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Corresponding Author: dr victor nunez, ph.d.

Corresponding Author's Institution: cimat

First Author: victor nunez, ph.d.

Order of Authors: victor nunez, ph.d.; mercedes jordan-santana, ph.d.

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CLASSICAL DRAWINGS OF BRANCHED COVERINGS

VÍCTOR NÚÑEZ AND MERCEDES JORDÁN–SANTANA

Abstract. For a branched covering $\varphi : M^3 \to (S^3, k)$, we give a description of how to embed $\varphi^{-1}(B)$ in M^3 to determine the link type of $\varphi^{-1}(k) \subset M^3$, where $B \subset S^3$ is a 3-ball in a bridge representation of k. We also relate, in the case $M^3 \cong S^3$, the bridge number of k with the bridge number of $\varphi^{-1}(k)$.

1. Introduction

The problem of the classification of the 3-manifolds starts with the construction of all 3-manifolds. One attractive way to construct all 3-manifolds is through the classic result on branched coverings: Each closed connected orientable 3-manifold is a branched covering of the 3-sphere branched along some link in S^3.

Given a link $k \subset S^3$, the equivalence classes of branched coverings $\varphi : M \to (S^3, k)$ are in 1-1 correspondence with the conjugacy classes of representations of the knot group of k into a finite symmetric group $\omega : \pi_1(S^3 - k) \to S_d$.

Two fundamental problems arise: given a combinatorial description $\omega : \pi_1(S^3 - k) \to S_d$ with associated covering $\varphi : M \to S^3$, first identify the manifold M, and second —a much more difficult and interesting problem—, compute the isotopy type of the link $\varphi^{-1}(k)$ in M.

Solutions for the first problem of identifying a covering manifold starting with combinatorial data, are well known by giving different descriptions of M. In this work we give a solution to the second problem.

Start with a branched covering $\varphi : S^3 \to (S^3, k)$. If k is drawn in an n-bridge representation, implying that there is a 3-ball $B \subset S^3$ such that k is the union of n unknotted properly embedded arcs in B and n arcs on ∂B, it is tempting to try to recover $\varphi^{-1}(k)$ from a drawing of $\varphi^{-1}(B)$. It is well known that this is possible if $\varphi^{-1}(B)$ is also a 3-ball (see [2]). If $\varphi^{-1}(B)$ is not a 3-ball, but a handlebody of positive genus, an arbitrary drawing (an arbitrary embedding) of $\varphi^{-1}(B)$ in S^3, is generally misleading.

We give a description of how to embed $\varphi^{-1}(B)$ in S^3 in the general case, and, therefore, we obtain a complete criterion to recover the link type of $\varphi^{-1}(k)$ from an embedding of $\varphi^{-1}(B)$ in S^3. In fact we describe how to embed 'faithfully' $\varphi^{-1}(B)$ in M for an arbitrary manifold M and an arbitrary branched covering $\varphi : M \to (S^3, k)$. Technically we describe how to extend an embedding $\varphi^{-1}(B) \subset M$ to both, a homeomorphism $f : M \to M$ and a branched covering $M \to (S^3, k)$ which is equivalent, through f, to the original covering φ. For this we impose mild sufficient (and necessary) conditions on the embedding (Theorem 2.1). Surprisingly enough, this general result is useful for actual computations (see Example 2.12).

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In the special case $\varphi : S^3 \to (S^3, k)$, we also relate the bridge number of k with the bridge number of $\varphi^{-1}(k)$ (Theorem 2.3).

It is known that there are universal links (see, for example, [2], [5] and [4]). A link $k \subset S^3$ is *universal* if for each 3-manifold M, there is a branched covering $\varphi : M \to S^3$ such that the branching of φ is exactly k.

It is interesting both to find universal links, and to decide if a given link is universal. Once one knows that some links are universal, given a link $k \subset S^3$, a possible strategy to decide if k is universal, is to find a branched covering $\varphi : S^3 \to (S^3, k)$ and, then, hopefully, to find a known universal link as a subset of $\varphi^{-1}(k) \subset S^3$.

Following this idea, we use the Main Theorem to prove that the pretzel knot $p(3, 3, 3)$ is universal (Example 2.13).

In Section 2 we prove our main theorems, and we also give some applications. Example 2.9 is a convenient account on how to construct coverings of 3-balls branched along arcs, giving a tool to draw $\varphi^{-1}(B)$ for B a 3-ball and $\varphi : M \to (S^3, k)$ a branched covering.

2. Coverings of S^3

We write S_d for the symmetric group on d symbols. If $\sigma \in S_d$, we write $|\sigma|$ for the number of cycles in the disjoint cycle decomposition of σ in S_d. Recall that a *trivial n-tangle* $(B, \{\alpha_i\}_{i=1}^n)$ consists of a 3-ball B and a set of n disjoint properly embedded arcs $\alpha_1, \alpha_2, \ldots, \alpha_n \subset B$ such that there exists a set of n disjoint *trivializing 2-disks* $D_1, D_2, \ldots, D_n \subset B$ for the arcs $\{\alpha_i\}$; that is, for each $i = 1, 2, \ldots, n$, D_i is an embedded 2-disk in B with $int(D_i) \subset int(B)$, and $\partial D_i = \alpha_i \cup a_i$, and $D_i \cap \partial B = a_i \subset \partial B$ an arc.

Let $k \subset S^3$ be a link, and assume that we have an n-bridge representation of k, that is, there is a 3-ball $B \subset S^3$ such that $(B, B \cap k)$ and $(\overline{S^3 - B}, (\overline{S^3 - B}) \cap k)$ are trivial n-tangles. Let $D_1, \ldots, D_n \subset \overline{S^3 - B}$ be a set of n trivializing disks with $D_i \cap \partial(\overline{S^3 - B}) = b_i$ an arc with endpoints in k $(i = 1, \ldots, n)$. We obtain a link $\ell = (B \cap k) \cup (\bigsqcup_{i=1}^n b_i)$ such that $\ell \subset B$, and $\ell \sim k$. The pair (B, ℓ) is called a *$2n$-gonal pillowcase* for k.

Let $\omega : \pi_1(S^3 - k) \to S_d$ be a transitive representation, and let $\varphi = \varphi_\omega : M \to (S^3, k)$ be the induced d-fold branched covering. The representation ω induces, by restriction, a transitive representation $\omega : \pi_1(B - B \cap k) \to S_d$, and we obtain the corresponding d-fold branched covering $\psi = \psi_\omega : B^\omega \to (B, B \cap k)$ as in Example 2.9 below. Notice that $B^\omega \cong \varphi^{-1}(B)$.

Write $B \cap k = \bigsqcup_{i=1}^n \alpha_i$, a disjoint union of properly embedded arcs in B. For $i = 1, \ldots, n$, let $\mu_i \in \pi_1(B - \bigsqcup_{i=1}^n \alpha_i)$ be the meridian around the arc α_i, and let us write $\omega(\mu_i) = \sigma_{i,1}\sigma_{i,2} \cdots \sigma_{i,|\omega(\mu_i)|} \in S_d$ for the disjoint cycle decomposition in S_d. In a $2n$-gonal pillowcase (B, ℓ) for k, if $b_j \subset \ell \cap \partial B$ is an arc component sharing an endpoint with an arc α_i, then the preimage $\psi^{-1}(b_j)$ is a disjoint union of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_{|\omega(\mu_i)|} \subset \partial B^\omega$ such that each Γ_m has just two vertices and as many edges as $order(\sigma_{i,m})$, each edge connecting both vertices. Let us call a *ramification graph* on ∂B^ω any such graph Γ_m. By drawing a small cycle on ∂B^ω around one of the vertices of Γ_m, we can order cyclically the edges of Γ_m, and we can talk (unambiguously) of pairs of adjacent edges of Γ_m on ∂B^ω. Let us call a *ramification cycle* on ∂B^ω the isotopy class on ∂B^ω of any pair of adjacent edges of a graph Γ_m. In case $order(\sigma_{i,m}) = 1$, implying that Γ_m consists of just one edge, a *ramification cycle* is the isotopy class of the boundary of a small 2-disk regular neighbourhood of Γ_m on ∂B^ω (though, in this latter case, we should talk more appropriately of a *pseudo-ramification cycle*).

For each ramification graph Γ on ∂B^ω choose an edge $\tilde{b}_\Gamma \subset \Gamma$ (any edge will serve); then $\tilde{\ell} = \psi^{-1}(B\cap k) \cup \bigcup\{\tilde{b}_\Gamma : \Gamma \text{ is a ramification graph}\}$ is a 1-manifold that we call a *cleansing* of $\psi^{-1}(\ell)$ in B^ω.

Now assume that $k \subset S^3$ has c components, $k = k_1 \sqcup \cdots \sqcup k_c$. Let us write n_m for the number of components $|k_m \cap B|$ $(m = 1, \ldots, c)$. Write again $B \cap k = \bigsqcup_{i=1}^n \alpha_i$, and let $\mu_i \in \pi_1(B - \bigsqcup_{i=1}^n \alpha_i)$ be the meridian around the arc α_i. If α_i and α_j are contained in the same component k_m of k, then the number of cycles $|\omega(\mu_i)| = |\omega(\mu_j)|$; let us write $|k_m|$ for this common number $(m = 1, \ldots, c)$.

Theorem 2.1. *Let $k \subset S^3$ be a link in an n-bridge representation and let (B, ℓ) be a $2n$-gonal pillowcase for k. Let $\omega : \pi_1(S^3 - k) \to S_d$ be a transitive representation, and let $\varphi : M \to (S^3, k)$ and $\psi : B^\omega \to (B, B \cap k)$ be the induced d-fold branched coverings.*

If there exists an embedding $\varepsilon : B^\omega \hookrightarrow M$ such that the ramification cycles on $\varepsilon(\partial B^\omega)$ bound disjoint 2-cells in $\overline{M - \varepsilon(B^\omega)}$, then any homeomorphism $\varepsilon(B^\omega) \cong \varphi^{-1}(B)$ can be extended to a homeomorphism of pairs $(M, \tilde{\ell}) \cong (M, \varphi^{-1}(k))$ for $\tilde{\ell}$ any cleansing of $\varepsilon(\psi^{-1}(\ell))$.

Proof. We identify $\varepsilon(B^\omega)$ with B^ω. First notice that, by hypothesis, any two cleansings of $\psi^{-1}(\ell)$ are of the same link type in M.

Write $\ell = (\bigsqcup_{i=1}^n \alpha_i) \cup (\bigsqcup_{i=1}^n b_i)$ with $\alpha_i \subset B$ a properly embedded arc, and $b_i \subset \partial B$, $i = 1, \ldots, n$. For $i \in \{1, \ldots, n\}$, let $T_i \subset \overline{S^3 - B}$ be the 2-handle attached to B along the boundary of a small regular neighbourhood e_i of the arc b_i on ∂B; T_i is defined by the boundary of a regular neighbourhood, in $S^3 - B$, of the trivializing 2-disk corresponding to b_i. Now for $j = 1, \ldots, d$, and $i = 1, \ldots, n$, write $\tilde{T}_i^j$ for the j-th lifting of T_i in $\varphi^{-1}(T_i)$; $\tilde{T}_i^j$ is a 2-handle attached to $\varphi^{-1}(B)$ along the j-th lifting ∂e_i^j of ∂e_i in $\varphi^{-1}(e_i) \subset \partial\varphi^{-1}(B)$. Also attach a 2-handle $\tilde{R}_i^j \subset \overline{M - B^\omega}$ to B^ω along the j-th lifting ∂e_i^j of ∂e_i in $\psi^{-1}(e_i)$; this is possible, for ∂e_i^j is parallel to a ramification cycle on B^ω, and, by hypothesis, it bounds a 2-cell in $\overline{M - B^\omega}$. We can then extend the homeomorphism $B^\omega \cong \varphi^{-1}(B)$ to a homeomorphism $B^\omega \cup \bigsqcup \tilde{R}_i^j \cong \varphi^{-1}(B) \cup \bigsqcup \tilde{T}_j^i$.

Since $\overline{S^3 - (B \cup \bigsqcup T_i)} = E_1 \sqcup \cdots \sqcup E_n \sqcup E_{n+1}$ is a disjoint union of 3-balls such that $E_{n+1} \cap k = \emptyset$, and $E_i \cap k$ is an arc β_i in $\overline{S^3 - B}$ for $i = 1, \ldots, n$, using Lemma 2.11, it follows that $\overline{M - (\varphi^{-1}(B) \cup \bigsqcup \tilde{T}_j^i)}$ is a disjoint union of $(n_1|k_1| + \cdots + n_c|k_c| + d)$ 3-balls.

It follows that $\overline{M - (B^\omega \cup \bigsqcup \tilde{R}_i^j)}$ is also a disjoint union of the same number of 3-balls; otherwise, if some component of $\overline{M - (B^\omega \cup \bigsqcup \tilde{R}_i^j)}$ is not a 3-ball, then we would be able to construct two prime decompositions of M with different lengths (one using $\varphi^{-1}(B) \cup \bigsqcup \tilde{T}_j^i \subset M$, and, the second, using $B^\omega \cup \bigsqcup \tilde{R}_i^j \subset M$), contradicting uniqueness of prime decompositions.

Therefore we can extend $B^\omega \cup \bigsqcup \tilde{R}_i^j \cong \varphi^{-1}(B) \cup \bigsqcup \tilde{T}_j^i$ to a homeomorphism $F : M \to M$. Now $F^{-1}(\varphi^{-1}(k))$ is of the same link type as a cleansing $\tilde{\ell}$ of $\psi^{-1}(\ell)$, for a component $\tilde{E}$ of $\varphi^{-1}(E_i)$ of a ball E_i intersecting k in one arc, intersects $\varphi^{-1}(k)$ also in just one unknotted arc $\tilde{\beta}$ (use a lifting of a trivializing disk for $E_i \cap k$ in E_i for unknottedness). The preimage $F^{-1}(\tilde{\beta})$ is also an unknotted arc in the ball

$F^{-1}(\tilde{E})$ which connects two ends of the arcs of $\psi^{-1}(\bigsqcup_{i=1}^{n}\alpha_i)$; therefore $F^{-1}(\tilde{\beta})$ can be pushed, with fixed endpoints, into an edge of a ramification graph. Therefore $F:(M,\tilde{\ell})\to(M,\varphi^{-1}(k))$ is a homeomorphism of pairs. □

Remark 2.2. Notice that in the proof of Theorem 2.1, we can at the same time extend $\psi : B^\omega \to B$ to a branched covering $\psi : M \to (S^3,k)$, and that the homeomorphism constructed in Theorem 2.1 is an equivalence of branched coverings between φ and ψ.

Theorem 2.3. *Let $k\subset S^3$ be a link of c components in an n-bridge representation and let (B,ℓ) be a $2n$-gonal pillowcase for k. Let $\omega:\pi_1(S^3-k)\to S_d$ be a transitive representation, and assume $\varphi:S^3\to(S^3,k)$ and $\psi:B^\omega\to(B,B\cap k)$ are the induced d-fold branched coverings.*

If there exists an embedding $\varepsilon : B^\omega \hookrightarrow S^3$ such that the ramification cycles on $\varepsilon(B^\omega)$ bound disjoint 2-cells in $\overline{S^3-\varepsilon(B^\omega)}$, then there are g disjoint 2-handles $T_1,T_2,\dots,T_g\subset\overline{S^3-\varepsilon(B^\omega)}$ attached along some ramification cycles on $\varepsilon(B^\omega)$ such that the pair

$$(\varepsilon(B^\omega)\cup\bigsqcup_{j=1}^{g}T_j,\varepsilon(\psi^{-1}(B\cap k)))$$

is a trivial $\left(\sum_{j=1}^{c}n_j|k_j|\right)$–tangle, where $g=1+d(n-1)-\sum_{j=1}^{c}n_j|k_j|$ is the genus of B^ω.

In particular $\varphi^{-1}(k)$ admits a $(\sum_{j=1}^{c}n_j|k_j|)$–bridge representation.

Proof. We identify again $\varepsilon(B^\omega)$ with B^ω. We compute, by the Riemann-Hurwitz formula, $genus(\partial B^\omega)=1+d(n-1)-\sum_{j=1}^{c}n_j|k_j|=g$.

In the proof of Theorem 2.1 we attached dn 2-handles $T_1,\dots,T_{d\cdot n}\subset\overline{S^3-B^\omega}$ to ∂B^ω; write $T_i=E_i\times I$ with E_i a 2-cell. The result $X=B^\omega\cup\bigsqcup T_i$ is the 3-sphere punctured $(d+\sum_{j=1}^{c}n_j|k_j|)$ times. Equivalently, X is a $(d+\sum_{j=1}^{c}n_j|k_j|-1)$ times punctured 3-ball. Each boundary component of X always contains disks of the boundaries of the 2-handles of the form $E_i\times\{0\}$ or $E_i\times\{1\}$, and sometimes contains pieces of ∂B^ω. Then if we take out $d+\sum_{j=1}^{c}n_j|k_j|-1$ 2-handles from X (one for each 'inner' 2-sphere of ∂X), we are left with a 3-ball $X_\circ=\overline{X-\bigsqcup_{i\in K}T_i}$ for some subset $K\subset\{1,2,\dots,dn\}$ of cardinality $d+\sum_{j=1}^{c}n_j|k_j|-1$. By renumbering the 2-handles we may assume that $K=\{g+1,g+2,\cdots,d\cdot n\}$. But then $X_\circ$ is the result of attaching $g=dn-(d+\sum_{j=1}^{c}n_j|k_j|-1)$ 2-handles to ∂B^ω. We conclude that $(B^\omega,\{T_i\}_{i=1}^{g})$ defines a Heegaard splitting of the 3-sphere.

Write $B\cap k=\bigsqcup_{i=1}^{n}\alpha_i$, and let $D_1,\dots,D_n\subset B$ be the trivializing 2-disks for the arcs $\alpha_1,\dots,\alpha_n\subset B$. For $i=1,\dots,n$, the preimage $\psi^{-1}(D_i)$ is a union of liftings of D_i, say, $\psi^{-1}(D_i)=\bigcup_{j=1}^{order(\sigma_{i,1})}D_{(i,1,j)}\sqcup\bigcup_{j=1}^{order(\sigma_{i,2})}D_{(i,2,j)}\sqcup\cdots\sqcup\bigcup_{j=1}^{order(\sigma_{i,|\omega(\mu_i)|})}D_{(i,|\omega(\mu_i)|,j)}$, and write $\psi^{-1}(\alpha_i)=\tilde{\alpha}_{i,1}\sqcup\cdots\sqcup\tilde{\alpha}_{i,|\omega(\mu_i)|}$; we are choosing numberings in such a way that $\bigcap_{j=1}^{order(\sigma_{i,m})}D_{(i,m,j)}=\tilde{\alpha}_{i,m}$; therefore each $D_{(i,m,j)}$ is a trivializing 2-disk for $\tilde{\alpha}_{i,m}$ in B^ω, for $\partial D_{(i,m,j)}=\tilde{\alpha}_{i,m}\cup a_{i,m}$ with $a_{i,m}\subset B^\omega$ an arc $(m=1,\dots,q_i)$. It follows that $H=\overline{B^\omega-\bigsqcup_{i,m}\mathcal{N}(\tilde{\alpha}_{i,m})}$ is a handlebody where $\mathcal{N}(\tilde{\alpha}_{i,m})$ is a small regular neighbourhood of $\tilde{\alpha}_{i,m}$ in B^ω $(i=1\dots,n;\ m=1,\dots,|\omega(\mu_i)|)$.

Now notice that $\mathcal{N}(\tilde{\alpha}_{i,m})$ is a 2-handle attached to ∂H, and write $\mathcal{N}(\tilde{\alpha}_{i,m})=N_{i,m}\times I$ with $N_{i,m}$ a 2-cell such that $N_{i,m}\cap\tilde{\alpha}_{i,m}$ is a single (transverse) point. Since for any triple (i, m, j), by construction, the 2-handle T_j does not intersect $\tilde{\alpha}_{i,m}$, we conclude that $(H, \{T_j\}_j \cup \{\mathcal{N}(\tilde{\alpha}_{i,m})\}_{i,m})$ also defines a Heegaard splitting for S^3. By Waldhausen ([6]), there is a set of meridians $F_1, \ldots, F_g, F_{1,1}, F_{1,2}, \ldots, F_{n,q_n} \subset H$ trivializing the Heegaard splitting; that is, $F_i \cap E_j = \delta_i^j S_i^j$, $F_i \cap N_{r,s} = \emptyset$, $F_{r,s} \cap E_j = \emptyset$, and $F_{r,s} \cap N_{u,v} = \delta_{r,s}^{u,v} S_{r,s}^{u,v}$ where S_i^j and $S_{r,s}^{u,v}$ are one-element sets, and the symbol $\delta_A^B Y$ is empty if $A \neq B$, and is Y otherwise. We see that the meridians $\{F_{i,m}\}_{i,m}$ define a set of $\sum_{j=1}^{c} n_j|k_j|$ trivializing disks for the 3-ball $B^\omega \cup \bigsqcup_{j=1}^{g} T_j$, giving us the conclusion of the theorem. □

Remark 2.4. In the context of Theorem 2.1, we see that the handlebody B^ω and the set of ramification cycles on ∂B^ω induce a Heegaard diagram for M: Just follow the first two paragraphs of the proof of Theorem 2.3 replacing S^3 for M. This is useful to identify the manifold M.

Remark 2.5. It is possible to obtain an analogous statement of Theorem 2.3 for arbitrary branched coverings $\varphi : M \to (S^3, k)$ and 'generalized' trivial tangles $(V, \{\alpha_i\})$ in M, where V is a handlebody. This seems to be interesting as in [1].

Remark 2.6. As in Remark 2.4, in the induced Heegaard diagram for M, if on the surface ∂B^ω we keep all ramification cycles and we add some meridians of B^ω, this induced diagram is an *admissible pointed Heegaard diagram compatible with the link* $\varphi^{-1}(k)$ as in [3].

Remark 2.7. If $k \subset S^3$ is a knot, then the conclusion of Theorem 2.3 is that $\varphi^{-1}(k)$ admits an $n|\omega(\mu)|$ bridge representation with μ a meridian of k.

Remark 2.8. By locating the different components $\varphi^{-1}(k) = \tilde{k}_1 \sqcup \tilde{k}_2 \sqcup \cdots$ in Theorem 2.3, the upper bound for the bridge number of each $\tilde{k}_i$ can be easily improved. For example if $k \subset S^3$ is an n-bridge knot and $\varphi : S^3 \to (S^3, k)$ is a 3-fold simple covering, then both the branch and the pseudo-branch components of $\varphi^{-1}(k)$ admit an n-bridge representation.

Example 2.9. *Coverings of trivial tangles.* Let $(B, \{\alpha_i\}_{i=1}^n)$ be a trivial n-tangle, and let $\omega : \pi_1(B - \bigsqcup \alpha_i) \to S_d$ be a representation. We will describe $\psi = \psi_\omega : B^\omega \to (B, \bigsqcup \alpha_i)$, the d-fold branched covering corresponding to the representation ω.

For $i = 1, \ldots, n$, let $\mu_i \in \pi_1(B - \bigsqcup \alpha_i)$ be the meridian that goes around the arc α_i. Assume $\omega(\mu_i) = \sigma_{i,1}\sigma_{i,2} \cdots \sigma_{i,|\omega(\mu_i)|} \in S_d$ is the disjoint cycle decomposition of $\omega(\mu_i)$ in S_d.

Let $D_1, \ldots, D_n \subset B$ be a set of disjoint trivializing 2-disks with $\partial D_i = \alpha_i \cup a_i$, and $a_i \subset \partial B$ $(i = 1, \ldots, n)$. Let $\hat{B}$ be the result of cutting B along the disks $D_1, \ldots, D_n$. For each $i = 1, \ldots, n$, we have two copies, D_i^+ and D_i^-, of D_i in $\partial \hat{B}$ such that $D_i^+ \cap D_i^-$ is a copy of α_i. We also have a quotient map $p : \hat{B} \to B$ which identifies D_i^+ with D_i^-, defining a homeomorphism $h_i : D_i^+ \to D_i^-$.

Now consider d copies, $\hat{B}_1, \ldots, \hat{B}_d$, of $\hat{B}$, and let $p_1 : \hat{B}_1 \to B, \ldots, p_d : \hat{B}_d \to B$ be d copies of the quotient map p. Fix $i \in \{1, \ldots, n\}$. For each $j \in \{1, 2, \ldots, q_i\}$, if $\sigma_{i,j} = (a_1, a_2, \ldots, a_r) \in S_d$, we identify the disk D_i^+ in $\partial \hat{B}_{a_m}$ with the disk D_i^- in $\partial \hat{B}_{a_{m+1}}$ (subindices of the a_m are taken modulo r) using the homeomorphism $h_i : D_i^+ \to D_i^-$ $(m = 1, \ldots, r)$.

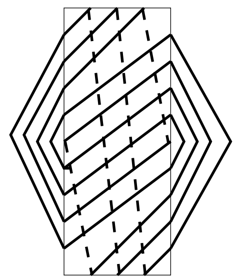

FIGURE 1. The Figure Eight Knot.

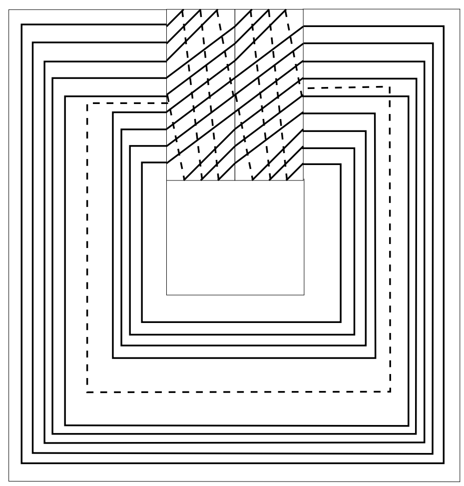

FIGURE 2

We call B^ω the resulting space of all these identifications ($i = 1, \ldots, n$), and we define $\psi : B^\omega \to B$ as the union $\psi = \bigcup_{j=1}^{d} p_j$. Then $\psi = \psi_\omega$ is the d-fold branched covering of $(B, \sqcup\alpha_i)$ corresponding to the representation ω.

The following remarks are upgraded to 'lemmas' just for reference purposes.

Lemma 2.10. *If $(B, \{\alpha_i\})$ is a trivial n-tangle, and $\omega : \pi_1(B - \bigsqcup \alpha_i) \to S_d$ is a representation, then B^ω is a disjoint union of handlebodies.*

Lemma 2.11. *If $(B, \{\alpha\})$ is a trivial 1-tangle, and μ is a meridian around the arc α, and $\omega : \pi_1(B - \alpha) \to S_d$ is a representation, then B^ω is a disjoint union of $|\omega(\mu)|$ 3-balls.*

Example 2.12. In Figure 1 appears the Figure Eight Knot in a square pillowcase, where the inner arcs of the ball B are orthogonal to the plane of the paper. For the double branched covering, that is known to be the lens space $L(5,3)$, we construct the handlebody B^ω depicted in Figure 2 with all its ramification graphs included (in this case $\omega(\mu) = (1,2) \in S_2$ for each meridian μ). A typical ramification cycle looks as drawn in Figure 3. We construct the embedding $B^\omega \hookrightarrow L(5,3)$ as depicted in Figure 4. This is a drawing in the 3-sphere where we have to perform surgery along the circle with attached surgery coefficient $5/3$. Going to the universal cover of

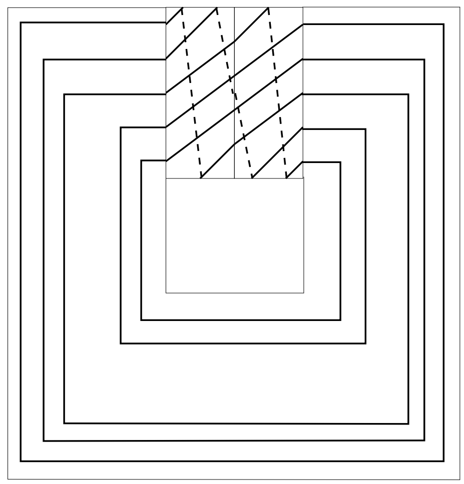

FIGURE 3

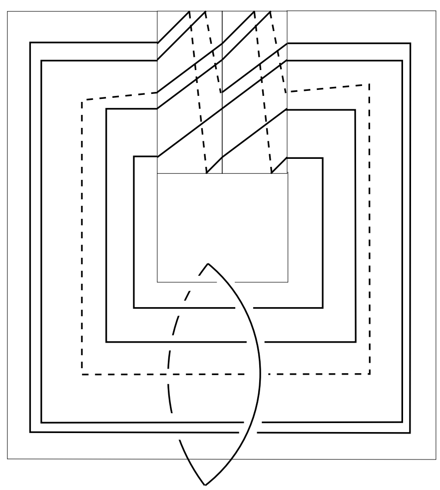

FIGURE 4

$L(5,3)$ we obtain Figure 5, where we still have to perform $1/3$ surgery. And finally we obtain the link in Figure 6 which is the preimage of the Figure Eight Knot under the regular dihedral covering of S^3 branched along this knot (*cf.* Figure 3 and 4 of [7]).

Example 2.13. In Figure 7 appears the pretzel knot $k = p(3,3,3)$ in an hexagonal pillowcase, where again the inner arcs of the ball B are orthogonal to the plane of the paper. We have the representation $\omega : \pi_1(S^3 - k) \to S_6$ such that $\omega(c_1) = (2,4,5)$, $\omega(c_4) = (1,6,4)$ and $\omega(c_7) = (1,2,3)$, where c_1, c_4, c_7 are the meridians of the inner arcs of B.

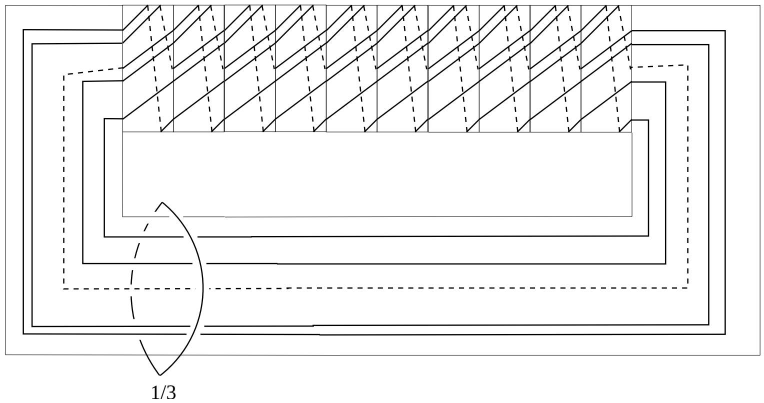

FIGURE 5

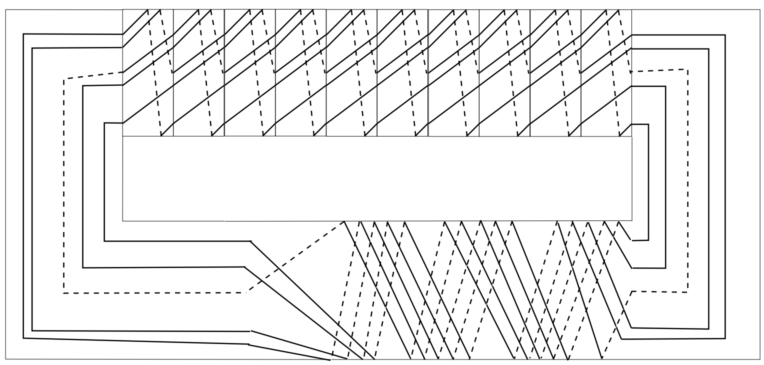

FIGURE 6

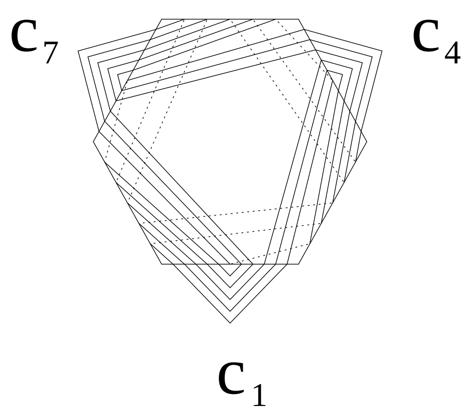

FIGURE 7. The Pretzel Knot $p(3, 3, 3)$.

It can be computed that the covering associated to ω is a homotopy 3-sphere, and from the drawing of B^ω in Figure 8, we see that it actually is the 3-sphere (it is a lens space).

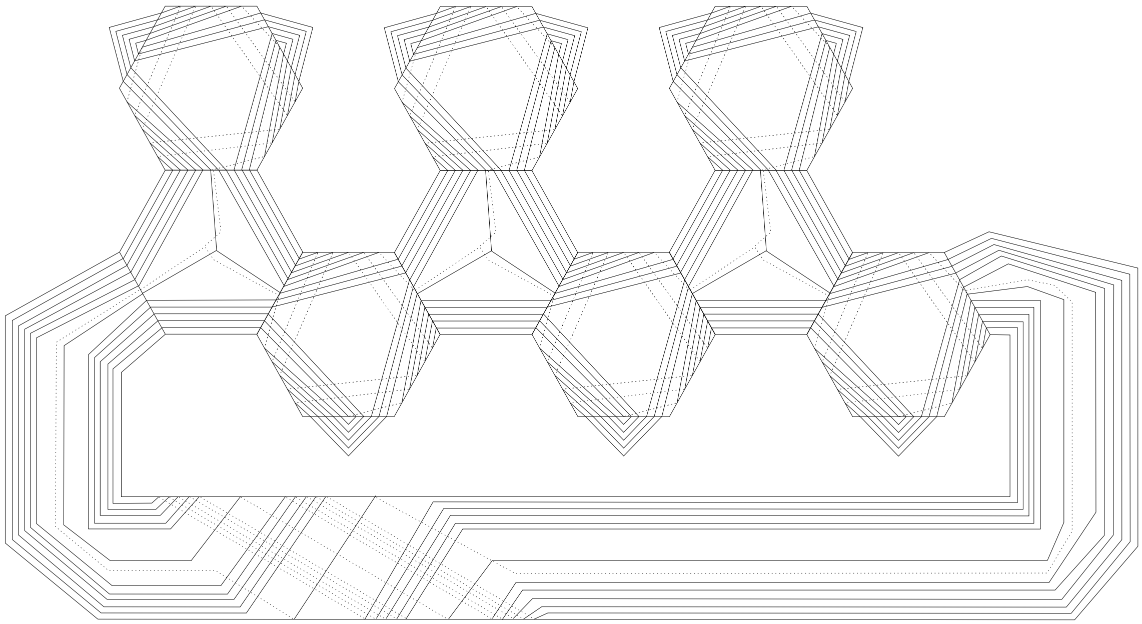

FIGURE 8

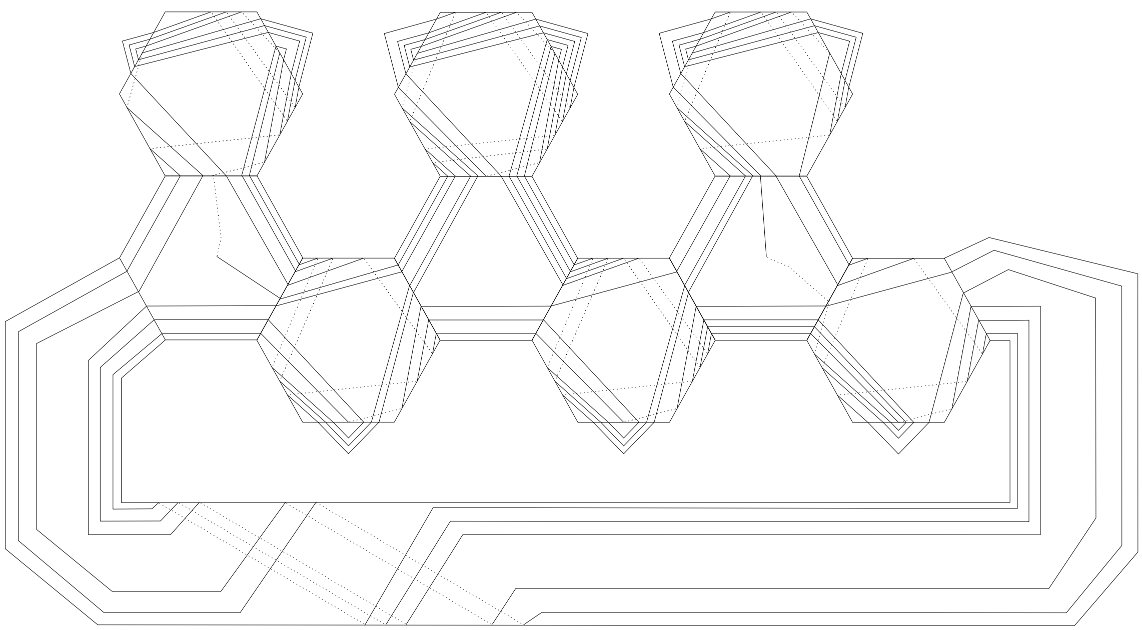

FIGURE 9

The drawing in Figure 8 satisfies the hypothesis of Theorem 2.1, and in Figure 9 we have a cleansing, and, therefore, an actual drawing of the preimage of k in S^3.

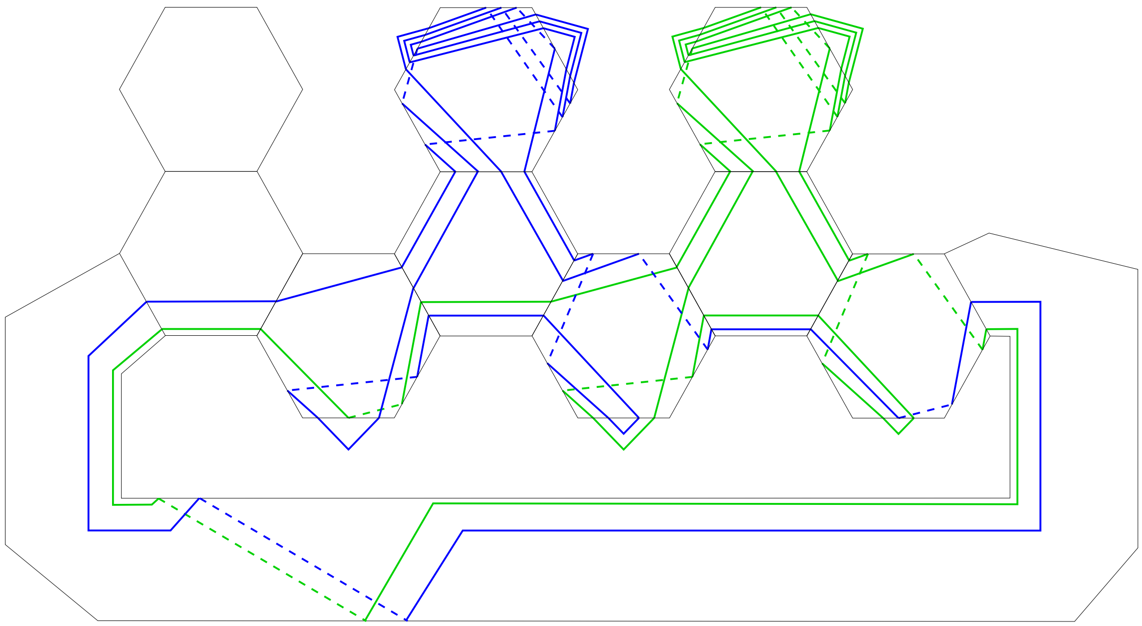

FIGURE 10

In Figure 10 are depicted only two components of the pseudo-branch that can be seen to be the Montesinos knot $m(2/7, 1, 2/7, 3) \sim m(9/7, 23/7) \sim m(-224/97)$; since this is a hyperbolic 2-bridge link, it is universal ([2]). That shows that *the pretzel knot* $p(3, 3, 3) = 9_{35}$ *is a universal knot.* From the results in [4], this reduces to nine the number of Montesinos knots up to 10 crossings that have so far undecided universality.

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CIMAT, A.P. 402, GUANAJUATO 36000, GTO., MÉXICO
E-mail address: `victor@cimat.mx`

CIMAT, A.P. 402, GUANAJUATO 36000, GTO., MÉXICO
E-mail address: `mjordan@cimat.mx`