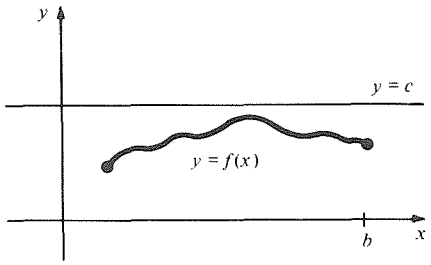


**Figure 3.1.5.** The graph starts below the line $y = c$ and never pierces the line, so it stays below the line.

In geometric language, this second version of the theorem says: "The graph of a continuous function which never meets a horizontal line must remain on one side of it." The first version says: "If the graph of a continuous function passes from one side of a horizontal line to the other, the graph must meet the line somewhere." You should convince yourself that these two statements really mean the same thing.

In practice, the second version of the intermediate value theorem can be useful for determining the sign of a function on intervals where it has no roots, as in the following example.

**Example 8** Suppose that $f$ is continuous on $[0, 3]$, that $f$ has no roots on the interval, and that $f(0) = 1$. Prove that $f(x) > 0$ for *all* $x$ in $[0, 3]$.

**Solution** Apply the intermediate value theorem (version 2), with $c = 0$, to $f$ on $[0, b]$ for each $b$ in $(0, 3]$. Since $f$ is continuous on $[0, 3]$, it is continuous on $[0, b]$; since $f(0) = 1 > 0$, we have $f(b) > 0$. But $b$ was anything in $(0, 3]$, so we have proved what was asked. ▲

## Exercises for Section 3.1

1. Decide where each of the functions whose graph is sketched in Fig. 3.1.6 is continuous.

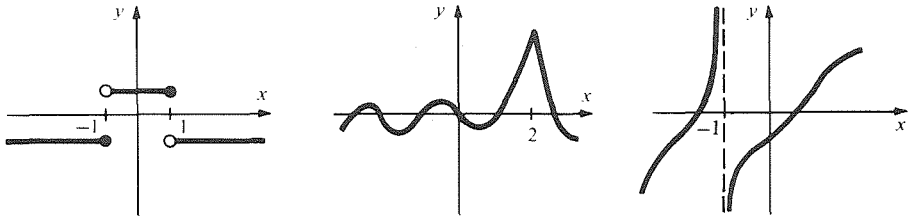


**Figure 3.1.6.** Where are these continuous?

2. Which of the functions in Fig. 3.1.7 are continuous at $x_0 = 1$?

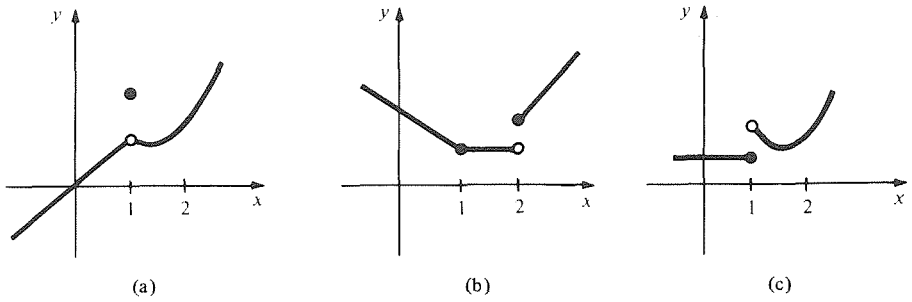


**Figure 3.1.7.** Which of these functions are continuous at $x_0 = 1$?

3. Show that $f(x) = (x^2 + 1)(x^2 - 1)$ is continuous at $x_0 = 0$.
4. Show that $f(x) = x^3 + 3x^2 - 2x$ is continuous at $x_0 = 3$.
5. Prove that $(x^2 - 1)/(x^3 + 3x)$ is continuous at $x_0 = 1$.
6. Prove that $(x^4 - 8)/(x^3 + 2)$ is continuous at $x_0 = 0$.
7. Where is $(x^2 - 1)/(x^4 + x^2 + 1)$ continuous?
8. Where is $(x^4 + 1)/(x^3 - 8)$ continuous?
9. Let $f(x) = (x^3 + 2)/(x^2 - 1)$. Show that $f$ is continuous on $[-\frac{1}{2}, \frac{1}{2}]$.
10. Is the function $(x^3 - 1)/(x^2 - 1)$ continuous at 1? Explain your answer.
11. Let $f(x)$ be the step function defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ -2 & \text{if } x \geqslant 0. \end{cases}$$

Show that $f$ is discontinuous at 0.

12. Let $f(x)$ be the *absolute value function*: $f(x) = |x|$; that is,

$$f(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Show that $f$ is continuous at $x_0 = 0$.

13. Using the laws of limits, show that if $f$ and $g$ are continuous at $x_0$, so is $f + g$.
14. If $f$ and $g$ are continuous at $x_0$ and $g(x_0) \neq 0$, show that $f/g$ is continuous at $x_0$.
15. Where is $f(x) = 8x^3/\sqrt{x^2 - 8}$ continuous?
16. Where is $f(x) = 9x^2 - 3x/\sqrt{x^4 - 2x^2 - 8}$ continuous?
17. Show that the equation $-s^5 + s^2 = 2s - 6$ has a solution.
18. Prove that the equation $x^3 + 2x - 1 = 7$ has a solution.
19. Prove that $f(x) = x^8 + 3x^4 - 1$ has at least two distinct zeros.
20. Show that $x^4 - 5x^2 + 1$ has at least two distinct zeros.
21. The roots of $f(x) = x^3 - 2x - x^2 + 2$ are $\sqrt{2}$, $-\sqrt{2}$, and 1. By evaluating $f(-3)$, $f(0)$, $f(1.3)$, and $f(2)$, determine the sign of $f(x)$ on each of the intervals between its roots.
22. Use the method of bisection to approximate $\sqrt{7}$ to within two decimal places. [*Hint:* Let $f(x) = x^2 - 7$. What should you use for $a$ and $b$?]
23. Find a solution of the equation $x^5 - x = 3$ to an accuracy of 0.01.
24. Find a solution of the equation $x^5 - x = 5$ to an accuracy of 0.1.
25. Suppose that $f$ is continuous on $[-1, 1]$ and that $f(x) - 2$ is never zero on $[-1, 1]$. If $f(0) = 0$, show that $f(x) < 2$ for all $x$ in $[-1, 1]$.
26. Suppose that $f$ is continuous on $[3, 5]$ and that $f(x) \neq 4$ for all $x$ in $[3, 5]$. If $f(3) = 3$, show that $f(5) < 4$.
27. Let $f(x)$ be 1 if a certain sample of lead is in the solid state at temperature $x$; let $f(x)$ be 0 if it is in the liquid state. Define $x_0$ to be the melting point of this lead sample. Is there any way to define $f(x_0)$ so as to make $f$ continuous? Give reasons for your answer, and supply a graph for $f$.
28. An empty bucket with a capacity of 10 liters is placed beneath a faucet. At time $t = 0$, the faucet is turned on, and water flows from the faucet at the rate of 5 liters per minute. Let $V(t)$ be the volume of the water in the bucket at time $t$. Present a plausible argument showing that $V$ is a continuous function on $(-\infty, \infty)$. Sketch a graph of $V$. Is $V$ differentiable on $(-\infty, \infty)$?
29. Let $f(x) = (x^2 - 4)/(x - 2)$, $x \neq 2$. Define $f(2)$ so that the resulting function is continuous at $x = 2$.
30. Let $f(x) = (x^3 - 1)/(x - 1)$, $x \neq 1$. How should $f(1)$ be defined in order that $f$ be continuous at each point?
31. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1, \\ ? & \text{if } 1 \leqslant x \leqslant 3, \\ x - 6 & \text{if } 3 < x. \end{cases}$$

How can you define $f(x)$ on the interval $[1, 3]$ in order to make $f$ continuous on $(-\infty, \infty)$? (A geometric argument will suffice.)

32. Let $f(x) = x + (4/x)$ for $x \leqslant -\frac{1}{2}$ and $x \geqslant 2$. Define $f(x)$ for $x$ in $(-\frac{1}{2}, 2)$ in such a way that the resulting function is continuous on the whole real line.
33. Let $f(x)$ be defined by $f(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$. How should you define $f(1)$ to make the resulting function continuous? [*Hint:* Plot a graph of $f(x)$ for $x$ near 1 by factoring the numerator.]
34. Let $f(x)$ be defined by $f(x) = 1/x$ for $x \neq 0$. Is there any way to define $f(0)$ so that the resulting function will be continuous?
35. Sketch the graph of a function which is continuous on the whole real line and differentiable everywhere except at $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
36. Is a function which is continuous at $x_0$ necessarily differentiable there? Prove or give a counterexample.
37. The function $f(x) = 1/(x - 1)$ never takes the value zero, yet $f(0) = -1$ is negative and $f(2) = 1$ is positive. Why isn't this a counterexample to the intermediate value theorem?
38. "Prove" that you were once exactly one meter tall.

In Exercises 39–42, use the method of bisection to find a root of the given function, on the given interval, to the given accuracy.

39. $x^3 - 11$ on $[2, 3]$; accuracy $\frac{1}{100}$.
40. $x^3 + 7$ on $[-3, -1]$; accuracy $\frac{1}{100}$.
41. $x^5 + x^2 + 1$ on $[-2, -1]$; accuracy $\frac{1}{1000}$.
42. $x^4 - 3x - 2$ on $[1, 2]$; accuracy $\frac{1}{500}$.

★43. In the method of bisection, each estimate of the solution of $f(x) = c$ is approximately twice as accurate as the previous one. Examining the list of powers of 2: 2, 4, 8, *16*, 32, 64, *128*, 256, 512, *1024*, 2048, 4096, 8192, *16384*, 32768, 65536, *131072*, . . . —suggests that we get one more decimal place of accuracy for every three or four repetitions of the procedure. Explain.

★44. Using the result of Exercise 43, determine how many times to apply the bisection procedure to guarantee the accuracy $A$ for the interval $[a, b]$ if:
(a) $A = \frac{1}{100}$; $[a, b] = [3, 4]$.
(b) $A = \frac{1}{1000}$; $[a, b] = [-1, 3]$.
(c) $A = \frac{1}{700}$; $[a, b] = [11, 23]$.
(d) $A = \frac{1}{15}$; $[a, b] = [0.1, 0.2]$.

★45. Can you improve the method of bisection by choosing a better point than the one halfway between the two previous points? [*Hint:* If $f(a)$ and $f(b)$ have opposite signs, choose the point where the line through $(a, f(a))$ and $(b, f(b))$ crosses the $x$ axis.] Is there a method of division more appropriate to the decimal system?

Do some experiments to see whether your method gives more accurate answers then the bisection method in the same number of steps. If so, does the extra accuracy justify the extra time involved in carrying out each step? You might wish to compare various methods on a competitive basis, either with friends, with yourself by timing the calculations, or by timing calculations done on a programmable calculator or microcomputer.

★46. Prove that if $f$ is continuous on an interval $I$ (not necessarily closed) and $f(x) \neq 0$ for all $x$ in $I$, then the sign of $f(x)$ is the same for all $x$ in $I$.

★47. Suppose that $f$ is continuous at $x_0$ and that in any open interval $I$ containing $x_0$ there are points $x_1$ and $x_2$ such that $f(x_1) < 0$ and $f(x_2) > 0$. Explain why $f(x_0)$ must equal 0.

★48. (a) Suppose that $f$ and $g$ are continuous on the real line. Show that $f - g$ is continuous. (b) Suppose that $f$ and $g$ are continuous functions on the whole real line. Prove that if $f(x) \neq g(x)$ for all $x$, and $f(0) > g(0)$, then $f(x) > g(x)$ for all $x$. Interpret your answer geometrically.

★49. Prove that any odd-degree polynomial has a root by following these steps.
1. Reduce the problem to showing that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has a root, where $a_i$ are constants and $n$ is odd.
2. Show that if $|x| > 1$ and $|x| > 2 \times \{|a_0| + \cdots + |a_{n-1}|\}$, then $f(x)/x^n$ is positive.
3. Conclude that $f(x) < 0$ if $x$ is large negative and $f(x) > 0$ if $x$ is large positive.
4. Apply the intermediate value theorem.

★50. Show that the polynomial $x^4 + bx + c = 0$ has a (real) root if $256c^3 < 27b^4$. Hint: use the fact that the tangent line is horizontal at the lowest point on the graph.

# 3.2 Increasing and Decreasing Functions

*The sign of the derivative indicates whether a function is increasing or decreasing.*

We begin this section by defining what it means for a function to be increasing or decreasing. Then we show that a function is increasing when its derivative is positive and is decreasing when its derivative is negative. Local maximum and minimum points occur where the derivative changes sign.

We can tell whether a function is increasing or decreasing at $x_0$ by seeing how its graph crosses the horizontal line $y = f(x_0)$ at $x_0$ (see Fig. 3.2.1). This

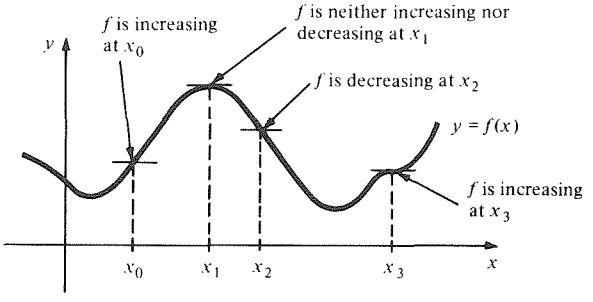


**Figure 3.2.1.** Places where the function $f$ is increasing and decreasing.