1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto $S^{n-1}$.

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

4. A deformation retraction in the weak sense of a space $X$ to a subspace $A$ is a homotopy $f_t : X \to X$ such that $f_0 = 1$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all $t$. Show that if $X$ deformation retracts to $A$ in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

5. Show that if a space $X$ deformation retracts to a point $x \in X$, then for each neighborhood $U$ of $x$ in $X$ there exists a neighborhood $V \subset U$ of $x$ such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

6. (a) Let $X$ be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0, 1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1 - r]$ for $r$ a rational number in $[0, 1]$. Show that $X$ deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point. [See the preceding problem.]

(b) Let $Y$ be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of $X$ arranged as in the figure below. Show that $Y$ is contractible but does not deformation retract onto any point.

(c) Let $Z$ be the zigzag subspace of $Y$ homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of $Y$ onto $Z$, but no true deformation retraction.

7. Fill in the details in the following construction from [Edwards 1999] of a compact space $Y \subset \mathbb{R}^3$ with the same properties as the space $Y$ in Exercise 6, that is, $Y$ is contractible but does not deformation retract to any point. To begin, let $X$ be the union of an infinite sequence of cones on the Cantor set arranged end-to-end, as in the figure. Next, form the one-point compactification of $X \times \mathbb{R}$. This embeds in $\mathbb{R}^3$ as a closed disk with curved ‘fins’ attached along
circular arcs, and with the one-point compactification of \( X \) as a cross-sectional slice. The desired space \( Y \) is then obtained from this subspace of \( \mathbb{R}^3 \) by wrapping one more cone on the Cantor set around the boundary of the disk.

8. For \( n > 2 \), construct an \( n \)-room analog of the house with two rooms.

9. Show that a retract of a contractible space is contractible.

10. Show that a space \( X \) is contractible iff every map \( f : X \to Y \), for arbitrary \( Y \), is nullhomotopic. Similarly, show \( X \) is contractible iff every map \( f : Y \to X \) is nullhomotopic.

11. Show that \( f : X \to Y \) is a homotopy equivalence if there exist maps \( g, h : Y \to X \) such that \( fg \simeq 1 \) and \( hf \simeq 1 \). More generally, show that \( f \) is a homotopy equivalence if \( fg \) and \( hf \) are homotopy equivalences.

12. Show that a homotopy equivalence \( f : X \to Y \) induces a bijection between the set of path-components of \( X \) and the set of path-components of \( Y \), and that \( f \) restricts to a homotopy equivalence from each path-component of \( X \) to the corresponding path-component of \( Y \). Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space \( X \) coincide with its path-components, then the same holds for any space \( Y \) homotopy equivalent to \( X \).

13. Show that any two deformation retractions \( r_t^0 \) and \( r_t^1 \) of a space \( X \) onto a subspace \( A \) can be joined by a continuous family of deformation retractions \( r_t^s \), \( 0 \leq s \leq 1 \), of \( X \) onto \( A \), where continuity means that the map \( X \times I \times I \to X \) sending \((x, s, t)\) to \( r_t^s(x) \) is continuous.

14. Given positive integers \( v, e, \) and \( f \) satisfying \( v - e + f = 2 \), construct a cell structure on \( S^2 \) having \( v \) 0-cells, \( e \) 1-cells, and \( f \) 2-cells.

15. Enumerate all the subcomplexes of \( S^\infty \), with the cell structure on \( S^\infty \) that has \( S^n \) as its \( n \)-skeleton.

16. Show that \( S^\infty \) is contractible.

17. (a) Show that the mapping cylinder of every map \( f : S^1 \to S^1 \) is a CW complex.
    (b) Construct a 2-dimensional CW complex that contains both an annulus \( S^1 \times I \) and a Möbius band as deformation retracts.

18. Show that \( S^1 \ast S^1 = S^3 \), and more generally \( S^m \ast S^n = S^{m+n+1} \).

19. Show that the space obtained from \( S^2 \) by attaching \( n \) 2-cells along any collection of \( n \) circles in \( S^2 \) is homotopy equivalent to the wedge sum of \( n + 1 \) 2-spheres.

20. Show that the subspace \( X \subset \mathbb{R}^3 \) formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to \( S^1 \vee S^1 \vee S^2 \).

21. If \( X \) is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that \( X \) is homotopy equivalent to a wedge sum of \( S^1 \)'s and \( S^2 \)'s.
22. Let \( X \) be a finite graph lying in a half-plane \( P \subset \mathbb{R}^3 \) and intersecting the edge of \( P \) in a subset of the vertices of \( X \). Describe the homotopy type of the ‘surface of revolution’ obtained by rotating \( X \) about the edge line of \( P \).

23. Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

24. Let \( X \) and \( Y \) be CW complexes with 0-cells \( x_0 \) and \( y_0 \). Show that the quotient spaces \( X \ast Y/(X \ast \{y_0\} \cup \{x_0\} \ast Y) \) and \( S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\}) \) are homeomorphic, and deduce that \( X \ast Y \cong S(X \wedge Y) \).

25. If \( X \) is a CW complex with components \( X_\alpha \), show that the suspension \( SX \) is homotopy equivalent to \( Y \setminus \bigcup_\alpha S X_\alpha \) for some graph \( Y \). In the case that \( X \) is a finite graph, show that \( SX \) is homotopy equivalent to a wedge sum of circles and 2-spheres.

26. Use Corollary 0.20 to show that if \( (X, A) \) has the homotopy extension property, then \( X \times I \) deformation retracts to \( X \times \{0\} \cup A \times I \). Deduce from this that Proposition 0.18 holds more generally for any pair \( (X_1, A) \) satisfying the homotopy extension property.

27. Given a pair \((X, A)\) and a homotopy equivalence \( f : A \to B \), show that the natural map \( X \to B \sqcup_f X \) is a homotopy equivalence if \((X, A)\) satisfies the homotopy extension property. [Hint: Consider \( X \cup M_f \) and use the preceding problem.] An interesting case is when \( f \) is a quotient map, hence the map \( X \to B \sqcup_f X \) is the quotient map identifying each set \( f^{-1}(b) \) to a point. When \( B \) is a point this gives another proof of Proposition 0.17.

28. Show that if \((X_1, A)\) satisfies the homotopy extension property, then so does every pair \((X_0 \sqcup_f X_1, X_0)\) obtained by attaching \( X_1 \) to a space \( X_0 \) via a map \( f : A \to X_0 \).

29. In case the CW complex \( X \) is obtained from a subcomplex \( A \) by attaching a single cell \( e^n \), describe exactly what the extension of a homotopy \( f_t : A \to Y \) to \( X \) given by the proof of Proposition 0.16 looks like. That is, for a point \( x \in e^n \), describe the path \( f_t(x) \) for the extended \( f_t \).