

For more details regarding the Owen value, see Owen (1977).

10.3 Additional Properties

In this section, we consider that (B,υ) and Φ are given and fixed. Given $S \subseteq N$, we assume that the members of S decide to play jointly and to leave the coalitions that they belong. Let $B'(S)$ be the *coalitional structure induced* by this behavior of the members of S, i.e.,

$$B'(S) = \{B'_1, B'_2, \cdots, B'_m, S\}, \quad \text{where } B'_k = B_k \backslash S, \text{ for all } B_k \in B.$$

We define (N, υ^*_Φ) where the characteristic function υ^*_Φ is given by

$$\upsilon^*_\Phi(S) = \Phi(B'(S), \upsilon)[S], \quad \text{for all } S \subseteq N.$$

This game is called the *gain game by association* because for all $S \subseteq N$, $\upsilon^*_\Phi(S)$ represent the amount obtain by S when the members of S decide to form one collaborative unit, leaving their original coalitions.

Now, we define the TU-game $(N, \upsilon^\diamond_\Phi)$, where $\upsilon^\diamond_\Phi$ is given by

$$\upsilon^\diamond_\Phi(S) = \sum_{B_k : B_k \cap S \neq \emptyset} \Big[\Phi(B, \upsilon)[B_k] - \Phi(B'(S), \upsilon)[B'_k]\Big], \quad \text{for all } S \subseteq N.$$

So, $\upsilon^\diamond_\Phi(S)$ represents the change in the payoffs of the groups B_k such that $B_k \cap S \neq \emptyset$ given some event that caused the formation of the new coalitional structure $B'(S)$. By this reason, $(N, \upsilon^\diamond_\varphi)$ is called the *lost game by association*.

Lemma 1. *Let $(B, \upsilon) \in CSG^N$ be a game with coalitional structure B. Then*

$$Ow(B, \upsilon^\diamond_{Ow})[B_k] = Ow(B, \upsilon)[B_k] = Ow(B, \upsilon^*_{Ow})[B_k], \quad \textit{for all } k \in M.$$

Proof. We will prove the left identity because for proving the right identity, we can use similar arguments. From the quotient game property, we have that $\mathrm{Ow}(B, \upsilon)[B_k] = \mathrm{Sh}_k(M, \upsilon_B)$. Then, calculating the worth of each $B_k \in B$ in the game $(N, \upsilon^\diamond_{\mathrm{Ow}})$, we can see that

$$\upsilon^\diamond_{\mathrm{Ow}}(B_k) = \mathrm{Ow}(B, \upsilon)[B_k].$$

We define $(M, \upsilon^B_{\mathrm{Ow}})$, where

$$\upsilon^B_{\mathrm{Ow}}(T) = \sum_{t \in T} \upsilon^\diamond_{\mathrm{Ow}}(B_t), \quad \text{for all } T \subseteq M.$$

Because the game $(M, \upsilon^B_{\mathrm{Ow}})$ is additive, we have that

$$\mathrm{Ow}(B, \upsilon^\diamond_{\mathrm{Ow}})[B_k] = \mathrm{Sh}_k(M, \upsilon^B_{\mathrm{Ow}}) = \mathrm{Ow}(B, \upsilon)[B_k].$$

Now, we are going to present the main result of the paper:

Theorem 2. *For all TU-game (N, υ) with coalitional structure B and for all $i \in N$, we have that*

$$Ow_i(B, \upsilon^\diamond_{Ow}) = Ow_i(B, \upsilon) = Ow_i(B, \upsilon^*_{Ow}).$$

Proof. Again, we will prove the left identity, because for proving the right identity, we can use similar arguments. We will prove that the result is valid for games in which $B = B^n$. For doing that, we will use the expression of the Owen value given in (10.1).

So, the Owen value for $i \in N$ in the game $(B, w_{\varphi}^{\diamond})$ is given by

$$\mathrm{Ow}_i(B, \upsilon_{\mathrm{Ow}}^{\diamond}) = \frac{1}{\mid \mathcal{A}(B,N) \mid} \sum_{\pi \in \mathcal{A}(B,N)} \left[\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi} \cup \{i\}) - \upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi})\right].$$

It is easy to check that

$$\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi} \cup \{i\}) = \sum_{j \in P_i^{\pi} \cup \{i\}} \mathrm{Ow}_j(B, \upsilon) \quad \text{and} \quad \upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi}) = \sum_{j \in P_i^{\pi}} \mathrm{Ow}_j(B, \upsilon).$$

Thus,

$$\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi} \cup \{i\}) - \upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi}) = \mathrm{Ow}_i(B, \upsilon),$$

and

$$\mathrm{Ow}_i(B, \upsilon_{\mathrm{Ow}}^{\diamond}) = \mathrm{Ow}_i(B, \upsilon).$$

Now we will prove that the result is valid when $B = B^N$. In this case we can see that

$$\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi} \cup \{i\}) = \upsilon(N) - \mathrm{Ow}(B'', \upsilon)[B_r \backslash \{i\}],$$

and

$$\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi}) = \upsilon(N) - \mathrm{Ow}(B', \upsilon)[B_r],$$

where $B'' = \{\{P_i^{\pi} \cup \{i\}\}, \{B_r \backslash \{i\}\}\}$ and $B' = \{\{P_i^{\pi}\}, \{B_r\}\}$ with $B_r = N \backslash P_i^{\pi}$.

Given that

$$\mathrm{Ow}(B'', \upsilon)[B_r \backslash \{i\}] = \upsilon(B_r \backslash \{i\}) + \frac{\upsilon(N) - \upsilon(B_r \backslash \{i\}) - \upsilon(P_i^{\pi} \cup \{i\})}{2},$$

$$\mathrm{Ow}(B', \upsilon)[B_r] = \upsilon(B_r) + \frac{\upsilon(N) - \upsilon(B_r) - \upsilon(P_i^{\pi})}{2},$$

then

$$\upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi} \cup \{i\}) - \upsilon_{\mathrm{Ow}}^{\diamond}(P_i^{\pi}) = \frac{\upsilon(B_r) - \upsilon(B_r \backslash \{i\}) + \upsilon(P_i^{\pi} \cup \{i\}) - \upsilon(P_i^{\pi})}{2}.$$

In this case it is easy to check that there is $\pi' \in \mathcal{A}(B, N)$ such that $P_i^{\pi'} \cup \{i\} = N \backslash \{P_i^{\pi}\} = B_r$ and $P_i^{\pi} = N \backslash \{P_i^{\pi'} \cup \{i\}\} = B_r \backslash \{i\}$. Thus, we conclude that

$$\begin{aligned}\mathrm{Ow}_i(B,\upsilon^{\diamond}_{\mathrm{Ow}}) &= \frac{1}{|\mathcal{A}(B,N)|}\sum_{\mathcal{A}(B,N)}\left[\frac{\upsilon(B_r)-\upsilon(B_r\backslash\{i\})}{2}+\frac{\upsilon(P_i^{\pi}\cup\{i\})-\upsilon(P_i^{\pi})}{2}\right]\\ &= \mathrm{Ow}_i(B,\upsilon).\end{aligned}$$

In general $m \geq 2$ and, when there is at least a coalition $B_k \in B$ such that $|B_k| \geq 2$, we will use the Lemma 1.

It is well known that the Owen value satisfies the axiom of balanced contributions within coalitions (Calvo et al. (1996)). That is, for any $i, j \in B_k \in B$, we have that

$$\mathrm{Ow}_i(B,\upsilon) - \mathrm{Ow}_i(B\setminus\{j\},\upsilon) = \mathrm{Ow}_j(B,\upsilon) - \mathrm{Ow}_j(B\setminus\{i\},\upsilon).$$

Adding over all players $j \in B_k \setminus \{i\}$ in the last expression, we have that

$$\begin{aligned}|B_k-1|\mathrm{Ow}_i(B,\upsilon) = &\sum_{j\in B_k\setminus\{i\}}\mathrm{Ow}_j(B,\upsilon) - \sum_{j\in B_k\setminus\{i\}}\mathrm{Ow}_j(B\setminus\{i\},\upsilon)\\ &+\sum_{j\in B_k\setminus\{i\}}\mathrm{Ow}_i(B\setminus\{j\},\upsilon),\end{aligned}$$

which can be written as follows:

$$|B_k|\,\mathrm{Ow}_i(B,\upsilon) = \mathrm{Ow}(B,\upsilon)[B_k] - \mathrm{Ow}(B\setminus\{i\},\upsilon)[B_k\setminus\{i\}] + \sum_{j\in B_k\setminus\{i\}}\mathrm{Ow}_i(B\setminus\{j\},\upsilon). \tag{10.5}$$

In the rest of the proof, we will use mathematical induction. Consider the case where $|B_k| = 2$. Then, we have that $B_k = \{i, j\}$ and thus

$$B\setminus\{j\} = \{B_1,\ldots,B_{k-1},B_k^*,B_{k+1},\ldots,B_m\}$$

where $B_k^* = \{i\}$. By using the Lemma 1, in Eq. (10.5) we have

$$\begin{aligned}2\,\mathrm{Ow}_i(B,\upsilon) &= \mathrm{Ow}(B,\upsilon)[B_k] - \mathrm{Ow}(B\setminus\{i\},\upsilon)[B_k\setminus\{i\}] + \mathrm{Ow}(B\setminus\{j\},\upsilon)[B_k^*],\\ &= \mathrm{Ow}(B,\upsilon^{\diamond}_{\mathrm{Ow}})[B_k] - \mathrm{Ow}(B\setminus\{i\},\upsilon^{\diamond}_{\mathrm{Ow}})[B_k\setminus\{i\}] + \mathrm{Ow}(B\setminus\{j\},\upsilon^{\diamond}_{\mathrm{Ow}})[B_k^*],\\ &= 2\,\mathrm{Ow}_i(B,\upsilon^{\diamond}_{\mathrm{Ow}}),\end{aligned}$$

thus

$$\mathrm{Ow}_i(B,\upsilon) = \mathrm{Ow}_i(B,\upsilon^{\diamond}_{\mathrm{Ow}}).$$

Suppose that the theorem is valid for coalitions $B_k \in B$ such that $|B_k| = s$. We will prove that it is valid when $|B_k| = s+1$. For doing that, we rewrite Eq. (10.5) where $|B_k| = s+1$. So

$$(s+1)\,\mathrm{Ow}_i(B,\upsilon) = \mathrm{Ow}(B,\upsilon)[B_k] - \mathrm{Ow}(B\setminus\{i\},\upsilon)[B_k\setminus\{i\}] + \sum_{j\in B_k\setminus\{i\}}\mathrm{Ow}_i(B\setminus\{j\},\upsilon).$$

The sum of the above equation involves the Owen value for $i \in B_k$. By induction hypothesis, it is equal to the Owen value in the TU-game $(N, w_\varphi^\diamond)$, and applying Lemma 1, we have

$$\begin{aligned}(s+1)\,\mathrm{Ow}_i(B,\upsilon) &= \mathrm{Ow}(B,\upsilon)[B_k] - \mathrm{Ow}(B\setminus\{i\},\upsilon)[B_k\setminus\{i\}] + \mathrm{Ow}(B\setminus\{j\},\upsilon)[B_k^*],\\ &= (s+1)\,\mathrm{Ow}_i(B,\upsilon_{\mathrm{Ow}}^\diamond).\end{aligned}$$

Then, we conclude that

$$\mathrm{Ow}_i(B,\upsilon) = \mathrm{Ow}_i(B,\upsilon_{\mathrm{Ow}}^\diamond).$$

The fact that the Owen value holds Theorem 2 can be seen as stability, because this value is immune to the manipulation that the games $\upsilon_{\mathrm{Ow}}^\diamond$ and υ_{Ow}^* mean.

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TEORÍA Y APLICACIONES

Campus Guanajuato

Economía Matemática y Econometría

Teoría y Aplicaciones

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Dos nuevas reglas para problemas de bancarrota

Oliver Juárez Romero, Jony Rojas Rojas, Francisco Sánchez Sánchez

Resumen. Una compañía va a la quiebra. Tiene varias deudas con diversos acreedores. El monto total de la deuda excede el valor de recuperación de la compañía. ¿Cómo se debe distribuir entre los acreedores el dinero disponible? Esta situación se conoce en la teoría de juegos como el problema de bancarrota.

En este artículo se hace una recapitulación de las principales reglas de división para problemas de bancarrota. Nuestro aporte consiste en proponer dos nuevas reglas para este tipo problemas. La primera de ellas está basada en el valor de Shapley de un juego cooperativo asociado al problema de bancarrota. La segunda regla es caracterizada en una subclase de problemas de bancarrota.

Introducción

Este artículo aborda el problema de dividir de forma justa un monto común entre varios agentes o el de distribuir un recurso escaso entre varios usos alternativos. Este problema constituye uno de los elementos que definen la naturaleza de la ciencia económica y la teoría de juegos cooperativos. Se formula en muy diversos contextos que van desde cuestiones de decisión personal, cómo la distribución de nuestra renta y la compra de distintos bienes y servicios, hasta los problemas de

decisión social más complejos, por ejemplo, cuánto dinero invertir en los programas de educación y cuánto en los de bienestar social.

En general los recursos de los que dispone una sociedad suelen ser limitados, por lo cual la escasez está siempre presente al momento de tomar decisiones. Nos ocuparemos aquí de un tipo de problemas de racionamiento extremadamente sencillo, pero que tienen una notable variedad de interpretaciones y una formulación matemática precisa. Se trata de repartir un bien entre los miembros de una sociedad cuando las demandas de ese bien exceden la cantidad disponible. Primero ilustraremos el problema con algunos escenarios similares al planteado y después lo precisaremos al definir el modelo.

- El Estado quiere definir su política de impuestos. Bajo el supuesto de que se conoce el ingreso de los contribuyentes y el monto que quiere recolectar el Estado, ¿cuánto debe pagar cada uno de los contribuyentes?
- Se desea repartir un bien escaso entre las familias de una población, por ejemplo agua en una ciudad del desierto. Las familias tienen diferentes necesidades de ese bien. Suponiendo que las demandas de las familias exceden la cantidad disponible de agua, se desea definir una política de racionamiento.
- Supongamos que varios agentes quieren contratar un servicio común. Supongamos además que se conoce el costo de contratación del servicio y el beneficio que obtendrá cada agente. ¿Cuánto debe pagar por el servicio cada uno de ellos?
- En su testamento un hombre dispone que a su muerte se le den montos específicos de dinero a cada una de sus esposas. Al morir el testador, no hay suficiente dinero para cubrir todos los montos. ¿Cuánto se le debe de dar a cada una de las esposas?

Nótese que estos cuatro problemas son matemáticamente equivalentes. Se quiere dividir un bien homogéneo entre un conjunto finito de agentes que lo reclaman de diferente manera, sin embargo no es posible satisfacer todos los reclamos o las ganancias de los agentes. Este problema es conocido como el problema de bancarrota. En la siguiente sección formalizaremos estas ideas.

El problema fue considerado inicialmente por O'Neil (1982), pero Aumman y Maschler (1985) fueron los primeros en desarrollar el análisis matemático y las soluciones más aceptadas a este problema. Aumman (2010) volvió a considerar el problema comparando las soluciones con las propuestas en el Talmud.

Las notas que aquí presentamos están organizadas como sigue: en la sección 1 se introduce el modelo formal y se presentan las cuatro reglas de división clásicas que actualmente se usan para resolver los problemas de bancarrota. En la sección 2 se introducen los conceptos básicos de la teoría de juegos cooperativos. En la sección 3 se definen dos nuevas reglas de división: se caracteriza axiomáticamente la primera y se muestran las propiedades que posee la segunda. Por último, en la sección 4 se comenta cómo puede elegirse una regla de división para un problema específico.

1. Problemas de bancarrota

Sea $N = \{1, \dots, \mathrm{n}\}$ un conjunto de agentes y supongamos que cada uno de ellos reclama una cantidad c_i con $i \in N$. También supongamos que se cuenta con una cantidad E para cubrir estos reclamos. Con el fin de que el problema sea de interés, supongamos que con esta cantidad no se alcanza a cubrir todos los reclamos, es decir, suponemos que $\sum_{i \in N} c_i > E$.

Definición. *Un problema de bancarrota es un par* (E, c)*, donde* $E \in R_+$ *representa la cantidad de bien disponible para repartir y* $c \in R^N$ *es el vector de reclamos, con* $\sum_{i \in N} c_i > E$.

Llamamos B a la familia de problemas de bancarrota con n agentes. Para simplificar la notación llamaremos C a la reclamación agregada, es decir, $C = \sum_{i \in N} c_i$. La diferencia $C - E > 0$ representa el déficit o la pérdida colectiva del problema de bancarrota $(E, c) \in B$.

Definición. *Una regla de división para un problema de bancarrota es una función* F *que a cada problema* $(E, c) \in B$*, asocia un único punto* $F(E, c) \in R^N$ *tal que*

(1) $0 \leq F_i(E, c) \leq c_i$*, para toda* $i \in N$.

(2) $\sum_{i \in N} F_i(E, c) = E$.

La regla de división $F(E, c)$ se interpreta como una forma de distribuir E. La condición (1) establece que ningún agente perciba una cantidad mayor a la que demanda ni menor que cero. La condición (2) establece el requisito de eficiencia: dado que no hay suficiente recurso para cubrir todas las reclamaciones, la cantidad de bien disponible debe distribuirse en su totalidad.

A continuación presentamos las principales reglas de división para problemas de bancarrota. La regla proporcional es probablemente la forma más conocida y utilizada para resolver problemas de esta naturaleza. Esta regla distribuye la cantidad de bien disponible proporcionalmente a las reclamaciones. Formalmente:

Definición. *La regla proporcional* P *es aquella regla que para cada problema* $(E, c) \in B$ *propone la división* $P(E, c) = \lambda c$*, donde* $\lambda = (E/C)$.

La regla igualitaria propone resolver los problemas de bancarrota igualando las cantidades percibidas por todos los agentes, bajo la restricción de que nadie obtenga más de lo que reclama. Formalmente se define del siguiente modo:

Definición. *La regla de igual ganancia* I *es aquella que para cada problema* $(E, c) \in B$*, y para cada agente* $i = 1,2, \dots, n$*, propone la siguiente división:*

$$I_i(E, c) = min\{\lambda, c_i\},$$

donde λ *resuelve la ecuación* $\sum_{i \in N} min\{\lambda, c_i\} = E$.

La regla de igual pérdida propone distribuir por igual el déficit entre los agentes. Ello supondrá dar a cada agente una cantidad

$$x_i = c_i - \frac{C-E}{n}$$

pero como esta cantidad puede ser negativa para algún agente, la solución se aplica con la restricción de que $x_i \geq 0$. Desde un punto de vista geométrico esta forma de resolver los problemas de bancarrota es muy natural, dado que equivale a elegir el punto del conjunto presupuestario que está más cerca del vector de reclamaciones c, en términos de la distancia euclideana. Formalmente:

Definición. *La regla de igual pérdida* IP *es aquella que para cada problema* $(E, c) \in B$*, y para cada agente* $i = 1,2, \dots, n$*, propone la división*

$$IP_i(E, c) = max\{0, c_\{i\} - \mu\},$$

donde $\mu > 0$ *resuelve la ecuación* $\sum_{i \in N} max\{0, c_\{i\} - \mu\} = E$.

La regla del Talmud puede identificarse como la aplicación de un principio de protección a los agentes según el cual la pérdida de cada individuo será del mismo tipo que la pérdida social. En particular, nadie obtendrá más de la mitad de su deuda cuando la cantidad disponible sea inferior a la mitad de la deuda agregada; y

nadie perderá más de la mitad de su reclamación cuando la cantidad disponible supere la mitad de la reclamación total. La regla del Talmud se define formalmente del siguiente modo:

Definición. *La solución del Talmud* T *es aquella que para cada problema* $(E, c) \in B$ *y para cada agente* $i = 1,2, \dots, n$*, propone la siguiente división:*

$$T_i(E, c) = min\left\{\frac{1}{2}c_i, \lambda\right\} \quad si \quad E \leq \frac{C}{2} \quad y$$

$$T_i(E, c) = min\left\{\frac{1}{2}c_i, c_i - \mu\right\} \quad si \quad E > \frac{C}{2},$$

donde $\lambda, \mu \geq 0$ *resuelven las ecuaciones*

$$\sum_{i \in N} min\left\{\frac{1}{2}c_i, \lambda\right\} = E \quad y \quad \sum_{i \in N} min\left\{\frac{1}{2}c_i, c_i - \mu\right\} = E.$$

Esta regla es una generalización de una regla encontrada en el Talmud Babilónico, atribuido al Rabí Natan, la cual presentamos en el siguiente ejemplo:

Ejemplo: Un hombre, que tiene deudas por 100, 200 y 300 (en algunas unidades), muere y lo que deja como herencia no es suficiente para pagar el total de sus deudas. En este caso el Talmud indica cómo se debe pagar a los acreedores en tres casos posibles: cuando la herencia es 100, 200 y 300. La solución sugerida por el Talmud se muestra en el siguiente cuadro:

	Acreedor 1	**Acreedor 2**	**Acreedor 3**
Deuda	100	200	300
100	33(1/3)	33(1/3)	33(1/3)
200	50	75	75
300	50	100	150

2. Juegos cooperativos

En esta sección recordaremos brevemente algunos conceptos y resultados de la teoría de juegos que son necesarios para plantear nuestra propuesta. Empezaremos con la definición de juego cooperativo.

Definición. *Un juego cooperativo con utilidad transferible (o simplemente un juego) es un par* (N, v)*, donde* $N = \{1, \dots, n\}$ *es un conjunto de jugadores, y* $v: 2^N \to R$ *es una función que asigna a cada coalición* S *de* N *un número real* $v(S)$ *con la condición de que* $v(\emptyset) = 0$*. El conjunto de los juegos cooperativos se denota por* G^N.

Las principales interrogantes que se abordan en los juegos cooperativos son las siguientes:

1. Dado el juego (N, v), ¿cuándo se forma la gran coalición N?
2. Dado que se forma N, ¿cómo se divide $v(N)$ entre los jugadores?

En general, en la teoría de juegos cooperativos se define una función f que asocia a cada juego (N, v) un vector de pagos $x \in R^N$. Luego se pide que f satisfaga ciertas propiedades deseables y que éstas la determinen unívocamente. Estas propiedades deben de ser sencillas, pocas y deseables. Implícitamente, se supone que si los jugadores están de acuerdo con las propiedades utilizadas también lo están con la solución que de ella se desprende. Cuando se consigue esto se dice que se ha caracterizado la solución f. El caso más conocido de juegos cooperativos es el valor de Shapley, caracterizado por Lloyd Stowell Shapley en (1953).

Teorema (Shapley, 1953). *Existe una única solución para los juegos cooperativos que satisface eficiencia, nulidad, simetría y linealidad. Además, esta solución está dada por*

$$Sh_i(N,v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!}\big(v(S \cup \{i\}) - v(S)\big)$$

donde s *representa la cardinalidad de* $S \subseteq N$.

Para conocer mayores detalles acerca de las propiedades que caracterizan el valor de Shapley (2013). Otra solución importante es la propuesta por Gillies (1953).

Definición. *El núcleo de un juego* $(N,v) \in G^N$ *es el subconjunto de* R^N, $C(v)$, *dado por*

$$C(v) = \{x \in R^N \,|\, \sum_{i \in S} x_i \geq v(S), \sum_{i \in N} x_i = v(N)\}.$$

3. Nuevas reglas

En la sección anterior presentamos cuatro reglas para el problema de bancarrota que proporcionan una distribución de la cantidad del bien a repartir como función de los reclamos de los agentes. Cada una de esas reglas fue diseñada para dividir el recurso disponible, pero pasan por alto que estas situaciones surgen debido a que la cantidad en conflicto es la suma de los reclamos menos el total del recurso. Por ello en esta sección proponemos dos nuevas reglas para los problemas de bancarrota. Con la primera regla distribuimos el monto en conflicto entre los agentes, de acuerdo con el valor de Shapley de un juego cooperativo, para obtener cuánto hay que restar de cada reclamación para eliminar el conflicto. Luego, cada agente recibe la cantidad que reclama menos el pago que le corresponde, de acuerdo con el valor de Shapley. Formalmente:

Dado un problema de bancarrota $(E,c) \in B$ *definimos el juego cooperativo* $(N, v_{\{E,C\}})$, *donde para cada coalición*

$$S \subseteq N,\ v_{\{E,C\}}(N,v) = max\,\{c(S) - E, 0\} \text{ con } c(S) = \sum_{i \in S} c_i.$$

Notemos que la cantidad en conflicto para los miembros de $S \subseteq N$ es precisamente $c(S) - E$, luego el valor de cada coalición en este juego es la cantidad en conflicto de los miembros de la coalición.

Proposición. *Para cada problema de bancarrota* $(E, c) \in B$, *el juego cooperativo* $(N, v_{\{E,C\}})$ *satisface las siguientes condiciones:*

1. $v(S) \leq v(T)$ *para todo* $S \subseteq T \subseteq N$.
2. $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ *para todo* $S \subseteq N$ *y todo* $i \notin S$.

Demostración. Sea

$$S \subseteq T \subseteq N.$$

Dado que

$$c(S) \leq c(T),$$

entonces

$$v(S) = max\{c(S) - E, 0\} \leq max\{c(T) - E, 0\} = v(T).$$

Sea

$$S \subseteq T \subseteq N. \text{ Si } v(T) = 0 \text{ o } v(\{i\}) = 0,$$

entonces por la parte 1 se cumple la parte 2.

Supongamos que

$$v(T) > 0 \text{ y } v(\{i\}) > 0,$$

entonces

$$v(T \cup \{i\}) > 0.$$

Luego,

$$v(T) + v(\{i\}) = c(T) + c_i - 2E < c(T) + c_i - E = v(T \cup \{i\}).$$

Lo cual concluye la prueba.

Proposición. *Para cada problema de bancarrota* $(E, c) \in B$ *el juego cooperativo* $(N, v_{\{E,C\}})$ *tiene un núcleo no vacío.*

Demostración. Dado que un juego convexo tiene un núcleo no vacío, solo necesitamos probar que el juego $v_{\{E,C\}}$ es convexo. Para esto necesitamos probar que para todo

$$S \subseteq T \subseteq N \text{ y todo } i \in N \backslash T,$$

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

$$Si\ v(T) = 0,$$

entonces

$$v(S) = 0$$

y la desigualdad anterior se cumple por la parte 1 de la proposición anterior.

Supongamos que

$$v(T) > 0 \text{ y } v(S) = 0,$$

entonces

$$max\ \{c_i + c(S) - E, 0\} \leq c_i = v(T \cup \{i\}) - v(T) \text{ porque } c(S) - E \leq 0.$$

Ahora, si

$$v(T) > 0 \text{ y } v(S) > 0,$$

entones

$$c_i = v(S \cup \{i\}) - v(S) = v(T \cup \{i\}) - v(T).$$

Todo esto implica que el juego $v_{\{E,C\}}(N, v)$ es convexo y por lo tanto tiene un núcleo no vacío, lo cual termina la prueba.

A continuación propondremos una forma de dividir el estado E entre los agentes: Primero, usando el valor de Shapley del juego $v_{\{E,c\}}$ dividimos entre los agentes la cantidad en conflicto, $C(N) - E$. Luego, cada jugador recibe su reclamación menos lo que él o ella haya obtenido de la cantidad en conflicto $C(N) - E$. Formalmente:

Para cada problema de bancarrota $(E, c) \in B$ *y cada* $i \in N$, *proponemos la división de* E *dado por* $\phi_i(E, c) = c_i - Sh_i(N, v_{\{E,C\}})$

Proposición. *La función* ϕ *es una regla de división.*

Demostración. Dado que el valor de Shapley es eficiente, entonces

$$\sum_{i \in N} \phi_i(E, c) = c(N) - (c(N) - E) = E.$$

Ahora sólo se necesita verificar que para todo $i \in N$ y todo problema de bancarrota (E, c) se cumple que $0 \leq \phi_i(E, c) \leq c_i$. Pero esto es equivalente a probar que

$$0 \leq Sh_i\big(N, v_{\{E,C\}}\big) \leq c_i.$$

Por la parte 2 de la proposición anterior tenemos que para todo

$$i \in N \; Sh_i(N, v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S))$$

$$= \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} c_i$$

$$= v(\{i\}) \geq 0.$$

Ahora, notemos que

$$Sh_i(N, v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S))$$

$$= \sum_{S \not\ni i, v(S)=0} \frac{s!(n-s-1)!}{n!} v(S \cup \{i\}) - \sum_{S \not\ni i, v(S)>0} \frac{s!(n-s-1)!}{n!} c_i$$

$$= \sum_{S \not\ni i, v(S)=0} \frac{s!(n-s-1)!}{n!} (c(S) + c_i - E) + \sum_{S \not\ni i, v(S)>0} \frac{s!(n-s-1)!}{n!} c_i$$

$$= \sum_{S \not\ni i, v(S)=0} \frac{s!(n-s-1)!}{n!} (c(S) - E) + \alpha c_i$$

donde

$$\alpha = \sum_{S \not\ni i, v(S)=0} \frac{s!(n-s-1)!}{n!} + \sum_{S \not\ni i, v(S)>0} \frac{s!(n-s-1)!}{n!}$$

Como $\alpha \in (0,1)$ y $c(S) - E \leq 0$ se sigue que $Sh_i(N, v) \leq c_i$.

Así $0 \leq Sh_i(N, v) \leq c_i$ que y por lo tanto, $0 \leq \phi_i(E, c) \leq c_i$ para todo $i \in N$ y todo problema de bancarrota (E, c).

En la siguiente proposición mostraremos que nuestra regla de división trata de la misma forma a los agentes que reclaman cantidades iguales del recurso, y que si pasamos la formulación de un problema de pesos a dólares entonces la solución en dólares debe ser exactamente la correspondiente al tipo de cambio.

Proposición. *La regla* $\phi_i(E,c) = c_i - Sh_i(N, v_{\{E,C\}})$ *satisface las siguientes propiedades:*
I. Tratamiento igualitario: Si $c_i = c_j$*, entonces* $F_i(E,c) = F_j(E,c)$ *para todo* $i, j \in N$.
II. Independencia de escala: $F(\lambda E, \lambda c) = F(E,c)$ *para todo* $\lambda > 0$.

Demostración.
Tratamiento igualitario. Si $c_i = c_j$, entonces i, j son jugadores simétricos en el juego $(N, v_{\{E,C\}})$, y por lo tanto, $Sh_i\big(N, v_{\{E,C\}}\big) = Sh_j(N, v_{\{E,C\}})$.
Luego, $\phi_i(E,c) = c_i - Sh_i\big(N, v_{\{E,C\}}\big) = c_j - Sh_j\big(N, v_{\{E,C\}}\big) = \phi_j(E,c)$.
Independencia de escala. Dado que $v_{\{\lambda E, \lambda C\}} = \lambda v_{\{E,C\}}$ y el valor de Shapley es lineal, entonces $Sh_i\big(N, v_{\{\lambda E, \lambda C\}}\big) = \lambda Sh_i\big(N, v_{\{E,C\}}\big)$ para $i \in N$. Esto implica que $\phi_i(\lambda E, \lambda c) = \lambda c_i - \lambda Sh_i\big(N, v_{\{E,C\}}\big) = \lambda \phi_i(E,c)$ para todo $i \in N$.

Ahora, consideraremos la subclase de problemas de bancarrota donde el recurso disponible más el monto total de las reclamaciones es al menos n-veces la máxima reclamación, y a lo más $2n$-veces la mínima reclamación. Para esta subclase, que la denotaremos como D', caracterizamos una regla utilizando la siguiente propiedad:

Definición. *Una regla* F *satisface Preservación de diferencias si para todo problema de bancarrota* $(E,c) \in D$ *se tiene que*

$$F_i(E,c) - F_j(E,c) = c_j - c_i$$

para todo $i, j \in N$.

Teorema. *Dado un problema de bancarrota* $(E, c) \in D'$*, existe una única regla* ψ *que preserva diferencias. Además, para toda* $i \in N$ *está proporciona la siguiente división del bien disponible:*

$$\psi_i(E, c) = \frac{E}{n} + \bar{C} - c_i,$$

donde $\bar{C} = \frac{C}{n}$.

Demostración.

Existencia. Nótese que para todo problema de bancarrota $(E, c) \in D'$, ya que:

$$\sum_{i \in N} \psi_i(E, c) = E + C - \sum_{i \in N} c_i = E.$$

Como

$$max\{nc_i | i \in N\} \leq E + C \leq min\{2nc_i | i \in N\},$$

entonces

$$c_i \geq \frac{E}{n} + \bar{C} - c_i = \frac{E+C}{n} - c_i \geq 0.$$

Unicidad. Supongamos que existen dos reglas ψ y φ que satisfacen preservación de diferencias. Entonces

$$\psi_i(E, c) - \psi_j(E, c) = \varphi_i(E, c) - \varphi_j(E, c)$$

para todo $i,\ j \in N$. Dado que ambas son reglas de división,

$$\sum_{i \in N} \psi_i(E, c) - \sum_{i \in N} \psi_j(E, c) = \sum_{i \in N} \varphi_i(E, c) - \sum_{i \in N} \varphi_j(E, c)$$

$$E - n\psi_j(E, c) = E - n\varphi_j(E, c)$$

Esta última igualdad implica que $\psi_j(E, c) = \varphi_j(E, c)$. Como $j \in N$ es arbitrario, entonces $\psi(E, c) = \varphi(E, c)$.

Un ejemplo numérico sencillo: con el fin de ilustrar nuestra regla consideremos la siguiente situación: Imaginemos a dos hermanos, María y Juan, que acuerdan comprar una casa que cuesta 60 000 dólares con la idea de venderla un año después por un precio superior. María aporta 40 000 dólares y Juan los 20 000 dólares restantes. Supongamos que cuando llega el momento de vender la casa para recuperar

la inversión con las posibles ganancias, resulta que el valor de la casa ha descendido, por ejemplo debido al incremento de la delincuencia en la zona, y que ahora es inferior al precio que pagaron Juan y María por su adquisición. Tenemos aquí un problema de bancarrota: hay que repartirse el valor de venta de la casa cuando no alcanza para cubrir el valor de la inversión. Consideremos tres escenarios distintos que corresponden a tres valores de venta diferentes. En el primer escenario la casa se vende por 58 000 dólares. En el segundo, el valor únicamente alcanza los 35 000 dólares. Por último, analizaremos el caso especialmente negativo en que la casa solo puede venderse por 28 000 dólares.

¿Cómo se reparte el valor de la casa en cada uno de estos casos, como función de las aportaciones iniciales, de acuerdo con las diferentes reglas de división? Veámoslo en el cuadro 1. Cada columna informa sobre el reparto correspondiente a un valor de venta de la casa (que juega el papel de E) para las diferentes soluciones propuestas. En cada celda hay dos números que describen lo que le corresponde a Juan (primer número) y a María (segundo número) en cada caso, siempre partiendo de la base de que Juan tiene una reclamación c_{J}= 20 000 dólares y María una reclamación c_{M}= 40 000 dólares.

Cuadro 1. Repartición de la casa

Regla	E = 58,000	E = 35,000	E = 28,000
Proporcional	(58,000/3,116,000/3)	(35,000/3, 70000/3)	(28,000/3,56.000/3)
Igual ganancia	(20,000, 38,000)	(17, 500, 17,500)	(14,000, 14,000)
Igual pérdida	(19,000, 39,000)	(7,500, 27,500)	(4,000, 24,000)
Talmud	(19,000, 39,000)	(10,000, 25,000)	(10,000,1 8,000)
Nuestra	(19,000, 39,000)	(10, 000, 25,000)	(10,000,18,000)

4. Cómo elegir una regla

Una pregunta natural que surge a la hora de resolver un problema de bancarrota es ¿cuál regla usar? o ¿cuál regla es superior a las otras? Estas interrogantes son difíciles de contestar, ya que dependen de la situación de la vida real en la que se quieren aplicar. Lo que podemos argumentar es que existen situaciones en las que una regla tiene una mejor interpretación que las otras. Por ejemplo, la regla proporcional se utiliza cuando una compañía de accionistas se va a la quiebra. La regla del Talmud del Rabí Natan puede ser considerada deseable cuando los agentes quieren repartir de forma igualitaria la parte reclamada de la deuda. Nuestra regla se puede aplicar cuando los agentes desean dividir el monto en conflicto, $C - E$. En conclusión, cada regla proporciona algo de luz con relación a la realidad.

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